AN INTRODUCTION TO LINEAR ALGEBRA

FOR SCIENCE AND ENGINEERING

DANIEL NORMAN • DAN WOLCZUK

THIRD EDITION

An Introduction to Linear Algebra for Science and Engineering

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Third Edition

Pearson

Pearson Canada Inc., 26 Prince Andrew Place, North York, Ontario M3C 2H4.

9780134682631

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Library and Archives Canada Cataloguing in Publication

Norman, Daniel, 1938-, author
Introduction to linear algebra for science and engineering / Daniel Norman, Dan Wolczuk, University of Waterloo. – Third edition.

ISBN 978-0-13-468263-1 (softcover)

1. Algebras, Linear–Textbooks. 2. Textbooks. I. Wolczuk, Dan, 1972-, author II. Title.

QA184.2.N67 2018 512'.5 C2018-906600-8

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Table of Contents

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A Note to Students

Linear Algebra – What Is It?

Welcome to the third edition of *An Introduction to Linear Algebra for Science and Engineering*! Linear algebra is essentially the study of vectors, matrices, and linear mappings, and is now an extremely important topic in mathematics. Its application and usefulness in a variety of different areas is undeniable. It encompasses technological innovation, economic decision making, industry development, and scientific research. We are literally surrounded by applications of linear algebra.

Most people who have learned linear algebra and calculus believe that the ideas of elementary calculus (such as limits and integrals) are more difficult than those of introductory linear algebra, and that most problems encountered in calculus courses are harder than those found in linear algebra courses. So, at least by this comparison, linear algebra is not hard. Still, some students find learning linear algebra challenging. We think two factors contribute to the difficulty some students have.

First, students do not always see what linear algebra is good for. This is why it is important to read the applications in the text–even if you do not understand them completely. They will give you some sense of where linear algebra fits into the broader picture.

Second, mathematics is often mistakenly seen as a collection of recipes for solving standard problems. Students are often uncomfortable with the fact that linear algebra is "abstract" and includes a lot of "theory." However, students need to realize that there will be no long-term payoff in simply memorizing the recipes–computers carry them out far faster and more accurately than any human. That being said, practicing the procedures on specific examples is often an important step towards a much more important goal: understanding the *concepts* used in linear algebra to formulate and solve problems, and learning to interpret the results of calculations. Such understanding requires us to come to terms with some theory. In this text, when working through the examples and exercises – which are often small – keep in mind that when you do apply these ideas later, you may very well have a million variables and a million equations, but the theory and methods remain constant. For example, Google's PageRank system uses a matrix that has thirty billion columns and thirty billion rows – you do not want to do that by hand! **When you are solving computational problems, always try to observe how your work relates to the theory you have learned.**

Mathematics is useful in so many areas because it is *abstract*: the same good idea can unlock the problems of control engineers, civil engineers, physicists, social scientists, and mathematicians because the idea has been abstracted from a particular setting. One technique solves many problems because someone has established a *theory* of how to deal with these kinds of problems. *Definitions* are the way we try to capture important ideas, and *theorems* are how we summarize useful general facts about the kind of problems we are studying. *Proofs* not only show us that a statement is true; they can help us understand the statement, give us practice using important ideas, and make it easier to learn a given subject. In particular, proofs show us how ideas are tied together, so we do not have to memorize too many disconnected facts.

Many of the concepts introduced in linear algebra are natural and easy, but some may seem unnatural and "technical" to beginners. Do not avoid these seemingly more difficult ideas; use examples and theorems to see how these ideas are an essential part of the story of linear algebra. By learning the "vocabulary" and "grammar" of linear algebra, you will be equipping yourself with concepts and techniques that mathematicians, engineers, and scientists find invaluable for tackling an extraordinarily rich variety of problems.

Linear Algebra – Who Needs It?

Mathematicians

Linear algebra and its applications are a subject of continuing research. Linear algebra is vital to mathematics because it provides essential ideas and tools in areas as diverse as abstract algebra, differential equations, calculus of functions of several variables, differential geometry, functional analysis, and numerical analysis.

Engineers

Suppose you become a control engineer and have to design or upgrade an automatic control system. The system may be controlling a manufacturing process, or perhaps an airplane landing system. You will probably start with a linear model of the system, requiring linear algebra for its solution. To include feedback control, your system must take account of many measurements (for the example of the airplane, position, velocity, pitch, etc.), and it will have to assess this information very rapidly in order to determine the correct control responses. A standard part of such a control system is a Kalman-Bucy filter, which is not so much a piece of hardware as a piece of mathematical machinery for doing the required calculations. Linear algebra is an essential part of the Kalman-Bucy filter.

If you become a structural engineer or a mechanical engineer, you may be concerned with the problem of vibrations in structures or machinery. To understand the problem, you will have to know about eigenvalues and eigenvectors and how they determine the normal modes of oscillation. Eigenvalues and eigenvectors are some of the central topics in linear algebra.

An electrical engineer will need linear algebra to analyze circuits and systems; a civil engineer will need linear algebra to determine internal forces in static structures and to understand principal axes of strain.

In addition to these fairly specific uses, engineers will also find that they need to know linear algebra to understand systems of differential equations and some aspects of the calculus of functions of two or more variables. Moreover, the ideas and techniques of linear algebra are central to numerical techniques for solving problems of heat and fluid flow, which are major concerns in mechanical engineering. Also, the ideas of linear algebra underlie advanced techniques such as Laplace transforms and Fourier analysis.

Physicists

Linear algebra is important in physics, partly for the reasons described above. In addition, it is vital in applications such as the inertia tensor in general rotating motion. Linear algebra is an absolutely essential tool in quantum physics (where, for example, energy levels may be determined as eigenvalues of linear operators) and relativity (where understanding change of coordinates is one of the central issues).

Life and Social Scientists

Input-output models, described by matrices, are often used in economics and other social sciences. Similar ideas can be used in modeling populations where one needs to keep track of sub-populations (generations, for example, or genotypes). In all sciences, statistical analysis of data is of a great importance, and much of this analysis uses linear algebra. For example, the method of least squares (for regression) can be understood in terms of projections in linear algebra.

Managers and Other Professionals

All managers need to make decisions about the best allocation of resources. Enormous amounts of computer time around the world are devoted to linear programming algorithms that solve such allocation problems. In industry, the same sorts of techniques are used in production, networking, and many other areas.

Who needs linear algebra? Almost every mathematician, engineer, scientist, economist, manager, or professional will find linear algebra an important and useful. So, who needs linear algebra? You do!

Will these applications be explained in this book?

Unfortunately, most of these applications require too much specialized background to be included in a first-year linear algebra book. To give you an idea of how some of these concepts are applied, a wide variety of applications are mentioned throughout the text. You will get to see many more applications of linear algebra in your future courses.

How To Make the Most of This Book: SQ3R

The SQ3R reading technique was developed by Francis Robinson to help students read textbooks more effectively. Here is a brief summary of this powerful method for learning. It is easy to learn more about this and other similar strategies online.

Survey: Quickly skim over the section. Make note of any heading or boldface words. Read over the definitions, the statement of theorems, and the statement of examples or exercises (do not read proofs or solutions at this time). Also, briefly examine the figures.

Question: Make a purpose for your reading by writing down general questions about the headings, boldface words, definitions, or theorems that you surveyed. For example, a couple of questions for Section 1.1 could be:
How do we use vectors in $\mathbb{R}^2$ *and* $\mathbb{R}^3$*?*
How does this material relate to what I have previously learned?
What is the relationship between vectors in $\mathbb{R}^2$ *and directed line segments?*
What are the similarities and differences between vectors and lines in $\mathbb{R}^2$ *and in* $\mathbb{R}^3$*?*

Read: Read the material in chunks of about one to two pages. Read carefully and look for the answers to your questions as well as key concepts and supporting details. *Take the time to solve the mid-section exercises before reading past them. Also, try to solve examples before reading the solutions, and try to figure out the proofs before you read them.* If you are not able to solve them, look carefully through the provided solution to figure out the step where you got stuck.

Recall: As you finish each chunk, put the book aside and summarize the important details of what you have just read. Write down the answers to any questions that you made and write down any further questions that you have. Think critically about how well you have understood the concepts, and if necessary, go back and reread a part or do some relevant end of section problems.

Review: This is an ongoing process. Once you complete an entire section, go back and review your notes and questions from the entire section. Test your understanding by trying to solve the end-of-section problems without referring to the book or your notes. Repeat this again when you finish an entire chapter and then again in the future as necessary.

Yes, you are going to find that this makes the reading go much slower for the first couple of chapters. However, students who use this technique consistently report that they feel that they end up spending a lot less time studying for the course as they learn the material so much better at the beginning, which makes future concepts much easier to learn.

A Note to Instructors

Welcome to the third edition of *An Introduction to Linear Algebra for Science and Engineering*! Thanks to the feedback I have received from students and instructors as well as my own research into the science of teaching and learning, I am very excited to present to you this new and improved version of the text. Overall, I believe the modifications I have made complement my overall approach to teaching. I believe in introducing the students slowly to difficult concepts and helping students learn these concepts more deeply by exposing them to the same concepts multiple times over spaced intervals.

One aspect of teaching linear algebra that I find fascinating is that so many different approaches can be used effectively. Typically, the biggest difference between most calculus textbooks is whether they have early or late transcendentals. However, linear algebra textbooks and courses can be done in a wide variety of orders. For example, in China it is not uncommon to begin an introductory linear algebra course with determinants and not cover solving systems of linear equations until after matrices and general vector spaces. Examination of the advantages and disadvantages of a variety of these methods has led me to my current approach.

It is well known that students of linear algebra typically find the computational problems easy but have great difficulty in understanding or applying the abstract concepts and the theory. However, with my approach, I find not only that very few students have trouble with concepts like general vector spaces but that they also retain their mastery of the linear algebra content in their upper year courses.

Although I have found my approach to be very successful with my students, I see the value in a multitude of other ways of organizing an introductory linear algebra course. Therefore, I have tried to write this book to accommodate a variety of orders. See Using This Text To Teach Linear Algebra below.

Changes to the Third Edition

- Some of the content has been reordered to make even better use of the spacing effect. The spacing effect is a well known and extensively studied effect from psychology, which states that students learn concepts better if they are exposed to the same concept multiple times over spaced intervals as opposed to learning it all at once. See:

  Dempster, F.N. (1988). *The spacing effect: A case study in the failure to apply the results of psychological research.* American Psychologist, 43(8), 627–634.

  Fain, R. J., Hieb, J. L., Ralston, P. A., Lyle, K. B. (2015, June), *Can the Spacing Effect Improve the Effectiveness of a Math Intervention Course for Engineering Students?* Paper presented at 2015 ASEE Annual Conference & Exposition, Seattle, Washington.

- The number and type of applications has been greatly increased and are used either to motivate the need for certain concepts or definitions in linear algebra, or to demonstrate how some linear algebra concepts are used in applications.

- A greater emphasis has been placed on the geometry of many concepts. In particular, Chapter 1 has been reorganized to focus on the geometry of linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ before exploring $\mathbb{R}^n$.

- Numerous small changes have been made to improve student comprehension.

Approach and Organization

Students of linear algebra typically have little trouble with computational questions, but they often struggle with abstract concepts and proofs. This is problematic because computers perform the computations in the vast majority of real world applications of linear algebra. Human users, meanwhile, must apply the theory to transform a given problem into a linear algebra context, input the data properly, and interpret the result correctly.

The approach of this book is both to use the spacing effect and to mix theory and computations throughout the course. Additionally, it uses real world applications to both motivate and explain the usefulness of some of the seemingly abstract concepts, and it uses the geometry of linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ to help students visualize many of the concepts. The benefits of this approach are as follows:

- It prevents students from mistaking linear algebra as very easy and very computational early in the course, and then getting overwhelmed by abstract concepts and theories later.

- It allows important linear algebra concepts to be developed and extended more slowly.

- It encourages students to use computational problems to help them understand the theory of linear algebra rather than blindly memorize algorithms.

- It helps students understand the concepts and why they are useful.

One example of this approach is our treatment of the concepts of spanning and linear independence. They are both introduced in Section 1.2 in $\mathbb{R}^2$ and $\mathbb{R}^3$, where they are motivated in a geometrical context. They are expanded to vectors in $\mathbb{R}^n$ in Section 1.4, and used again for matrices in Section 3.1 and polynomials in Section 4.1, before they are finally extended to general vector spaces in Section 4.2.

Other features of the text's organization include

- The idea of linear mappings is introduced early in a geometrical context, and is used to explain aspects of matrix multiplication, matrix inversion, features of systems of linear equations, and the geometry of eigenvalues and eigenvectors. Geometrical transformations provide intuitively satisfying illustrations of important concepts.

- Topics are ordered to give students a chance to work with concepts in a simpler setting before using them in a much more involved or abstract setting. For example, before reaching the definition of a vector space in Section 4.2, students will have seen the ten vector space axioms and the concepts of linear independence and spanning for three different vectors spaces, and will have had some experience in working with bases and dimensions. Thus, instead of being bombarded with new concepts at the introduction of general vector spaces, students will just be generalizing concepts with which they are already familiar.

Pedagogical Features

Since mathematics is best learned by doing, the following pedagogical elements are included in the text:

- A selection of routine mid-section exercises are provided, with answers included in the back of the book. These allow students to use and test their understanding of one concept before moving onto other concepts in the section.
- Practice problems are provided for students at the end of each section. See "A Note on the Exercises and Problems" below.

Applications

Often the applications of linear algebra are not as transparent, concise, or approachable as those of elementary calculus. Most convincing applications of linear algebra require a fairly lengthy buildup of background, which would be inappropriate in a linear algebra text. However, without some of these applications, many students would find it difficult to remain motivated to learn linear algebra. An additional difficultly is that the applications of linear algebra are so varied that there is very little agreement on which applications should be covered.

In this text we briefly discuss a few applications to give students some exposure to how linear algebra is applied.

List of Applications

- Force vectors in physics (Sections 1.1, 1.3)
- Bravais lattice (Section 1.2)
- Graphing quadratic forms (Sections 1.2, 6.2, 8.3)
- Acceleration due to forces (Section 1.3)
- Area and volume (Sections 1.3, 1.5, 5.4)
- Minimum distance from a point to a plane (Section 1.5)
- Best approximation (Section 1.5)
- Forces and moments (Section 2.1)
- Flow through a network (Sections 2.1, 2.4, 3.1)
- Spring-mass systems (Sections 2.4, 3.1, 3.5, 6.1)
- Electrical circuits (Sections 2.4, 9.2)
- Partial fraction decompositions (Section 2.4)
- Balancing chemical equations (Section 2.4)
- Planar trusses (Section 2.4)
- Linear programming (Section 2.4)
- Magic squares (Chapter 4 Review)
- Systems of Linear Difference Equations (Section 6.2)
- Markov processes (Section 6.3)
- Differential equations (Section 6.3)
- Curve of best fit (Section 7.3)

- Overdetermined systems (Section 7.3)
- Fourier series (Section 7.5)
- Small deformations (Sections 6.2, 8.4)
- Inertia tensor (Section 8.4)
- Effective rank (Section 8.5)
- Image compression (Section 8.5)

A wide variety of additional applications are mentioned throughout the text.

A Note on the Exercises and Problems

Most sections contain mid-section exercises. The purpose of these exercises is to give students a way of checking their understanding of some concepts before proceeding to further concepts in the section. Thus, when reading through a chapter, a student should always complete each exercise before continuing to read the rest of the chapter.

At the end of each section, problems are divided into A, B, and C Problems.

The A Problems are practice problems and are intended to provide a sufficient variety and number of standard computational problems and the odd theoretical problem for students to master the techniques of the course; answers are provided at the back of the text. Full solutions are available in the Student Solutions Manual.

The B Problems are homework problems. They are generally identical to the A Problems, with no answers provided, and can be used by by instructors for homework. In a few cases, the B Problems are not exactly parallel to the A Problems.

The C Problems usually require students to work with general cases, to write simple arguments, or to invent examples. These are important aspects of mastering mathematical ideas, and all students should attempt at least some of these–and not get discouraged if they make slow progress. With effort most students will be able to solve many of these problems and will benefit greatly in the understanding of the concepts and connections in doing so.

In addition to the mid-section exercises and end-of-section problems, there is a sample Chapter Quiz in the Chapter Review at the end of each chapter. Students should be aware that their instructors may have a different idea of what constitutes an appropriate test on this material.

At the end of each chapter, there are some Further Problems; these are similar to the C Problems and provide an extended investigation of certain ideas or applications of linear algebra. Further Problems are intended for advanced students who wish to challenge themselves and explore additional concepts.

Using This Text To Teach Linear Algebra

There are many different approaches to teaching linear algebra. Although we suggest covering the chapters in order, the text has been written to try to accommodate a variety of approaches.

Early Vector Spaces We believe that it is very beneficial to introduce general vector spaces immediately after students have gained some experience in working with a few specific examples of vector spaces. Students find it easier to generalize the concepts of spanning, linear independence, bases, dimension, and linear mappings while the earlier specific cases are still fresh in their minds. Additionally, we feel that it can be unhelpful to students to have determinants available too soon. Some students are far too eager to latch onto mindless algorithms involving determinants (for example, to check linear independence of three vectors in three-dimensional space), rather than actually come to terms with the defining ideas. Lastly, this allows eigenvalues, eigenvectors, and diagonalization to be focused on later in the course. I personally find that if diagonalization is taught too soon, students will focus mainly on being able to diagonalize small matrices by hand, which causes the importance of diagonalization to be lost.

Early Systems of Linear Equations For courses that begin with solving systems of linear questions, the first two sections of Chapter 2 may be covered prior to covering Chapter 1 content.

Early Determinants and Diagonalization Some reviewers have commented that they want to be able to cover determinants and diagonalization before abstract vectors spaces and that in some introductory courses abstract vector spaces may be omitted entirely. Thus, this text has been written so that Chapter 5, Chapter 6, most of Chapter 7, and Chapter 8 may be taught prior to Chapter 4 (note that all required information about subspaces, bases, and dimension for diagonalization of matrices over $\mathbb{R}$ is covered in Chapters 1, 2, and 3). Moreover, we have made sure that there is a very natural flow from matrix inverses and elementary matrices at the end of Chapter 3 to determinants in Chapter 5.

Early Complex Numbers Some introductory linear algebra courses include the use of complex numbers from the beginning. We have written Chapter 9 so that the sections of Chapter 9 may be covered immediately after covering the relevant material over $\mathbb{R}$.

A Matrix-Oriented Course For both options above, the text is organized so that sections or subsections involving linear mappings may be omitted without loss of continuity.

MyLab Math

MyLab Math and MathXL are online learning resources available to instructors and students using *An Introduction to Linear Algebra for Science and Engineering*.

MyLab Math provides engaging experiences that personalize, stimulate, and measure learning for each student. MyLab's comprehensive **online gradebook** automatically tracks your students' results on tests, quizzes, homework, and in the study plan. The homework and practice exercises in MyLab Math are correlated to the exercises in the textbook, and MyLab provides **immediate, helpful feedback** when students enter incorrect answers. The **study plan** can be assigned or used for individual practice and is personalized to each student, tracking areas for improvement as students navigate problems. With over 100 questions (all algorithmic) added to the third edition, MyLab Math for *An Introduction to Linear Algebra for Science and Engineering* is a well-equipped resource that can help improve individual students' performance.

To learn more about how MyLab combines proven learning applications with powerful assessment, visit www.pearson.com/mylab or contact your Pearson representative.

A Personal Note

The third edition of *An Introduction to Linear Algebra for Science and Engineering* is meant to engage students and pique their curiosity, as well as provide a template for instructors. I am constantly fascinated by the countless potential applications of linear algebra in everyday life, and I intend for this textbook to be approachable to all. I will not pretend that mathematical prerequisites and previous knowledge are not required. However, the approach taken in this textbook encourages the reader to explore a variety of concepts and provides exposure to an extensive amount of mathematical knowledge. Linear algebra is an exciting discipline. My hope is that those reading this book will share in my enthusiasm.

Acknowledgments

Thanks are expressed to:

Agnieszka Wolczuk for her support and encouragement.

Mike La Croix for all of the amazing figures in the text, and for his assistance in editing, formatting, and LaTeX'ing.

Peiyao Zeng, Daniel Yu, Adam Radek Martinez, Bruno Verdugo Paredes, and Alex Liao for proof-reading and their many valuable comments and suggestions.

Stephen New, Paul McGrath, Ken McCay, Paul Kates, and many other of my colleagues who have helped me become a better instructor.

To all of the reviewers whose comments, corrections, and recommendations have resulted in many positive improvements.

Charlotte Morrison-Reed for all of her hard work in making the third edition of this text possible and for her suggestions and editing.

A very special thank you to Daniel Norman and all those who contributed to the first and second editions.

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CHAPTER 1

Euclidean Vector Spaces

CHAPTER OUTLINE

Some of the material in this chapter will be familiar to many students, but some ideas that are introduced here will be new to most. In this chapter we will look at operations on and important concepts related to vectors. We will also look at some applications of vectors in the familiar setting of Euclidean space. Most of these concepts will later be extended to more general settings. A firm understanding of the material from this chapter will help greatly in understanding the topics in the rest of this book.

1.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

We begin by considering the two-dimensional plane in Cartesian coordinates. Choose an origin O and two mutually perpendicular axes, called the x_1-axis and the x_2-axis, as shown in Figure 1.1.1. Any point P in the plane can be uniquely identified by the 2-tuple (p_1, p_2), called the **coordinates** of P. In particular, p_1 is the distance from P to the x_2-axis, with p_1 positive if P is to the right of this axis and negative if P is to the left, and p_2 is the distance from P to the x_1-axis, with p_2 positive if P is above this axis and negative if P is below. You have already learned how to plot graphs of equations in this plane.

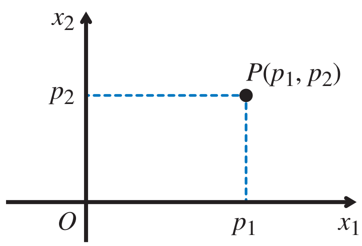


Figure 1.1.1 Coordinates in the plane.

For applications in many areas of mathematics, and in many subjects such as physics, chemistry, economics, and engineering, it is useful to view points more abstractly. In particular, we will view them as **vectors** and provide rules for adding them and multiplying them by constants.

Definition
$\mathbb{R}^2$

We let $\mathbb{R}^2$ denote the set of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are real numbers called the **components** of the vector. Mathematically, we write

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$

We say two vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ are **equal** if $x_1 = y_1$ and $x_2 = y_2$. We write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Although we are viewing the elements of $\mathbb{R}^2$ as vectors, we can still interpret these geometrically as points. That is, the vector $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ can be interpreted as the point $P(p_1, p_2)$. Graphically, this is often represented by drawing an arrow from $(0, 0)$ to (p_1, p_2), as shown in Figure 1.1.2. Note, that the point $(0, 0)$ and the points between $(0, 0)$ and (p_1, p_2) should not be thought of as points "on the vector." The representation of a vector as an arrow is particularly common in physics; force and acceleration are vector quantities that can conveniently be represented by an arrow of suitable magnitude and direction.

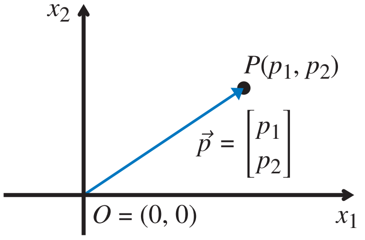


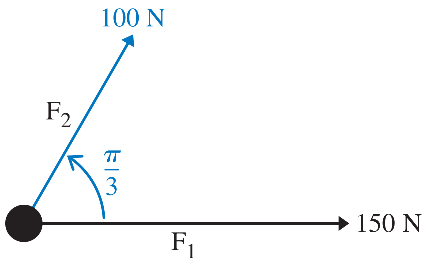
Figure 1.1.2 Graphical representation of a vector.

EXAMPLE 1.1.1

An object on a frictionless surface is being pulled by two strings with force and direction as given in the diagram.

(a) Represent each force as a vector in $\mathbb{R}^2$.
(b) Represent the net force being applied to the object as a vector in $\mathbb{R}^2$.

Solution: (a) The force F_1 has $150N$ of horizontal force and $0N$ of vertical force. Thus, we can represent this with the vector



$$\vec{F}_1 = \begin{bmatrix} 150 \\ 0 \end{bmatrix}$$

The force F_2 has horizontal component $100\cos\frac{\pi}{3} = 50$ N and vertical component $100\sin\frac{\pi}{3} = 50\sqrt{3}$ N. Therefore, we can represent this with the vector

$$\vec{F}_2 = \begin{bmatrix} 50 \\ 50\sqrt{3} \end{bmatrix}$$

(b) We know from physics that to get the net force we add the horizontal components of the forces together and we add the vertical components of the forces together. Thus, the net horizontal component is $150N + 50N = 200N$. The net vertical force is $0N + 50\sqrt{3}N = 50\sqrt{3}N$. We can represent this as the vector

$$\vec{F} = \begin{bmatrix} 200 \\ 50\sqrt{3} \end{bmatrix}$$

The example shows that in physics we add vectors by adding their corresponding components. Similarly, we find that in physics we multiply a vector by a scalar by multiplying each component of the vector by the scalar.

Since we want our generalized concept of vectors to be able to help us solve physical problems like these and more, we define addition and scalar multiplication of vectors in $\mathbb{R}^2$ to match.

Definition

Addition and Scalar Multiplication in $\mathbb{R}^2$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$. We define **addition** of vectors by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

We define **scalar multiplication** of $\vec{x}$ by a factor of $t \in \mathbb{R}$, called a **scalar**, by

$$t\vec{x} = t\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix}$$

Remark

It is important to note that $\vec{x} - \vec{y}$ is to be interpreted as $\vec{x} + (-1)\vec{y}$.

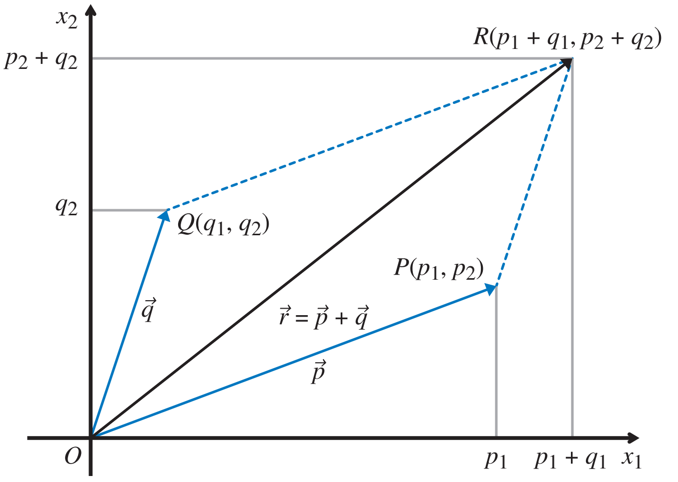


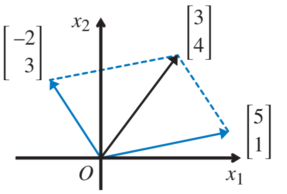
Figure 1.1.3 Addition of vectors $\vec{p}$ and $\vec{q}$.

The addition of two vectors is illustrated in Figure 1.1.3: construct a parallelogram with vectors $\vec{p}$ and $\vec{q}$ as adjacent sides; then $\vec{p} + \vec{q}$ is the vector corresponding to the vertex of the parallelogram opposite to the origin. Observe that the components really are added according to the definition. This is often called the **parallelogram rule for addition**.

EXAMPLE 1.1.2

Let $\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \vec{y} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Calculate $\vec{x} + \vec{y}$.

Solution: We have $\vec{x} + \vec{y} = \begin{bmatrix} -2+5 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.



Scalar multiplication is illustrated in Figure 1.1.4. Observe that multiplication by a negative scalar reverses the direction of the vector.

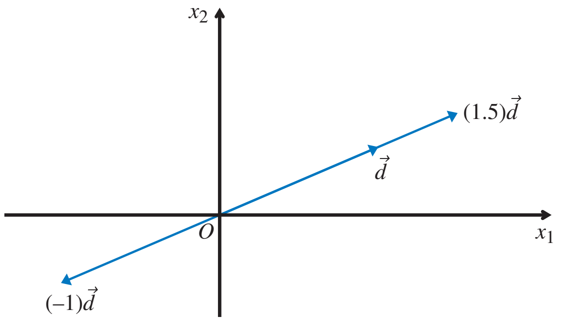


Figure 1.1.4 Scalar multiplication of the vector $\vec{d}$.

EXAMPLE 1.1.3

Let $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in \mathbb{R}^2$. Calculate $\vec{u} + \vec{v}$, $3\vec{w}$, and $2\vec{v} - \vec{w}$.

Solution: We get

$$\begin{aligned}
\vec{u} + \vec{v} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
3\vec{w} &= 3\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \\
2\vec{v} - \vec{w} &= 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}
\end{aligned}$$

EXERCISE 1.1.1

Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Calculate each of the following and illustrate with a sketch.

(a) $\vec{u} + \vec{w}$ (b) $-\vec{v}$ (c) $(\vec{u} + \vec{w}) - \vec{v}$

We will frequently look at sums of scalar multiples of vectors. So, we make the following definition.

Definition
Linear Combination

Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^2$ and $c_1, \dots, c_k \in \mathbb{R}$. We call the sum $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ a **linear combination** of the vectors $\vec{v}_1, \dots, \vec{v}_k$.

It is important to observe that $\mathbb{R}^2$ has the property that any linear combination of vectors in $\mathbb{R}^2$ is a vector in $\mathbb{R}^2$ (combining properties V1, V6 in Theorem 1.1.1 below). Although this property is clear for $\mathbb{R}^2$, it does not hold for most subsets of $\mathbb{R}^2$. As we will see in Section 1.4, in linear algebra, we are mostly interested in sets that have this property.

Theorem 1.1.1

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$ we have

V1 $\vec{x} + \vec{y} \in \mathbb{R}^2$ (closed under addition)
V2 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (addition is commutative)
V3 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ (addition is associative)
V4 There exists a vector $\vec{0} \in \mathbb{R}^2$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{R}^2$ (zero vector)
V5 For each $\vec{x} \in \mathbb{R}^2$ there exists a vector $-\vec{x} \in \mathbb{R}^2$ such that $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverses)
V6 $s\vec{x} \in \mathbb{R}^2$ (closed under scalar multiplication)
V7 $s(t\vec{x}) = (st)\vec{x}$ (scalar multiplication is associative)
V8 $(s + t)\vec{x} = s\vec{x} + t\vec{x}$ (a distributive law)
V9 $s(\vec{x} + \vec{y}) = s\vec{x} + s\vec{y}$ (another distributive law)
V10 $1\vec{x} = \vec{x}$ (scalar multiplicative identity)

Observe that the zero vector from property V4 is the vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and the additive inverse of $\vec{x}$ from V5 is $-\vec{x} = (-1)\vec{x}$.

The Vector Equation of a Line in $\mathbb{R}^2$

In Figure 1.1.4, it is apparent that the set of all multiples of a non-zero vector $\vec{d}$ creates a line through the origin. We make this our definition of a line in $\mathbb{R}^2$: a **line through the origin in** $\mathbb{R}^2$ is a set of the form

$$\{t\vec{d} \mid t \in \mathbb{R}\}$$

Often we do not use formal set notation but simply write a **vector equation** of the line:

$$\vec{x} = t\vec{d}, \quad t \in \mathbb{R}$$

The non-zero vector $\vec{d}$ is called a **direction vector** of the line.

Similarly, we define a **line through** $\vec{p}$ **with direction vector** $\vec{d} \neq \vec{0}$ to be the set

$$\{\vec{p} + t\vec{d} \mid t \in \mathbb{R}\}$$

which has vector equation

$$\vec{x} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R}$$

This line is parallel to the line with equation $\vec{x} = t\vec{d}, t \in \mathbb{R}$ because of the parallelogram rule for addition. As shown in Figure 1.1.5, each point on the line through $\vec{p}$ can be obtained from a corresponding point on the line $\vec{x} = t\vec{d}, t \in \mathbb{R}$ by adding the vector $\vec{p}$. We say that the line has been **translated** by $\vec{p}$. More generally, two lines are parallel if the direction vector of one line is a non-zero scalar multiple of the direction vector of the other line.

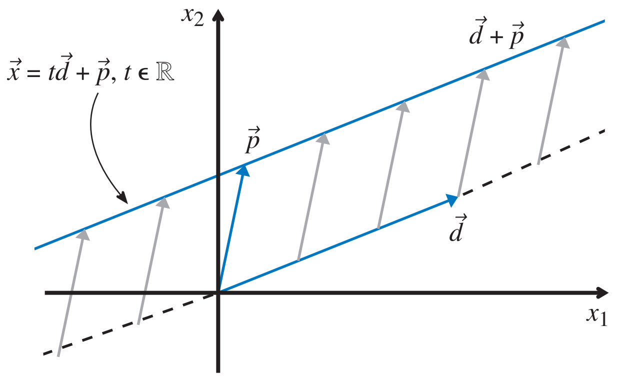


Figure 1.1.5 The line with vector equation $\vec{x} = t\vec{d} + \vec{p}, t \in \mathbb{R}$.

EXAMPLE 1.1.4

A vector equation of the line through the point $P(2, -3)$ with direction vector $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$ is

$$\vec{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + t\begin{bmatrix} -4 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

EXAMPLE 1.1.5

Write a vector equation of the line through $P(1,2)$ parallel to the line with vector equation

$$\vec{x} = t\begin{bmatrix}2\\3\end{bmatrix}, \quad t \in \mathbb{R}$$

Solution: Since they are parallel, we can choose the same direction vector. Hence, a vector equation of the line is

$$\vec{x} = \begin{bmatrix}1\\2\end{bmatrix} + t\begin{bmatrix}2\\3\end{bmatrix}, \quad t \in \mathbb{R}$$

EXERCISE 1.1.2

Write a vector equation of a line through $P(0,0)$ parallel to the line

$$\vec{x} = \begin{bmatrix}4\\-3\end{bmatrix} + t\begin{bmatrix}3\\-1\end{bmatrix}, \quad t \in \mathbb{R}$$

Sometimes the components of a vector equation are written separately. In particular, expanding a vector equation $\vec{x} = \vec{p} + t\vec{d}, t \in \mathbb{R}$ we get

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}p_1\\p_2\end{bmatrix} + t\begin{bmatrix}d_1\\d_2\end{bmatrix} = \begin{bmatrix}p_1 + td_1\\p_2 + td_2\end{bmatrix}$$

Comparing entries, we get **parametric equations** of the line:

$$\begin{cases} x_1 = p_1 + td_1 \\ x_2 = p_2 + td_2, \end{cases} \quad t \in \mathbb{R}$$

The familiar **scalar equation** of the line is obtained by eliminating the parameter t. Provided that $d_1 \neq 0$ we solve the first equation for t to get

$$\frac{x_1 - p_1}{d_1} = t$$

Substituting this into the second equation gives the scalar equation

$$x_2 = p_2 + \frac{d_2}{d_1}(x_1 - p_1) \tag{1.1}$$

What can you say about the line if $d_1 = 0$?

EXAMPLE 1.1.6

Write a vector equation, a scalar equation, and parametric equations of the line passing through the point $P(3,4)$ with direction vector $\begin{bmatrix}-5\\1\end{bmatrix}$.

Solution: A vector equation is $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} + t\begin{bmatrix}-5\\1\end{bmatrix}, t \in \mathbb{R}$.

So, parametric equations are $\begin{cases} x_1 = 3 - 5t \\ x_2 = 4 + t, \end{cases} \quad t \in \mathbb{R}.$

Hence, a scalar equation is $x_2 = 4 - \frac{1}{5}(x_1 - 3)$.

Directed Line Segments

For dealing with certain geometrical problems, it is useful to introduce **directed line segments**. We denote the directed line segment from point P to point Q by $\vec{PQ}$ as in Figure 1.1.6. We think of it as an "arrow" starting at P and pointing towards Q. We shall identify directed line segments from the origin O with the corresponding vectors; we write $\vec{OP} = \vec{p}$, $\vec{OQ} = \vec{q}$, and so on. A directed line segment that starts at the origin is called the **position vector** of the point.

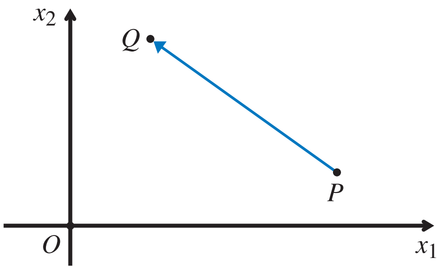


Figure 1.1.6 The directed line segment $\vec{PQ}$ from P to Q.

For many problems, we are interested only in the direction and length of the directed line segment; we are not interested in the point where it is located. For example, in Figure 1.1.3 on page 4, we may wish to treat the line segment $\vec{QR}$ as if it were the same as $\vec{OP}$. Taking our cue from this example, for arbitrary points P, Q, R in $\mathbb{R}^2$, we define $\vec{QR}$ to be **equivalent** to $\vec{OP}$ if $\vec{r} - \vec{q} = \vec{p}$. In this case, we have used one directed line segment $\vec{OP}$ starting from the origin in our definition.

More generally, for arbitrary points Q, R, S, and T in $\mathbb{R}^2$, we define $\vec{QR}$ to be equivalent to $\vec{ST}$ if they are both equivalent to the same $\vec{OP}$ for some P. That is, if

$$\vec{r} - \vec{q} = \vec{p} \text{ and } \vec{t} - \vec{s} = \vec{p} \text{ for the same } \vec{p}$$

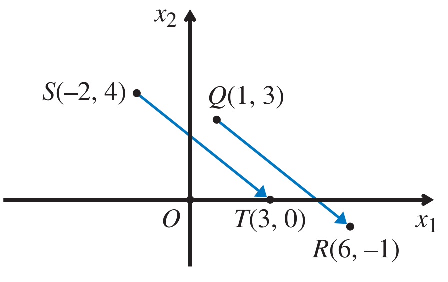
We can abbreviate this by simply requiring that

$$\vec{r} - \vec{q} = \vec{t} - \vec{s}$$

EXAMPLE 1.1.7

For points $Q(1,3)$, $R(6,-1)$, $S(-2,4)$, and $T(3,0)$, we have that $\vec{QR}$ is equivalent to $\vec{ST}$ because

$$\vec{r} - \vec{q} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \vec{t} - \vec{s}$$



In some problems, where it is not necessary to distinguish between equivalent directed line segments, we "identify" them (that is, we treat them as the same object) and write $\vec{PQ} = \vec{RS}$. Indeed, we identify them with the corresponding line segment starting at the origin, so in Example 1.1.7 we write $\vec{QR} = \vec{ST} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$.

Remark

Writing $\vec{QR} = \vec{ST}$ is a bit sloppy—an abuse of notation—because $\vec{QR}$ is not really the same object as $\vec{ST}$. However, introducing the precise language of "equivalence classes" and more careful notation with directed line segments is not helpful at this stage. By introducing directed line segments, we are encouraged to think about vectors that are located at arbitrary points in space. This is helpful in solving some geometrical problems, as we shall see below.

EXAMPLE 1.1.8

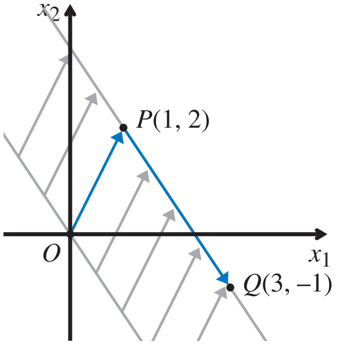
Find a vector equation of the line through $P(1,2)$ and $Q(3,-1)$.

Solution: A direction vector of the line is

$$\vec{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence, a vector equation of the line with direction $\vec{PQ}$ that passes through $P(1,2)$ is

$$\vec{x} = \vec{p} + t\vec{PQ} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad t \in \mathbb{R}$$



Observe in the example above that we would have the same line if we started at the second point and "moved" toward the first point—or even if we took a direction vector in the opposite direction. Thus, the same line is described by the vector equations

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + r\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad r \in \mathbb{R}$$

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad s \in \mathbb{R}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad t \in \mathbb{R}$$

In fact, there are infinitely many descriptions of a line: we may choose any point on the line, and we may use any non-zero scalar multiple of the direction vector.

EXERCISE 1.1.3

Find a vector equation of the line through $P(1,1)$ and $Q(-2,2)$.

Vectors, Lines, and Planes in $\mathbb{R}^3$

Everything we have done so far works perfectly well in three dimensions. We choose an origin O and three mutually perpendicular axes, as shown in Figure 1.1.7. The x_1-axis is usually pictured coming out of the page (or screen), the x_2-axis to the right, and the x_3-axis towards the top of the picture.

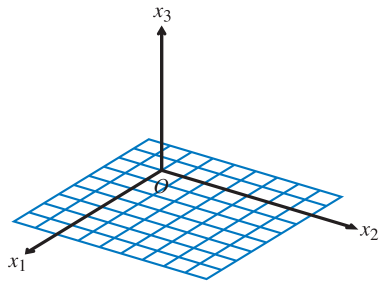


Figure 1.1.7 The positive coordinate axes in $\mathbb{R}^3$.

It should be noted that we are adopting the convention that the coordinate axes form a **right-handed system**. One way to visualize a right-handed system is to spread out the thumb, index finger, and middle finger of your right hand. The thumb is the x_1-axis; the index finger is the x_2-axis; and the middle finger is the x_3-axis. See Figure 1.1.8.

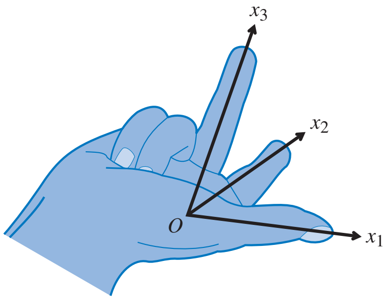


Figure 1.1.8 Identifying a right-handed system.

We now define $\mathbb{R}^3$ to be the three-dimensional analog of $\mathbb{R}^2$.

Definition
$\mathbb{R}^3$

We define $\mathbb{R}^3$ to be the set of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, with $x_1, x_2, x_3 \in \mathbb{R}$.

Mathematically,

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

We say two vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are **equal** and write $\vec{x} = \vec{y}$ if $x_i = y_i$, for $i = 1, 2, 3$.

Definition
Addition and Scalar Multiplication in $\mathbb{R}^3$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$. We define **addition** of vectors by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

We define the **scalar multiplication** of $\vec{x}$ by a **scalar** $t \in \mathbb{R}$ by

$$t\vec{x} = t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \end{bmatrix}$$

Addition still follows the parallelogram rule. It may help you to visualize this if you realize that two vectors in $\mathbb{R}^3$ must lie within a plane in $\mathbb{R}^3$ so that the two-dimensional picture is still valid. See Figure 1.1.9.

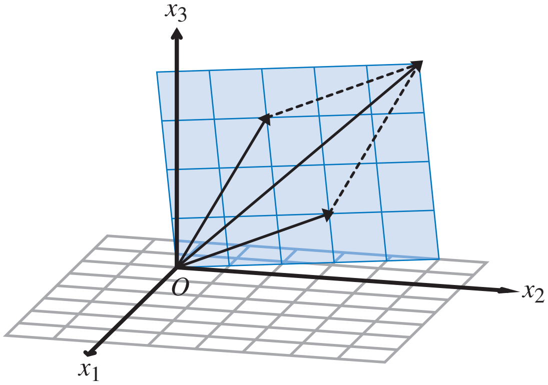


Figure 1.1.9 Two-dimensional parallelogram rule in $\mathbb{R}^3$.

EXAMPLE 1.1.9

Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. Calculate $\vec{v} + \vec{u}$, $-\vec{w}$, and $-\vec{v} + 2\vec{w} - \vec{u}$.

Solution: We have

$$\vec{v} + \vec{u} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$-\vec{w} = -\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$-\vec{v} + 2\vec{w} - \vec{u} = -\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

EXERCISE 1.1.4

Let $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}^3$. Calculate $2\vec{x} - \vec{y} + 3\vec{z}$.

As before, we call a sum of scalar multiples of vectors in $\mathbb{R}^3$ a linear combination. Moreover, of course, vectors in $\mathbb{R}^3$ satisfy all the same properties in Theorem 1.1.1 replacing $\mathbb{R}^2$ by $\mathbb{R}^3$ in properties V1, V4, V5, and V6.

The zero vector in $\mathbb{R}^3$ is $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and the additive inverse of $\vec{x} \in \mathbb{R}^3$ is $-\vec{x} = (-1)\vec{x}$.

Directed line segments are the same in three-dimensional space as in the two-dimensional case.

The line through the point P in $\mathbb{R}^3$ (corresponding to a vector $\vec{p}$) with direction vector $\vec{d} \neq \vec{0}$ can be described by a vector equation:

$$\vec{x} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R}$$

CONNECTION

It is important to realize that a line in $\mathbb{R}^3$ cannot be described by a single scalar equation, as in $\mathbb{R}^2$. We shall see in Section 1.3 that a single scalar equation in $\mathbb{R}^3$ describes a plane in $\mathbb{R}^3$.

EXAMPLE 1.1.10

Find a vector equation and parametric equations of the line that passes through the points $P(1, 5, -2)$ and $Q(4, -1, 3)$.

Solution: A direction vector is $\vec{PQ} = \begin{bmatrix} 4-1 \\ -1-5 \\ 3-(-2) \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$. Hence, a vector equation of the line is

$$\vec{x} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

Hence, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1+3t \\ 5-6t \\ -2+5t \end{bmatrix}, \quad t \in \mathbb{R}$$

Consequently, corresponding parametric equations are

$$\begin{cases} x_1 = 1 + 3t \\ x_2 = 5 - 6t \qquad t \in \mathbb{R} \\ x_3 = -2 + 5t, \end{cases}$$

EXERCISE 1.1.5 Find a vector equation and parametric equations of the line that passes through the points $P(1, 2, 2)$ and $Q(1, -2, 3)$.

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$ that are not scalar multiples of each other. This implies that the sets $\{t\vec{u} \mid t \in \mathbb{R}\}$ and $\{s\vec{v} \mid s \in \mathbb{R}\}$ are both lines in $\mathbb{R}^3$ through the origin in different directions. Thus, the set of all possible linear combinations of $\vec{u}$ and $\vec{v}$ forms a two-dimensional plane. That is, the set

$$\{t\vec{u} + s\vec{v} \mid s, t \in \mathbb{R}\}$$

is a **plane through the origin in** $\mathbb{R}^3$. As we did with lines, we could say that

$$\{\vec{p} + t\vec{u} + s\vec{v} \mid s, t \in \mathbb{R}\}$$

is a **plane through** $\vec{p}$ **in** $\mathbb{R}^3$ and that

$$\vec{x} = \vec{p} + t\vec{u} + s\vec{v}, \qquad s, t \in \mathbb{R}$$

is a **vector equation** for the plane. It is very important to note that if either $\vec{u}$ or $\vec{v}$ is a scalar multiple of the other, then the set $\{t\vec{u} + s\vec{v} \mid s, t \in \mathbb{R}\}$ would *not* be a plane.

EXAMPLE 1.1.11 Determine which of the following vectors are in the plane with vector equation

$$\vec{x} = s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}1\\2\\2\end{bmatrix}, \qquad s, t \in \mathbb{R}$$

(a) $\vec{p} = \begin{bmatrix}5/2\\1\\3\end{bmatrix}$ (b) $\vec{q} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Solution: (a) The vector $\vec{p}$ is in the plane if and only if there are scalars $s, t \in \mathbb{R}$ such that

$$\begin{bmatrix}5/2\\1\\3\end{bmatrix} = s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Performing the linear combination on the right-hand side gives

$$\begin{bmatrix}5/2\\1\\3\end{bmatrix} = \begin{bmatrix}s+t\\2t\\s+2t\end{bmatrix}$$

For these vectors to be equal we must have

$$s + t = 5/2, \qquad 2t = 1, \qquad s + 2t = 3$$

EXAMPLE 1.1.11 (continued)

We find that the values $t = 1/2$ and $s = 2$ satisfies all three equations. Hence, we have found that $\vec{p}$ is in the plane. In particular,

$$\begin{bmatrix} 5/2 \\ 1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(b) We need to determine whether there exists $s, t \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Performing the linear combination on the right-hand side gives

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s + t \\ 2t \\ s + 2t \end{bmatrix}$$

Since vectors are equal only if they have equal components, for these vectors to be equal we must have

$$s + t = 1, \qquad 2t = 1, \qquad s + 2t = 1$$

The middle equation shows us that we must have $t = 1/2$. However, then we would need $s = 1/2$ for the first equation and $s = 0$ for the third equation. Therefore, there is no value of s and t that satisfies all three equations. Thus, $\vec{q}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, so $\vec{q}$ is not in the plane.

EXERCISE 1.1.6

Consider the plane in $\mathbb{R}^3$ with vector equation

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}$$

Find two vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ that are in the plane, and find a vector $\vec{w} \in \mathbb{R}^3$ that is not in the plane.

EXERCISE 1.1.7

Find two non-zero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ such that $\{s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}\}$ is a line in $\mathbb{R}^3$.

PROBLEMS 1.1

Practice Problems

In Problems A1–A4, compute the given linear combination in $\mathbb{R}^2$ and illustrate with a sketch.

A1 $\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}$

A2 $\begin{bmatrix}3\\2\end{bmatrix}-\begin{bmatrix}4\\1\end{bmatrix}$

A3 $3\begin{bmatrix}-1\\4\end{bmatrix}$

A4 $2\begin{bmatrix}2\\1\end{bmatrix}-2\begin{bmatrix}3\\-1\end{bmatrix}$

In Problems A5–A10, compute the given linear combination in $\mathbb{R}^2$.

A5 $\begin{bmatrix}4\\-2\end{bmatrix}+\begin{bmatrix}-1\\3\end{bmatrix}$

A6 $\begin{bmatrix}-3\\-4\end{bmatrix}-\begin{bmatrix}-2\\5\end{bmatrix}$

A7 $-2\begin{bmatrix}3\\-2\end{bmatrix}$

A8 $\frac{1}{2}\begin{bmatrix}2\\6\end{bmatrix}+\frac{1}{3}\begin{bmatrix}4\\3\end{bmatrix}$

A9 $\frac{2}{3}\begin{bmatrix}3\\1\end{bmatrix}-2\begin{bmatrix}1/4\\1/3\end{bmatrix}$

A10 $\sqrt{2}\begin{bmatrix}\sqrt{2}\\\sqrt{3}\end{bmatrix}+3\begin{bmatrix}1\\\sqrt{6}\end{bmatrix}$

In Problems A11–A16, compute the given linear combination in $\mathbb{R}^3$.

A11 $\begin{bmatrix}2\\3\\4\end{bmatrix}-\begin{bmatrix}5\\1\\-2\end{bmatrix}$

A12 $\begin{bmatrix}2\\1\\-6\end{bmatrix}+\begin{bmatrix}-3\\1\\-4\end{bmatrix}$

A13 $-6\begin{bmatrix}4\\-5\\-6\end{bmatrix}$

A14 $-2\begin{bmatrix}-5\\1\\1\end{bmatrix}+3\begin{bmatrix}-1\\0\\-1\end{bmatrix}$

A15 $2\begin{bmatrix}2/3\\-1/3\\2\end{bmatrix}+\frac{1}{3}\begin{bmatrix}3\\-2\\1\end{bmatrix}$

A16 $\sqrt{2}\begin{bmatrix}1\\1\\1\end{bmatrix}+\pi\begin{bmatrix}-1\\0\\1\end{bmatrix}$

A17 Let $\vec{v}=\begin{bmatrix}1\\2\\-2\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}2\\-1\\3\end{bmatrix}$. Determine

(a) $2\vec{v}-3\vec{w}$

(b) $-3(\vec{v}+2\vec{w})+5\vec{v}$

(c) $\vec{u}$ such that $\vec{w}-2\vec{u}=3\vec{v}$

(d) $\vec{u}$ such that $\vec{u}-3\vec{v}=2\vec{u}$

A18 Let $\vec{v}=\begin{bmatrix}3\\1\\1\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}5\\-1\\-2\end{bmatrix}$. Determine

(a) $\frac{1}{2}\vec{v}+\frac{1}{2}\vec{w}$

(b) $2(\vec{v}+\vec{w})-(2\vec{v}-3\vec{w})$

(c) $\vec{u}$ such that $\vec{w}-\vec{u}=2\vec{v}$

(d) $\vec{u}$ such that $\frac{1}{2}\vec{u}+\frac{1}{3}\vec{v}=\vec{w}$

A19 Consider the points $P(2,3,1)$, $Q(3,1,-2)$, $R(1,4,0)$, and $S(-5,1,5)$. Determine $\vec{PQ}$, $\vec{PR}$, $\vec{PS}$, $\vec{QR}$, and $\vec{SR}$. Verify that $\vec{PQ}+\vec{QR}=\vec{PR}=\vec{PS}+\vec{SR}$.

For Problems A20–A23, write a vector equation for the line passing through the given point with the given direction vector.

A20 $P(3,4)$, $\vec{d}=\begin{bmatrix}-5\\1\end{bmatrix}$

A21 $P(2,3)$, $\vec{d}=\begin{bmatrix}-4\\-6\end{bmatrix}$

A22 $P(2,0,5)$, $\vec{d}=\begin{bmatrix}4\\-2\\-11\end{bmatrix}$

A23 $P(4,1,5)$, $\vec{d}=\begin{bmatrix}-2\\1\\2\end{bmatrix}$

For Problems A24–A28, write a vector equation for the line that passes through the given points.

A24 $P(-1,2)$, $Q(2,-3)$

A25 $P(4,1)$, $Q(-2,-1)$

A26 $P(1,3,-5)$, $Q(-2,1,0)$

A27 $P(-2,1,1)$, $Q(4,2,2)$

A28 $P\left(\frac{1}{2},\frac{1}{4},1\right)$, $Q\left(-1,1,\frac{1}{3}\right)$

For Problems A29–A32, determine parametric equations and a scalar equation for the line that passes through the given points.

A29 $P(-1,2)$, $Q(2,-3)$

A30 $P(1,1)$, $Q(2,2)$

A31 $P(1,0)$, $Q(3,0)$

A32 $P(1,3)$, $Q(-1,5)$

A33 (a) A set of points is **collinear** if all the points lie on the same line. By considering directed line segments, give a general method for determining whether a given set of three points is collinear.

(b) Determine whether the points $P(1,2)$, $Q(4,1)$, and $R(-5,4)$ are collinear. Show how you decide.

(c) Determine whether the points $S(1,0,1)$, $T(3,-2,3)$, and $U(-3,4,-1)$ are collinear. Show how you decide.

A34 Prove properties V2 and V8 of Theorem 1.1.1.

A35 Consider the object from Example 1.1.1. If the force F_1 is tripled to $450N$ and the force F_2 is halved to $50N$, then what is the vector representing the net force being applied to the object?

Homework Problems

In Problems B1–B4, compute the given linear combination and illustrate with a sketch.

B1 $\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}$

B2 $\begin{bmatrix}-3\\4\end{bmatrix}-\begin{bmatrix}2\\1\end{bmatrix}$

B3 $2\begin{bmatrix}1\\3\end{bmatrix}$

B4 $3\begin{bmatrix}2\\5\end{bmatrix}-\begin{bmatrix}1\\-1\end{bmatrix}$

In Problems B5–B9, compute the given linear combination in $\mathbb{R}^2$.

B5 $\begin{bmatrix}7\\11\end{bmatrix}+\begin{bmatrix}-2\\4\end{bmatrix}$

B6 $\begin{bmatrix}2\\-5\end{bmatrix}-\begin{bmatrix}3\\-3\end{bmatrix}$

B7 $5\begin{bmatrix}3\\-2\end{bmatrix}$

B8 $\frac{3}{4}\begin{bmatrix}2\\6\end{bmatrix}-\frac{1}{4}\begin{bmatrix}3\\-1\end{bmatrix}$

B9 $\sqrt{2}\begin{bmatrix}\sqrt{2}\\2\end{bmatrix}+\sqrt{6}\begin{bmatrix}1\\-\sqrt{3}\end{bmatrix}-\begin{bmatrix}\sqrt{6}\\\sqrt{2}\end{bmatrix}$

In Problems B10–B16, compute the given linear combination in $\mathbb{R}^3$.

B10 $\begin{bmatrix}3\\2\\-1\end{bmatrix}-\begin{bmatrix}3\\5\\8\end{bmatrix}$

B11 $\begin{bmatrix}4\\2\\2\end{bmatrix}+\begin{bmatrix}-3\\2\\-3\end{bmatrix}$

B12 $\begin{bmatrix}1\\2\\1\end{bmatrix}+\begin{bmatrix}3\\1\\0\end{bmatrix}+\begin{bmatrix}0\\1\\1\end{bmatrix}$

B13 $3\begin{bmatrix}1\\2\\5\end{bmatrix}$

B14 $0\begin{bmatrix}2\\-5\\-1\end{bmatrix}$

B15 $\frac{2}{3}\begin{bmatrix}3\\5\\-1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}-6\\-10\\2\end{bmatrix}$

B16 $(1+\sqrt{2})\begin{bmatrix}1+\sqrt{2}\\0\\\sqrt{2}-1\end{bmatrix}-\begin{bmatrix}\sqrt{2}\\0\\2\end{bmatrix}$

B17 Let $\vec{v}=\begin{bmatrix}2\\4\\3\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}2\\-2\\-1\end{bmatrix}$. Determine

(a) $3\vec{v}-2\vec{w}$
(b) $-2(\vec{v}-2\vec{w})+3\vec{w}$
(c) $\vec{u}$ such that $\vec{w}+\vec{u}=2\vec{v}$
(d) $\vec{u}$ such that $2\vec{u}+3\vec{w}=-\vec{v}$

B18 Let $\vec{v}=\begin{bmatrix}1\\3\\-1\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}2\\4\\3\end{bmatrix}$. Determine

(a) $3\vec{v}+4\vec{w}$
(b) $-\frac{1}{2}\vec{v}+\frac{3}{4}\vec{w}$
(c) $\vec{u}$ such that $2\vec{v}+\vec{u}=\vec{v}$
(d) $\vec{u}$ such that $3\vec{u}-2\vec{w}=\vec{v}$

For Problems B19 and B20, determine $\vec{PQ}$, $\vec{PR}$, $\vec{PS}$, $\vec{QR}$, and $\vec{SR}$, and verify that $\vec{PQ}+\vec{QR}=\vec{PR}=\vec{PS}+\vec{SR}$.

B19 $P(2,3,2)$, $Q(5,4,1)$, $R(-2,3,-1)$, $S(7,-3,4)$

B20 $P(-2,3,-1)$, $Q(4,5,1)$, $R(-2,-1,0)$, and $S(3,1,-1)$

For Problems B21–B26, write a vector equation for the line passing through the given point with the given direction vector.

B21 $P(2,-1)$, $\vec{d}=\begin{bmatrix}-3\\2\end{bmatrix}$

B22 $P(0,0,0)$, $\vec{d}=\begin{bmatrix}2\\2\\1\end{bmatrix}$

B23 $P(3,1)$, $\vec{d}=\begin{bmatrix}1\\2\end{bmatrix}$

B24 $P(1,-1,2)$, $\vec{d}=\begin{bmatrix}1\\1\\0\end{bmatrix}$

B25 $P(1,1,1)$, $\vec{d}=\begin{bmatrix}1\\0\\1\end{bmatrix}$

B26 $P(-2,3,1)$, $\vec{d}=\begin{bmatrix}2\\3\\1\end{bmatrix}$

For Problems B27–B32, write a vector equation for the line that passes through the given points.

B27 $P(2,4)$, $Q(1,2)$

B28 $P(-2,5)$, $Q(-1,-1)$

B29 $P(1,3,2)$, $Q(0,0,0)$

B30 $P(0,1,4)$, $Q(-1,2,2)$

B31 $P(-2,6,1)$, $Q(-2,5,1)$

B32 $P\left(1,2,\frac{1}{2}\right)$, $Q\left(\frac{1}{2},\frac{1}{3},0\right)$

For Problems B33–B38, determine parametric equations and a scalar equation for the line that passes through the given points.

B33 $P(2,5)$, $Q(3,3)$

B34 $P(3,-1)$, $Q(6,1)$

B35 $P(0,3)$, $Q(1,-5)$

B36 $P(-3,1)$, $Q(4,1)$

B37 $P(2,0)$, $Q(0,-3)$

B38 $P(5,-2)$, $Q(6,3)$

For Problems B39–B41, use the solution from Problem A33 (a) to determine whether the given points are collinear. Show how you decide.

B39 $P(1,1)$, $Q(4,3)$, $R(-5,-3)$

B40 $P(2,-1,2)$, $Q(3,2,3)$, $R(1,-4,0)$

B41 $S(0,4,4)$, $T(-1,5,6)$, $U(4,0,-4)$

Conceptual Problems

C1 Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Find real numbers t_1 and t_2 such that $t_1\vec{v} + t_2\vec{w} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Illustrate with a sketch.

(b) Find real numbers t_1 and t_2 such that $t_1\vec{v} + t_2\vec{w} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for any $x_1, x_2 \in \mathbb{R}$.

(c) Use your result in part (b) to find real numbers t_1 and t_2 such that $t_1\vec{v}_1 + t_2\vec{v}_2 = \begin{bmatrix} \sqrt{2} \\ \pi \end{bmatrix}$.

C2 Let P, Q, and R be points in $\mathbb{R}^2$ corresponding to vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$ respectively.

(a) Explain in terms of directed line segments why

$$\vec{PQ} + \vec{QR} + \vec{RP} = \vec{0}$$

(b) Verify the equation of part (a) by expressing $\vec{PQ}$, $\vec{QR}$, and $\vec{RP}$ in terms of $\vec{p}$, $\vec{q}$, and $\vec{r}$.

For Problems C3 and C4, let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^3$ and $s, t \in \mathbb{R}$.

C3 Prove that $s(t\vec{x}) = (st)\vec{x}$

C4 Prove that $s(\vec{x} + \vec{y}) = s\vec{x} + s\vec{y}$

C5 Let $\vec{p}$ and $\vec{d} \neq \vec{0}$ be vectors in $\mathbb{R}^2$. Prove that $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$, is a line in $\mathbb{R}^2$ passing through the origin if and only if $\vec{p}$ is a scalar multiple of $\vec{d}$.

C6 Let $\vec{p}, \vec{u}, \vec{v} \in \mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ are not scalar multiples of each other. Prove that $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$, $s, t \in \mathbb{R}$ is a plane in $\mathbb{R}^3$ passing through the origin if and only if $\vec{p}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

C7 Let O, Q, P, and R be the corner points of a parallelogram (see Figure 1.1.3). Prove that the two diagonals of the parallelogram $\vec{OR}$ and $\vec{PQ}$ bisect each other.

C8 Let $A(a_1, a_2)$ and $B(b_1, b_2)$ be points in $\mathbb{R}^2$. Find the coordinates of the point 1/3 of the way from the point A to the point B.

C9 Consider the plane in $\mathbb{R}^3$ with vector equation

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

(a) Find parametric equations for the plane.

(b) Use the parametric equations you found in (a) to find a scalar equation for the plane.

C10 We have seen how to use a vector equation of a line to find parametric equations, and how to use parametric equations to find a scalar equation of the line. In this exercise, we will perform these steps in reverse. Let $ax_1 + bx_2 = c$ be the scalar equation of a line in $\mathbb{R}^2$ with a and b both non-zero.

(a) Find parametric equations for the line by setting $x_2 = t$ and solving for x_1.

(b) Substitute the parametric equations into the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and use operations on vectors to write $\vec{x}$ in the form $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$.

(c) Find a vector equation of the line $2x_1 + 3x_2 = 5$.

(d) Find a vector equation of the line $x_1 = 3$.

C11 Let L be a line in $\mathbb{R}^2$ with vector equation $\vec{x} = t\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, $t \in \mathbb{R}$. Prove that a point $P(p_1, p_2)$ is on the line L if and only if $p_1 d_2 = p_2 d_1$.

C12 Show that if two lines in $\mathbb{R}^2$ are not parallel to each other, then they must have a point of intersection. (Hint: Use the result of Problem C11.)

1.2 Spanning and Linear Independence in $\mathbb{R}^2$ and $\mathbb{R}^3$

In this section we will give a preview of some important concepts in linear algebra. We will use the geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$ to help you visualize and understand these concepts.

Spanning in $\mathbb{R}^2$ and $\mathbb{R}^3$

We saw in the previous section that lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ that pass through the origin have the form

$$\vec{x} = t\vec{d}, \quad t \in \mathbb{R}$$

for some non-zero direction vector $\vec{d}$. That is, such a line is the set of all possible scalar multiples of $\vec{d}$.

Similarly, we saw that planes in $\mathbb{R}^3$ that pass through the origin have the form

$$\vec{x} = s\vec{u} + t\vec{v}, \quad s, t \in \mathbb{R}$$

where neither $\vec{u}$ nor $\vec{v}$ is a scalar multiple of the other. Hence, such a plane is the set of all possible linear combinations of $\vec{u}$ and $\vec{v}$.

Sets of all possible linear combinations (or scalar multiples in the case of a single vector) are extremely important in linear algebra. We make the following definition.

Definition
Span in $\mathbb{R}^2$

Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^2$ or a set of vectors in $\mathbb{R}^3$. We define the **span** of B, denoted Span B, to be the set of all possible linear combinations of the vectors in B. Mathematically,

$$\operatorname{Span} B = \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

A **vector equation** for Span B is

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k, \quad c_1, \dots, c_k \in \mathbb{R}$$

If $S = \operatorname{Span} B$, then we say that B **spans** S, that B is a **spanning set** for S, and that S is **spanned** by B.

EXAMPLE 1.2.1

Describe the set spanned by $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ geometrically.

Solution: A vector equation for the spanned set is

$$\vec{x} = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad s \in \mathbb{R}$$

Thus, the spanned set is a line in $\mathbb{R}^2$ through the origin with direction vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

EXAMPLE 1.2.2

Is the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$?

Solution: Using the definition of span, the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is in the spanned set if it can be written as a linear combination of the vectors in the spanning set. That is, we need to determine whether there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Performing operations on vectors on the right-hand side gives

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 \\ 2c_1 + c_2 \end{bmatrix}$$

Since vectors are equal if and only if their corresponding entries are equal, we get that this vector equation implies

$$\begin{aligned} 3 &= c_1 - c_2 \\ 1 &= 2c_1 + c_2 \end{aligned}$$

Adding the equations gives $4 = 3c_1$ and so $c_1 = \frac{4}{3}$.
Substituting this into either equation gives $c_2 = -\frac{5}{3}$. Hence, we have that

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus, by definition, $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

EXAMPLE 1.2.3

Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Show that $\text{Span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2$.

Solution: We need to show that every vector in $\mathbb{R}^2$ can be written as a linear combination of the vectors $\vec{e}_1$ and $\vec{e}_2$. We pick a general vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$. We need to determine whether there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We observe that we can take $c_1 = x_1$ and $c_2 = x_2$. That is, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, $\text{Span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2$.

We have just shown that every vector in $\mathbb{R}^2$ can be written as a unique linear combination of the vectors $\vec{e}_1$ and $\vec{e}_2$. This is not surprising since Span$\{\vec{e}_1\}$ is the x_1-axis and Span$\{\vec{e}_2\}$ is the x_2-axis. In particular, when we write $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2$ we are really just writing the vector $\vec{x}$ in terms of its standard coordinates. We call the set $\{\vec{e}_1, \vec{e}_2\}$ the **standard basis** for $\mathbb{R}^2$.

Remark

In physics and engineering, it is common to use the notation $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ instead of $\vec{e}_1$ and $\vec{e}_2$.

Using the definition of span, we have that if $\vec{d} \in \mathbb{R}^2$ with $\vec{d} \neq \vec{0}$, then geometrically Span$\{\vec{d}\}$ is a line through the origin in $\mathbb{R}^2$. If $\vec{u}, \vec{v} \in \mathbb{R}^2$ with $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, then what is Span$\{\vec{u}, \vec{v}\}$ geometrically? It is tempting to say that the set $\{\vec{u}, \vec{v}\}$ would span $\mathbb{R}^2$. However, as demonstrated in the next example, this does not have to be true.

EXAMPLE 1.2.4

Describe Span $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$ geometrically.

Solution: Using the definition of span, a vector equation of the spanned set is

$$\vec{x} = s\begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 6 \\ 4 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Observe that we can rewrite this as

$$\begin{aligned} \vec{x} &= s\begin{bmatrix} 3 \\ 2 \end{bmatrix} + (2t)\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad s, t \in \mathbb{R} \\ &= (s + 2t)\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad s, t \in \mathbb{R} \end{aligned}$$

Since $c = s + 2t$ can take any real value, the spanned set is a line through the origin with direction vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Hence, before we can describe a spanned set geometrically, we must first see if we can simplify the spanning set.

EXAMPLE 1.2.5

Describe Span $\left\{\begin{bmatrix}3\\1\\-3\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\-2\end{bmatrix}\right\}$ geometrically.

Solution: By definition, a vector equation for the spanned set is

$$\vec{x} = c_1\begin{bmatrix}3\\1\\-3\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\1\end{bmatrix} + c_3\begin{bmatrix}3\\0\\-2\end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

We observe that $\begin{bmatrix}3\\1\\-3\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\0\\-2\end{bmatrix}$. Hence, we can rewrite the vector equation as

$$\begin{aligned}\vec{x} &= c_1\begin{bmatrix}3\\1\\-3\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\1\end{bmatrix} + c_3\left(\begin{bmatrix}3\\1\\-3\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix}\right), \quad c_1, c_2, c_3 \in \mathbb{R}\\ &= (c_1 + c_3)\begin{bmatrix}3\\1\\-3\end{bmatrix} + (c_2 + c_3)\begin{bmatrix}0\\-1\\1\end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}\end{aligned}$$

Since $\begin{bmatrix}3\\1\\-3\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ are not scalar multiples of each other, we cannot simplify the vector equation any more. Thus, the set is a plane with vector equation

$$\vec{x} = s\begin{bmatrix}3\\1\\-3\end{bmatrix} + t\begin{bmatrix}0\\-1\\1\end{bmatrix}, \quad s, t \in \mathbb{R}$$

In Example 1.2.4, we used the fact that the second vector was a scalar multiple of the first to simplify the vector equation. In Example 1.2.5, we used the fact that the third vector could be written as a linear combination of the first two vectors to simplify the spanning set. Rather than having to perform these steps each time, we create a theorem to help us.

Theorem 1.2.1

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^2$ or a set of vectors in $\mathbb{R}^3$. Some vector $\vec{v}_i$, $1 \leq i \leq k$, can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k$ if and only if

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$$

This theorem shows that if one vector $\vec{v}_i$ in the spanning set can be written as a linear combination of the other vectors, then $\vec{v}_i$ can be removed from the spanning set without changing the set that is being spanned.

EXERCISE 1.2.1

Use Theorem 1.2.1 to find a simplified spanning set for each of the following sets.

(a) $\operatorname{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ (b) $\operatorname{Span}\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\4\\2\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}\right\}$

Linear Independence and Bases in $\mathbb{R}^2$ and $\mathbb{R}^3$

Examples 1.2.4 and 1.2.5 show that it is important to identify if a spanning set is as simple as possible. For example, if $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$, then it is impossible to determine the geometric interpretation of $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ without knowing if one of the vectors in the spanning set can be removed using Theorem 1.2.1. We now look at a mathematical way of determining if one vector in a set can be written as a linear combination of the others.

Definition
Linearly Dependent
Linearly Independent

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^2$ or a set of vectors in $\mathbb{R}^3$. The set $\mathcal{B}$ is said to be **linearly dependent** if there exist real coefficients $c_1, \dots, c_k$ not all zero such that

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

The set $\mathcal{B}$ is said to be **linearly independent** if the only solution to

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

is $c_1 = c_2 = \cdots = c_k = 0$ (called the **trivial solution**).

EXAMPLE 1.2.6

Determine whether the set $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ in $\mathbb{R}^3$ is linearly independent.

Solution: By definition, we need to find all solutions of the equation

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Performing the linear combination on the right-hand side gives

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}c_1 + c_3\\c_2 + c_3\\c_1 + c_2\end{bmatrix}$$

Comparing entries gives the system of equations

$$c_1 + c_3 = 0, \qquad c_2 + c_3 = 0, \qquad c_1 + c_2 = 0$$

Adding the first to the second and then subtracting the third gives $2c_3 = 0$. Hence, $c_3 = 0$ which then implies $c_1 = c_2 = 0$ from the first and second equations.

Since $c_1 = c_2 = c_3 = 0$ is the only solution, the set is linearly independent.

EXAMPLE 1.2.7

Determine whether the set $C = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ is linearly independent or linearly dependent.

Solution: We consider the equation

$$\begin{bmatrix}0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} + c_3\begin{bmatrix}2\\2\end{bmatrix}$$

We observe that taking $c_1 = -2$, $c_2 = 0$, and $c_3 = 1$ satisfies the equation. Hence, by definition, the set C is linearly dependent.

As desired, the definition of linear independence/linear dependence gives us the following theorem.

Theorem 1.2.2

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^2$ or a set of vectors in $\mathbb{R}^3$. The set $\mathcal{B}$ is linearly dependent if and only if $\vec{v}_i \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$ for some i, $1 \leq i \leq k$.

Theorem 1.2.2 tells us that a set $\mathcal{B}$ is linearly independent if and only if none of the vectors in $\mathcal{B}$ can be written as a linear combination of the others. That is, the simplest spanning set $\mathcal{B}$ for a given set S is one that is linearly independent. Hence, we make the following definition.

Definition

Basis of $\mathbb{R}^2$

Basis of $\mathbb{R}^3$

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a set in $\mathbb{R}^2$. If $\mathcal{B}$ is linearly independent and $\operatorname{Span}\mathcal{B} = \mathbb{R}^2$, then the set $\mathcal{B}$ is called a **basis** of $\mathbb{R}^2$.

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a set in $\mathbb{R}^3$. If $\mathcal{B}$ is linearly independent and $\operatorname{Span}\mathcal{B} = \mathbb{R}^3$, then the set $\mathcal{B}$ is called a **basis** of $\mathbb{R}^3$.

Remarks

1. The plural of basis is **bases**. As we will see, both $\mathbb{R}^2$ and $\mathbb{R}^3$ have infinitely many bases.
2. Here we are relying on our geometric intuition to say that all bases of $\mathbb{R}^2$ have exactly two vectors and all bases of $\mathbb{R}^3$ have exactly three vectors. In Chapter 2, we will mathematically prove this assertion.

We saw in Example 1.2.3 that the set $\{\vec{e}_1, \vec{e}_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is the standard basis for $\mathbb{R}^2$. We now look at the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ for $\mathbb{R}^3$.

EXAMPLE 1.2.8

Prove that the set $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis for $\mathbb{R}^3$.

Solution: To show that $\mathcal{B}$ is a basis, we need to prove that it is linearly independent and spans $\mathbb{R}^3$.

Linear Independence: Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Comparing entries, we get that $c_1 = c_2 = c_3 = 0$. Therefore, $\mathcal{B}$ is linearly independent.

Spanning: Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in $\mathbb{R}^3$. Observe that we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\operatorname{Span}\mathcal{B} = \mathbb{R}^3$.

Since $\mathcal{B}$ is a linearly independent spanning set for $\mathbb{R}^3$, it is a basis for $\mathbb{R}^3$.

EXAMPLE 1.2.9

Prove that the set $C = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

Solution: We need to show that $\operatorname{Span} C = \mathbb{R}^2$ and that C is linearly independent.

Spanning: Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ and consider

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ 2c_1 + c_2 \end{bmatrix} \tag{1.2}$$

Comparing entries, we get two equations in two unknowns

$$\begin{aligned} -c_1 + c_2 &= x_1 \\ 2c_1 + c_2 &= x_2 \end{aligned}$$

Solving gives $c_1 = \frac{1}{3}(-x_1 + x_2)$ and $c_2 = \frac{1}{3}(2x_1 + x_2)$. Hence, we have that

$$\frac{1}{3}(-x_1 + x_2) \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{1}{3}(2x_1 + x_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, $\operatorname{Span} C = \mathbb{R}^2$.

Linear Independence: Take $x_1 = x_2 = 0$ in equation (1.2). Our general solution to that equation says that the only solution is $c_1 = \frac{1}{3}(-0 + 0) = 0$ and $c_2 = \frac{1}{3}(2(0) + 0) = 0$. So, C is also linearly independent. Therefore, C is a basis for $\mathbb{R}^2$.

Graphically, the basis in Example 1.2.9 represents a different set of coordinate axes. Figure 1.2.1 shows how these vectors still form a grid covering all points in $\mathbb{R}^2$.

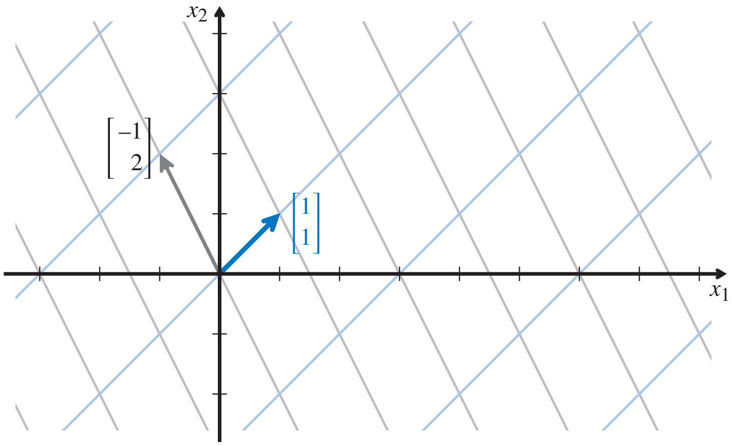


Figure 1.2.1 Geometric representation of the basis $\mathcal{B}$ in Example 1.2.9.

Observe that the coefficients $c_1 = \frac{1}{3}(-x_1 + x_2)$ and $c_2 = \frac{1}{3}(2x_1 + x_2)$ represent the location of the vector $\vec{x}$ on this grid system in the same way that the values x_1 and x_2 represent the location of the vector $\vec{x}$ on the standard coordinate axes.

Definition
Coordinates in $\mathbb{R}^2$

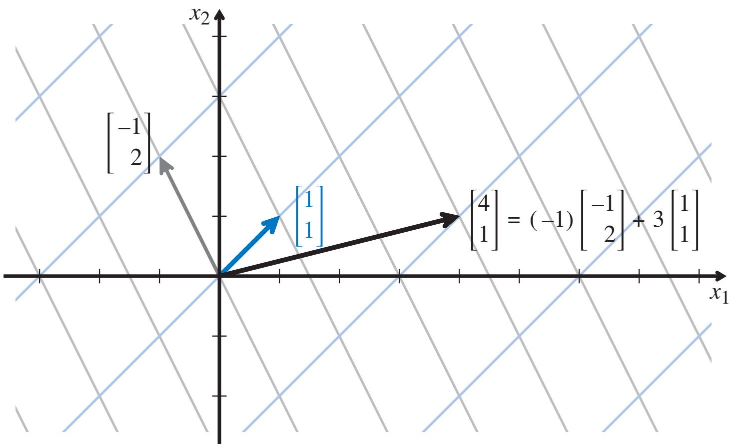
Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$ and let $\vec{x} \in \mathbb{R}^2$. If $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$, then the scalars c_1 and c_2 are called the **coordinates of $\vec{x}$ with respect to the basis $\mathcal{B}$**.

EXAMPLE 1.2.10

Find the coordinates of $\vec{x} = \begin{bmatrix}4\\1\end{bmatrix}$ with respect to the basis $C = \left\{\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$.

Solution: From our work in Example 1.2.9 we have that the coordinates are

$$c_1 = \frac{1}{3}(-4 + 1) = -1, \qquad c_2 = \frac{1}{3}(2(4) + 1) = 3$$



EXAMPLE 1.2.11 Given that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, find the coordinates of $\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ with respect to $\mathcal{B}$.

Solution: We need to find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{bmatrix}$$

Comparing entries gives the system of two equations in two unknowns

$$\begin{aligned} 4 &= c_1 + 2c_2 \\ 1 &= 2c_1 + c_2 \end{aligned}$$

Solving, we find that the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B}$ are $c_1 = -2/3$ and $c_2 = 7/3$.

Applications of Non-Standard Bases

Although we generally use the standard basis in $\mathbb{R}^2$ and $\mathbb{R}^3$, there are applications in which the naturally occurring grid system does not line up with the standard coordinate axes. For example, in crystallography, atoms in a monoclinic crystal are depicted in a Bravais lattice such that the angle between two of the axes is not 90°. Figure 1.2.2 shows a two-dimensional Bravais lattice depicting a monoclinic crystal.

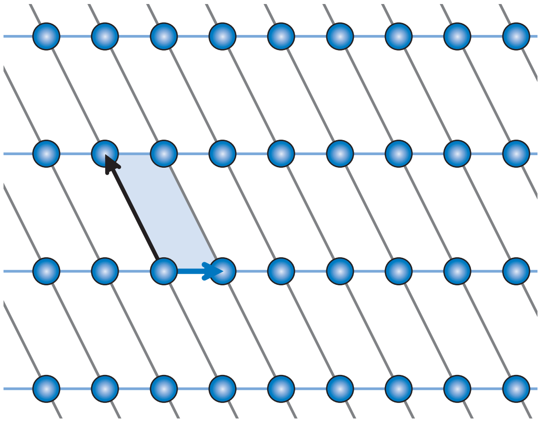


Figure 1.2.2 Two dimensional Bravais lattice of a monoclinic crystal.

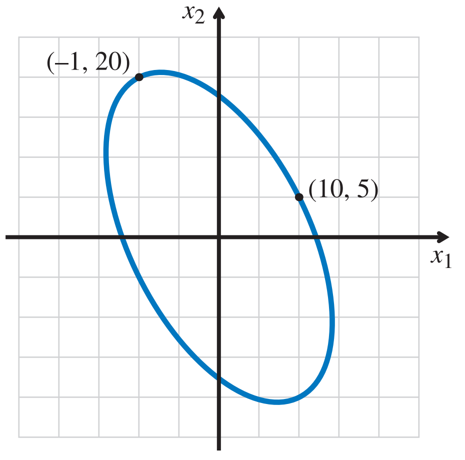
Here is another example demonstrating when it can be advantageous to use an alternate coordinate system.

EXAMPLE 1.2.12

Consider the ellipse

$$17x_1^2 + 12x_1x_2 + 8x_2^2 = 2500$$

This equation, which is written in terms of standard coordinates, does not look easy to use.



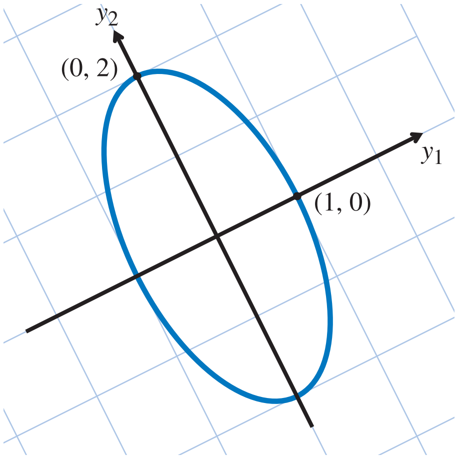
However, if we sketch the ellipse on a new grid system based on the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ 10 \end{bmatrix} \right\}$$

then we see that in this grid system the equation of the ellipse is

$$\frac{y_1^2}{1^2} + \frac{y_2^2}{2^2} = 1 \Rightarrow 4y_1^2 + y_2^2 = 4$$

which is much easier to work with.



CONNECTION

A natural question to ask is how we found the basis $\mathcal{B}$ in Example 1.2.12 that corresponded to the ellipse. This procedure, covered in Section 8.3, requires many of the concepts from Chapters 2, 3, 5, 6, and 8 to understand.

EXERCISE 1.2.2

(a) If $\vec{u}$ is a non-zero vector in $\mathbb{R}^3$, then what is the geometric interpretation of $S_1 = \text{Span}\{\vec{u}\}$?

(b) If $\vec{u}, \vec{v}$ are non-zero vectors in $\mathbb{R}^3$, then what is the geometric interpretation of $\text{Span}\{\vec{u}, \vec{v}\}$?

(c) If $\vec{u}, \vec{v}$ are non-zero vectors in $\mathbb{R}^3$ such that $\{\vec{u}, \vec{v}\}$ is linearly independent, then what is the geometric interpretation of $\text{Span}\{\vec{u}, \vec{v}\}$?

(d) If $\vec{u}, \vec{v}, \vec{w}$ are non-zero vectors in $\mathbb{R}^3$, then what is the geometric interpretation of $\text{Span}\{\vec{u}, \vec{v}, \vec{w}\}$?

PROBLEMS 1.2

Practice Problems

For Problems A1–A6, determine whether the vector $\vec{x}$ is in Span $\mathcal{B}$. If so, write $\vec{x}$ as a linear combination of the vectors in $\mathcal{B}$.

A1 $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

A2 $\vec{x} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

A3 $\vec{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

A4 $\vec{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

A5 $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

A6 $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}$

For Problems A7–A14, determine whether the set is linearly independent. If it is linearly dependent, write one of the vectors in the set as a linear combination of the others.

A7 $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

A8 $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$

A9 $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

A10 $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \end{bmatrix} \right\}$

A11 $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

A12 $\left\{ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

A13 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} \right\}$

A14 $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$

For Problems A15–A20, describe the set geometrically and write a simplified vector equation for the set.

A15 $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

A16 $\operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$

A17 $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix} \right\}$

A18 $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix} \right\}$

A19 $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$

A20 $\operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

For Problems A21–A26, determine whether the set forms a basis for $\mathbb{R}^2$.

A21 $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

A22 $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

A23 $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

A24 $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \end{bmatrix} \right\}$

A25 $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

A26 $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$

For Problems A27–A30, determine whether the set forms a basis for $\mathbb{R}^3$.

A27 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right\}$

A28 $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

A29 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

A30 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

For Problems A31–A33:

(a) Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(b) Find the coordinates of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with respect to $\mathcal{B}$.

A31 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

A32 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

A33 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

A34 Let $\{\vec{v}_1, \vec{v}_2\}$ be a set of vectors in $\mathbb{R}^3$. Prove that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent if and only if neither $\vec{v}_1$ nor $\vec{v}_2$ is a scalar multiple of the other.

A35 Let $\{\vec{v}_1, \vec{v}_2\}$ be a set of vectors in $\mathbb{R}^3$. Prove that $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$ for any $t \in \mathbb{R}$, $t \neq 0$.

Homework Problems

For Problems B1–B6, determine whether the vector $\vec{x}$ is in Span $\mathcal{B}$. If so, write $\vec{x}$ as a linear combination of the vectors in $\mathcal{B}$.

B1 $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

B2 $\vec{x} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

B3 $\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 3 \end{bmatrix} \right\}$

B4 $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

B5 $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

B6 $\vec{x} = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}$

For Problems B7–B14, determine whether the set is linearly independent. If it is linearly dependent, write one of the vectors in the set as a linear combination of the others.

B7 $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

B8 $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \end{bmatrix} \right\}$

B9 $\left\{ \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

B10 $\left\{ \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \end{bmatrix} \right\}$

B11 $\left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$

B12 $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$

B13 $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$

B14 $\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right\}$

For Problems B15–B20, describe the set geometrically and write a simplified vector equation for the set.

B15 Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

B16 Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

B17 Span $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right\}$

B18 Span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

B19 Span $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \\ -2 \end{bmatrix} \right\}$

B20 Span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

For Problems B21–B24, determine whether the set forms a basis for $\mathbb{R}^2$.

B21 $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

B22 $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}$

B23 $\left\{ \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$

B24 $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$

For Problems B25–B28, determine whether the set forms a basis for $\mathbb{R}^3$.

B25 $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$

B26 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

B27 $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

B28 $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

B29 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

(a) Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(b) Find the coordinates of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with respect to $\mathcal{B}$.

For Problems B30–B33:

(a) Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(b) Find the coordinates of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with respect to $\mathcal{B}$.

B30 $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

B31 $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

B32 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

B33 $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$

1.3 Length and Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$

In many physical applications, we are given measurements in terms of angles and magnitudes. We must convert this data into vectors so that we can apply the tools of linear algebra to solve problems. For example, we may need to find a vector representing the path (and speed) of a plane flying northwest at 1300 km/h. To do this, we need to identify the length of a vector and the angle between two vectors. In this section, we see how we can calculate both of these quantities with the dot product operator.

Length, Angles, and Dot Products in $\mathbb{R}^2$

The length of a vector in $\mathbb{R}^2$ is defined by the usual distance formula (that is, Pythagoras' Theorem), as in Figure 1.3.1.

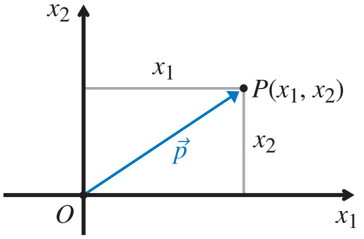


Figure 1.3.1 Length in $\mathbb{R}^2$.

Definition
Length in $\mathbb{R}^2$

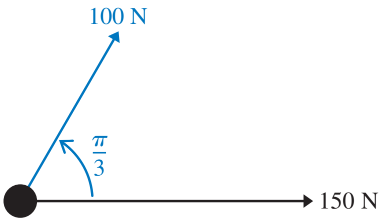
If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, its **length** is defined to be $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$.

EXAMPLE 1.3.1

An object that weighs 10kg is being pulled by two strings with force and direction as given in the diagram. Newton's second law of motion says that the force F acting on the object is equal to its mass m times its acceleration a. What is the resulting acceleration of the object?

Solution: In Example 1.1.1 we found that the net force being applied to the object is $\vec{F} = \begin{bmatrix} 200 \\ 50\sqrt{3} \end{bmatrix}$. The total amount of force being applied will be the length of this vector:

$$\begin{aligned} \left\|\vec{F}\right\| &= \sqrt{(200)^2 + (50\sqrt{3})^2} \\ &= \sqrt{47500} \\ &\approx 218 \end{aligned}$$



Hence, the total amount of force is about $F = 218$ N. Thus, we find that the acceleration is approximately

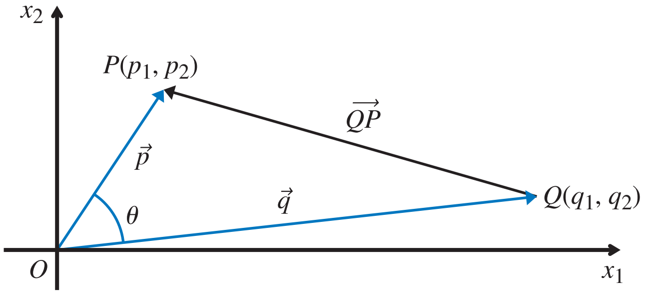
$$a = \frac{F}{m} = \frac{218}{10} = 21.8 m/s^2$$

In the example above, we may also be interested in finding the angle from the horizontal at which the object is moving.

Theorem 1.3.1 If $\vec{p}, \vec{q} \in \mathbb{R}^2$ and θ is an angle between $\vec{p}$ and $\vec{q}$, then

$$p_1q_1 + p_2q_2 = \|\vec{p}\|\,\|\vec{q}\|\cos\theta$$

Proof: Consider the figure below.



The Law of Cosines gives

$$\|\vec{QP}\|^2 = \|\vec{p}\|^2 + \|\vec{q}\|^2 - 2\|\vec{p}\|\,\|\vec{q}\|\cos\theta \tag{1.3}$$

Substituting $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, $\vec{QP} = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \end{bmatrix}$ into (1.3) and simplifying gives

$$p_1q_1 + p_2q_2 = \|\vec{p}\|\,\|\vec{q}\|\cos\theta$$

■

Remark

When solving for θ we usually choose θ such that $0 \leq \theta \leq \pi$.

EXAMPLE 1.3.2 Find the angle θ at which the object in Example 1.3.1 is moving.

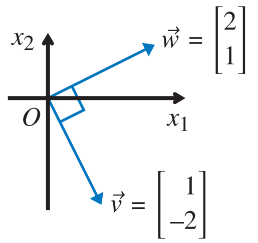
Solution: We need to calculate the angle between the net force vector and either of the initial force vectors. We have $\vec{F}_1 = \begin{bmatrix} 150 \\ 0 \end{bmatrix}$ and $\vec{F} = \begin{bmatrix} 200 \\ 50\sqrt{3} \end{bmatrix}$. Thus, an angle θ between them satisfies

$$\begin{aligned} 150(200) + 0(50\sqrt{3}) &= \|\vec{F}_1\|\,\|\vec{F}\|\cos\theta \\ 150(200) &\approx 150(218)\cos\theta \\ \frac{100}{109} &\approx \cos\theta \end{aligned}$$

We get that $\theta \approx 0.41$ radians.

EXAMPLE 1.3.3

Find the angle in $\mathbb{R}^2$ between $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.



Solution: We have $v_1w_1 + v_2w_2 = 1(2) + (-2)(1) = 0$. Hence, $\cos\theta = \dfrac{0}{\|\vec{v}\|\,\|\vec{w}\|} = 0$. Thus, $\theta = \dfrac{\pi}{2}$ radians. That is, $\vec{v}$ and $\vec{w}$ are perpendicular to each other.

The formula $p_1q_1+p_2q_2$ on the left-hand side of the equation for the angle between vectors in Theorem 1.3.1 matches with what we use in the equation for the length of a vector. So, we make the following definition.

Definition
Dot Product in $\mathbb{R}^2$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. The **dot product** of $\vec{x}$ and $\vec{y}$, denoted $\vec{x} \cdot \vec{y}$, is defined by

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2$$

Using this definition, we can rewrite our formulas for angles and length as

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$
$$\vec{x} \cdot \vec{y} = \|\vec{x}\|\,\|\vec{y}\| \cos\theta$$

Using the dot product and what we saw in Example 1.3.3, we make the following definition.

Definition
Orthogonal Vectors in $\mathbb{R}^2$

Two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$ are **orthogonal** to each other if and only if $\vec{x} \cdot \vec{y} = 0$.

EXAMPLE 1.3.4

Show that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are orthogonal.

Solution: We have $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1(-2) + 2(1) = 0$. Hence, by definition, they are orthogonal.

Remark

Notice that the definition of orthogonality implies that the zero vector $\vec{0}$ is orthogonal to every vector in $\mathbb{R}^2$.

Length, Angles, and Dot Products in $\mathbb{R}^3$

To define length and angles in $\mathbb{R}^3$, we repeat what we did in $\mathbb{R}^2$. This is easiest if we begin by defining the dot product in $\mathbb{R}^3$.

Definition
Dot Product in $\mathbb{R}^3$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The **dot product** of $\vec{x}$ and $\vec{y}$, denoted $\vec{x} \cdot \vec{y}$, is defined by

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3$$

EXAMPLE 1.3.5

Calculate the dot product of the following pairs of vectors.

(a) $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (b) $\vec{u} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$

Solution: (a) We have $\vec{x} \cdot \vec{y} = 1(2) + (-2)(1) + 1(3) = 3$.

(b) We have $\vec{u} \cdot \vec{v} = \frac{1}{2}(1) + 0(5) + \frac{1}{2}(-1) = 0$.

The dot product has the following important properties.

Theorem 1.3.2

If $\vec{x}, \vec{y}, \vec{z}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $s, t \in \mathbb{R}$, then

(1) $\vec{x} \cdot \vec{x} \geq 0$
(2) $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$
(3) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
(4) $\vec{x} \cdot (s\vec{y} + t\vec{z}) = s(\vec{x} \cdot \vec{y}) + t(\vec{x} \cdot \vec{z})$

For vectors in $\mathbb{R}^3$, the formula for the length can be obtained from a two-step calculation using the formula in $\mathbb{R}^2$. Consider the points $X(x_1, x_2, x_3)$ and $P(x_1, x_2, 0)$. Observe that OPX is a right triangle, so that

$$\|\vec{x}\|^2 = \|\vec{OP}\|^2 + \|\vec{PX}\|^2 = (x_1^2 + x_2^2) + x_3^2$$

Definition
Length in $\mathbb{R}^3$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$, its **length** is defined to be

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$

One immediate application of this formula is to calculate the distance between two points. In particular, if we have points P and Q, then the distance between them is the length of the directed line segment $\vec{PQ}$.

EXAMPLE 1.3.6

Find the distance between the points $P(-1,3,4)$ and $Q(2,-5,1)$ in $\mathbb{R}^3$.

Solution: We have $\vec{PQ} = \begin{bmatrix} 2-(-1) \\ -5-3 \\ 1-4 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -3 \end{bmatrix}$. Hence, the distance between the two points is

$$\|\vec{PQ}\| = \sqrt{\vec{PQ}\cdot\vec{PQ}} = \sqrt{3^2+(-8)^2+(-3)^2} = \sqrt{82}$$

We also find that if $\vec{p}$ and $\vec{q}$ are vectors in $\mathbb{R}^3$ and θ is an angle between them, then

$$\vec{p}\cdot\vec{q} = \|\vec{p}\|\,\|\vec{q}\|\cos\theta$$

EXAMPLE 1.3.7

Find the angle in $\mathbb{R}^3$ between $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$.

Solution: We have

$$\begin{aligned} \vec{v}\cdot\vec{w} &= 1(3)+4(-1)+(-2)(4) = -9 \\ \|\vec{v}\| &= \sqrt{1^2+4^2+(-2)^2} = \sqrt{21} \\ \|\vec{w}\| &= \sqrt{3^2+(-1)^2+4^2} = \sqrt{26} \end{aligned}$$

Hence,

$$\cos\theta = \frac{-9}{\sqrt{21}\sqrt{26}} \approx -0.38516$$

So, $\theta \approx 1.966$ radians.

EXERCISE 1.3.1

Find the angle in $\mathbb{R}^3$ between $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

If the angle θ between $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$ is $\frac{\pi}{2}$ radians, then, as in $\mathbb{R}^2$, we get

$$\vec{x}\cdot\vec{y} = \|\vec{x}\|\,\|\vec{y}\|\cos\frac{\pi}{2} = 0$$

Definition
Orthogonal Vectors in $\mathbb{R}^3$

Two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$ are **orthogonal** to each other if and only if $\vec{x}\cdot\vec{y} = 0$.

EXAMPLE 1.3.8

Show that $\vec{x} = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 9 \\ 2 \\ -3 \end{bmatrix}$ are orthogonal.

Solution: We have $\vec{x} \cdot \vec{y} = 3(9) + (-6)(2) + 5(-3) = 0$ as required.

Scalar Equations of Planes

We saw in Section 1.1 that a plane can be described by the vector equation $\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2, t_1, t_2 \in \mathbb{R}$, where $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. In many problems, it is more useful to have a **scalar equation** that represents the plane. We now look at how to use the dot product and orthogonality to find such an equation.

Suppose that we want to find an equation of the plane that passes through the point $P(p_1, p_2, p_3)$. Suppose that we can find a non-zero vector $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$, called a **normal vector** of the plane, that is orthogonal to any directed line segment $\vec{PQ}$ lying in the plane. (That is, $\vec{n}$ is orthogonal to $\vec{PQ}$ for any point Q in the plane; see Figure 1.3.2.) To find the equation of this plane, let $X(x_1, x_2, x_3)$ be any point on the plane. Then $\vec{n}$ is orthogonal to $\vec{PX}$, so

$$0 = \vec{n} \cdot \vec{PX} = \vec{n} \cdot (\vec{x} - \vec{p}) = n_1(x_1 - p_1) + n_2(x_2 - p_2) + n_3(x_3 - p_3)$$

This equation, which must be satisfied by the coordinates of a point X in the plane, can be written in the form

$$n_1x_1 + n_2x_2 + n_3x_3 = d, \qquad \text{where} \qquad d = n_1p_1 + n_2p_2 + n_3p_3 = \vec{n} \cdot \vec{p}$$

This is a scalar equation of this plane. For computational purposes, the form $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ is often easiest to use.

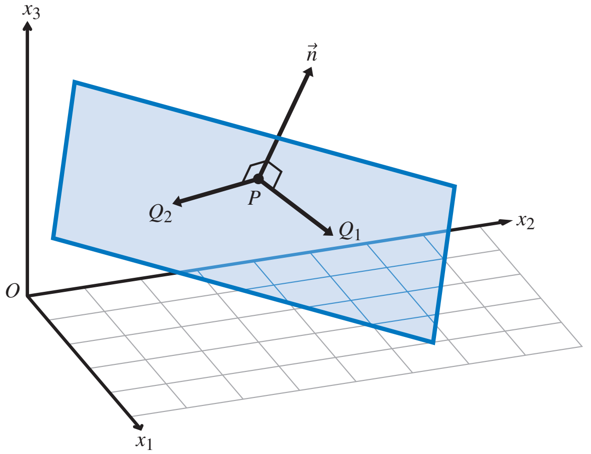


Figure 1.3.2 The normal $\vec{n}$ is orthogonal to every directed line segment lying in the plane.

EXAMPLE 1.3.9

Find a scalar equation of the plane that passes through the point $P(2,3,-1)$ and has normal vector $\vec{n} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$.

Solution: The equation is

$$\vec{n} \cdot (\vec{x} - \vec{p}) = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 2 \\ x_2 - 3 \\ x_3 + 1 \end{bmatrix} = 0$$

or

$$x_1 - 4x_2 + x_3 = 1(2) + (-4)(3) + 1(-1) = -11$$

Note that any non-zero scalar multiple of $\vec{n}$ will also be a normal vector for the plane. Moreover, our argument above works in reverse. That is, if $n_1x_1 + n_2x_2 + n_3x_3 = d$ is a scalar equation of the plane, then $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is a normal vector for the plane.

EXAMPLE 1.3.10

Find a normal vector of the plane with scalar equation $x_1 - 2x_3 = 5$.

Solution: We should think of this equation as $x_1 + 0x_2 - 2x_3 = 5$. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

Two planes are defined to be **parallel** if the normal vector to one plane is a non-zero scalar multiple of the normal vector of the other plane. Thus, for example, the plane $x_1 + 2x_2 - x_3 = 1$ is parallel to the plane $2x_1 + 4x_2 - 2x_3 = 7$.

Two planes are **orthogonal** to each other if their normal vectors are orthogonal. For example, the plane $x_1 + x_2 + x_3 = 0$ is orthogonal to the plane $x_1 + x_2 - 2x_3 = 0$ since

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 0$$

EXAMPLE 1.3.11

Find a scalar equation of the plane that contains the point $P(2,4,-1)$ and is parallel to the plane $2x_1 + 3x_2 - 5x_3 = 6$.

Solution: A scalar equation of the desired plane can have the form $2x_1 + 3x_2 - 5x_3 = d$ since the planes are parallel. The plane must pass through P, so we find that a scalar equation of the plane is

$$2x_1 + 3x_2 - 5x_3 = 2(2) + 3(4) - 5(-1) = 21$$

EXAMPLE 1.3.12 Find a scalar equation of a plane that contains the point $P(3,-1,3)$ and is orthogonal to the plane $x_1 - 2x_2 + 4x_3 = 2$.

Solution: Any normal vector of the desired plane must be orthogonal to $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. We pick $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. Thus, a scalar equation of this plane can have the form $2x_2 + x_3 = d$. Since the plane passes through P, we find that a scalar equation of this plane is

$$2x_2 + x_3 = 0(3) + 2(-1) + 1(3) = 1$$

EXERCISE 1.3.2 Find a scalar equation of the plane that contains the point $P(1,2,3)$ and is parallel to the plane $x_1 - 3x_2 - 2x_3 = -5$.

It is important to note that we can reverse the reasoning that leads to the scalar equation of the plane in order to identify the set of points that satisfies the equation

$$n_1x_1 + n_2x_2 + n_3x_3 = d$$

If $n_1 \neq 0$, we can solve this equation for x_1 to get

$$x_1 = \frac{d}{n_1} - \frac{n_2}{n_1}x_2 - \frac{n_3}{n_1}x_3$$

Hence, every vector $\vec{x}$ in the plane satisfies the vector equation

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{d}{n_1} - \frac{n_2}{n_1}x_2 - \frac{n_3}{n_1}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d/n_1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -n_2/n_1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -n_3/n_1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

Observe that this is a vector equation of a plane through the point $P(d/n_1, 0, 0)$. If $n_2 \neq 0$ or $n_3 \neq 0$, we could instead solve the scalar equation of the plane for x_2 or x_3 respectively to get alternate vector equations for the plane.

EXAMPLE 1.3.13 Find a vector equation of the plane in $\mathbb{R}^3$ that satisfies $5x_1 - 6x_2 + 7x_3 = 11$.

Solution: Solving the scalar equation of the plane for x_1 gives

$$x_1 = \frac{11}{5} + \frac{6}{5}x_2 - \frac{7}{5}x_3$$

Hence, a vector equation is

$$\vec{x} = \begin{bmatrix} \frac{11}{5} + \frac{6}{5}x_2 - \frac{7}{5}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/5 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 6/5 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -7/5 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

EXERCISE 1.3.3 Find a vector equation of the plane in $\mathbb{R}^3$ that satisfies $2x_1 + 3x_2 - x_3 = 0$.

At first, one may wonder why we would want both a scalar equation and a vector equation of a plane. Depending on what we need, both representations have their uses. For example, the scalar equation makes it very easy to check if a given point $Q(q_1, q_2, q_3)$ is in the plane. We just have to verify whether or not

$$n_1q_1 + n_2q_2 + n_3q_3 = d$$

On the other hand, if we want to generate 100 points in the plane, the vector equation

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}, \quad s, t \in \mathbb{R}$$

is more suitable. We just need to pick 100 different pairs of s and t.

Cross Products

Given a pair of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, how can we find a third vector $\vec{w}$ that is orthogonal to both $\vec{u}$ and $\vec{v}$? This problem arises naturally in many ways. For example, in physics, it is observed that the force on an electrically charged particle moving in a magnetic field is in the direction orthogonal to the velocity of the particle and to the vector describing the magnetic field.

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. If $\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, it must satisfy the equations

$$\begin{aligned} \vec{u} \cdot \vec{w} &= u_1w_1 + u_2w_2 + u_3w_3 = 0 \\ \vec{v} \cdot \vec{w} &= v_1w_1 + v_2w_2 + v_3w_3 = 0 \end{aligned}$$

In Chapter 2, we shall develop systematic methods for solving such equations for w_1, w_2, w_3. For the present, we simply give a solution:

$$\vec{w} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Definition
Cross Product

The **cross product** of vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$ is defined by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

EXAMPLE 1.3.14

Calculate the cross product of $\begin{bmatrix}2\\3\\5\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\2\end{bmatrix}$ in $\mathbb{R}^3$.

Solution: $\begin{bmatrix}2\\3\\5\end{bmatrix} \times \begin{bmatrix}-1\\1\\2\end{bmatrix} = \begin{bmatrix}6-5\\-5-4\\2-(-3)\end{bmatrix} = \begin{bmatrix}1\\-9\\5\end{bmatrix}$.

Remarks

1. Unlike the dot product of two vectors, which is a scalar, the cross product of two vectors in $\mathbb{R}^3$ is itself a new vector.
2. The cross product is a construction that is defined only in $\mathbb{R}^3$. (There is a generalization to higher dimensions, but it is considerably more complicated, and it will not be considered in this book.)

The formula for the cross product is a little awkward to remember, but there are many tricks for remembering it. One way is to write the components of $\vec{u}$ in a row above the components of $\vec{v}$:

$$\begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{matrix}$$

Then, for the first entry in $\vec{u} \times \vec{v}$, we cover the first column and calculate the difference of the products of the cross-terms:

$$\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \Rightarrow u_2v_3 - u_3v_2$$

For the second entry in $\vec{u} \times \vec{v}$, we cover the second column and take the negative of the difference of the products of the cross-terms:

$$-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \Rightarrow -(u_1v_3 - u_3v_1)$$

Similarly, for the third entry, we cover the third column and calculate the difference of the products of the cross-terms:

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \Rightarrow u_1v_2 - u_2v_1$$

Note carefully that the second term must be given a minus sign in order for this procedure to provide the correct answer. Since the formula can be difficult to remember, we recommend checking the answer by verifying that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

EXERCISE 1.3.4

Calculate the cross product of $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$.

By construction, $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$, so the direction of $\vec{u} \times \vec{v}$ is known except for the sign: does it point "up" or "down"? The general rule is as follows: the three vectors $\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$, taken in this order, form a right-handed system (see page 10). Let us see how this works for simple cases.

EXERCISE 1.3.5

Let $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ be the standard basis vectors in $\mathbb{R}^3$. Verify that

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \quad \vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \quad \vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

but

$$\vec{e}_2 \times \vec{e}_1 = -\vec{e}_3, \quad \vec{e}_3 \times \vec{e}_2 = -\vec{e}_1, \quad \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$$

Check that in every case, the three vectors taken in order form a right-handed system.

These simple examples also suggest some of the general properties of the cross product.

Theorem 1.3.3

For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we have

(1) $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$
(2) $\vec{x} \times \vec{x} = \vec{0}$
(3) $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$
(4) $(t\vec{x}) \times \vec{y} = t(\vec{x} \times \vec{y}) = \vec{x} \times (t\vec{y})$
(5) $\vec{x} \times \vec{y} = \vec{0}$ if and only if either $\vec{x} = \vec{0}$ or $\vec{y}$ is a scalar multiple of $\vec{x}$
(6) If $\vec{n} = \vec{x} \times \vec{y}$, then for any $\vec{w} \in \operatorname{Span}\{\vec{x}, \vec{y}\}$ we have $\vec{w} \cdot \vec{n} = 0$

Proof: These properties follow easily from the definition of the cross product and are left to the reader.

One rule we might expect does not in fact hold. In general,

$$\vec{x} \times (\vec{y} \times \vec{z}) \neq (\vec{x} \times \vec{y}) \times \vec{z}$$

This means that the parentheses cannot be omitted in a cross product. (There are formulas available for these triple-vector products, but we shall not need them. See Problem F3 in Further Problems at the end of this chapter.)

Applications of the Cross Product

Finding the Normal to a Plane

In Section 1.1, the vector equation of a plane was given in the form $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$, where $\{\vec{u}, \vec{v}\}$ is linearly independent. By definition, a normal vector $\vec{n}$ must be a non-zero vector orthogonal to both $\vec{u}$ and $\vec{v}$. Therefore, we can take $\vec{n} = \vec{u} \times \vec{v}$.

EXAMPLE 1.3.15

The lines $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, s \in \mathbb{R}$ and $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$ must lie in a common plane since they have the point $(1, 3, 2)$ in common. Find a scalar equation of the plane that contains these lines.

Solution: To find a normal vector for the plane we take the cross product of the direction vectors for the lines. We get

$$\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$$

Therefore, since the plane passes through $P(1, 3, 2)$, we find that a scalar equation of the plane is

$$-4x_1 - 3x_2 + 2x_3 = (-4)(1) + (-3)(3) + 2(2) = -9$$

EXAMPLE 1.3.16

Find a scalar equation of the plane that contains the three points $P(1, -2, 1)$, $Q(2, -2, -1)$, and $R(4, 1, 1)$.

Solution: Since P, Q, and R lie in the plane, then so do the directed line segments $\vec{PQ}$ and $\vec{PR}$. Hence, the normal to the plane is given by

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$$

Since the plane passes through P, we find that a scalar equation of the plane is

$$6x_1 - 6x_2 + 3x_3 = (6)(1) + (-6)(-2) + 3(1) = 21, \quad \text{or} \quad 2x_1 - 2x_2 + x_3 = 7$$

EXERCISE 1.3.6

Find a scalar equation of the plane with vector equation

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

The Length of the Cross Product

Given $\vec{u}$ and $\vec{v}$, the direction of their cross product is known. What is the length of the cross product of $\vec{u}$ and $\vec{v}$? We give the answer in the following theorem.

Theorem 1.3.4 Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ and θ be the angle between $\vec{u}$ and $\vec{v}$, then $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\,\|\vec{v}\| \sin\theta$.

Proof: We give an outline of the proof. We have

$$\|\vec{u} \times \vec{v}\|^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$

Expand by the binomial theorem and then add and subtract the term $(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2)$. The resulting terms can be arranged so as to be seen to be equal to

$$(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$$

Thus,

$$\begin{aligned} \|\vec{u} \times \vec{v}\|^2 &= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2 \cos^2\theta \\ &= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta) = \|\vec{u}\|^2\|\vec{v}\|^2 \sin^2\theta \end{aligned}$$

and the result follows. ■

To interpret this formula, consider Figure 1.3.3. Assuming $\{\vec{u}, \vec{v}\}$ is linearly independent, then the vectors $\vec{u}$ and $\vec{v}$ determine a parallelogram. Take the length of $\vec{u}$ to be the base of the parallelogram. From trigonometry, we know that the length of the altitude is $\|\vec{v}\| \sin\theta$. So, the area of the parallelogram is

$$(\text{base}) \times (\text{altitude}) = \|\vec{u}\|\,\|\vec{v}\| \sin\theta = \|\vec{u} \times \vec{v}\|$$

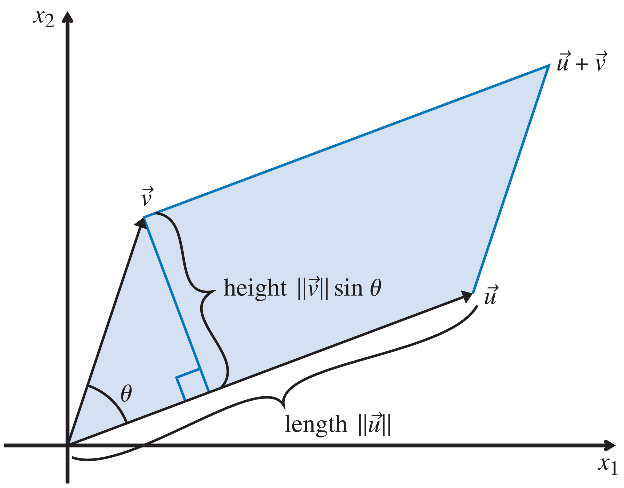


Figure 1.3.3 The area of the parallelogram is $\|\vec{u}\|\,\|\vec{v}\| \sin\theta$.

EXAMPLE 1.3.17

Find the area of the parallelogram determined by $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: By Theorem 1.3.4 and our work above, we find that the area is

$$\|\vec{u} \times \vec{v}\| = \left\| \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} \right\| = \sqrt{(-3)^2 + (-2)^2 + (-1)^2} = \sqrt{14}$$

EXERCISE 1.3.7

Find the area of the parallelogram determined by $\vec{u} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

Finding the Line of Intersection of Two Planes

Unless two planes in $\mathbb{R}^3$ are parallel, their intersection will be a line. The direction vector of this line lies in both planes, so it is perpendicular to both of the normals. It can therefore be obtained as the cross product of the two normals. Once we find a point that lies on this line, we can write the vector equation of the line.

EXAMPLE 1.3.18

Find a vector equation of the line of intersection of the two planes $x_1 + x_2 - 2x_3 = 3$ and $2x_1 - x_2 + 3x_3 = 6$.

Solution: The normal vectors of the planes are $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Hence, a direction vector of the line of intersection is

$$\vec{d} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -3 \end{bmatrix}$$

One easy way to find a point on the line is to let $x_3 = 0$ and then solve the remaining equations $x_1 + x_2 = 3$ and $2x_1 - x_2 = 6$. The solution is $x_1 = 3$ and $x_2 = 0$. Hence, a vector equation of the line of intersection is

$$\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -7 \\ -3 \end{bmatrix}, \quad t \in \mathbb{R}$$

EXERCISE 1.3.8

Find a vector equation of the line of intersection of the two planes $-x_1 - 2x_2 + x_3 = -2$ and $2x_1 + x_2 - 2x_3 = 1$.

PROBLEMS 1.3

Practice Problems

For Problems A1–A6, calculate the length of the vector.

A1 $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ **A2** $\begin{bmatrix} 2/\sqrt{29} \\ -5/\sqrt{29} \end{bmatrix}$ **A3** $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

A4 $\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$ **A5** $\begin{bmatrix} 1 \\ 1/5 \\ -3 \end{bmatrix}$ **A6** $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$

For Problems A7–A10, calculate the distance from P to Q.

A7 $P(2,3), Q(-4,1)$ **A8** $P(1,1,-2), Q(-3,1,1)$

A9 $P(4,-6,1), Q(-3,5,1)$ **A10** $P(2,1,1), Q(4,6,-2)$

For Problems A11–A16, determine whether the pair of vectors is orthogonal.

A11 $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ **A12** $\begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

A13 $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$ **A14** $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$

A15 $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **A16** $\begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \end{bmatrix}$

For Problems A17–A20, determine all values of k for which the pair of vectors is orthogonal.

A17 $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ k \end{bmatrix}$ **A18** $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} k \\ k^2 \end{bmatrix}$

A19 $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -k \\ k \end{bmatrix}$ **A20** $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} k \\ k \\ -k \end{bmatrix}$

For Problems A21–A23, find a scalar equation of the plane that contains the given point with the given normal vector.

A21 $P(-1,2,-3), \vec{n} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ **A22** $P(2,5,4), \vec{n} = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$

A23 $P(1,-1,1), \vec{n} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$

For Problems A24–A29, calculate the cross product.

A24 $\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ **A25** $\begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} \times \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

A26 $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$ **A27** $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$

A28 $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ **A29** $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

A30 Let $\vec{u} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$. Check by calculation that the following general properties hold.

(a) $\vec{u} \times \vec{u} = \vec{0}$

(b) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

(c) $\vec{u} \times 3\vec{w} = 3(\vec{u} \times \vec{w})$

(d) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(e) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times \vec{v})$

(f) $\vec{u} \cdot (\vec{v} \times \vec{w}) = -\vec{v} \cdot (\vec{u} \times \vec{w})$

For Problems A31–A34, determine a scalar equation of the plane with the given vector equation.

A31 $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + s\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + t\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad s,t \in \mathbb{R}$

A32 $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad s,t \in \mathbb{R}$

A33 $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}$

A34 $\vec{x} = s\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix}, \quad s,t \in \mathbb{R}$

For Problems A35–A40, determine a vector equation of the plane with the given scalar equation.

A35 $2x_1 - 3x_2 + x_3 = 0$ **A36** $4x_1 + x_2 - 2x_3 = 5$

A37 $x_1 + 2x_2 + 2x_3 = 1$ **A38** $3x_1 + 5x_2 - 4x_3 = 7$

A39 $6x_1 - 3x_2 + 9x_3 = 0$ **A40** $2x_1 + x_2 + 3x_3 = 3$

For Problems A41–A46, determine a scalar equation of the plane that contains the set of points.

A41 $P(2,1,5), Q(4,-3,2), R(2,6,-1)$

A42 $P(3,1,4), Q(-2,0,2), R(1,4,-1)$

A43 $P(-1,4,2), Q(3,1,-1), R(2,-3,-1)$

A44 $P(1,0,1), Q(-1,0,1), R(0,0,0)$

A45 $P(0,2,1), Q(3,-1,1), R(1,3,0)$

A46 $P(1,5,-3), Q(2,6,-1), R(1,0,1)$

For Problems A47–A49, find a scalar equation of the plane through the given point and parallel to the given plane.

A47 $P(1,-3,-1)$, $2x_1 - 3x_2 + 5x_3 = 17$

A48 $P(0,-2,4)$, $x_2 = 0$

A49 $P(1,2,1)$, $x_1 - x_2 + 3x_3 = 5$

For Problems A50–A53, determine a vector equation of the line of intersection of the given planes.

A50 $x_1 + 3x_2 - x_3 = 5$ and $2x_1 - 5x_2 + x_3 = 7$

A51 $2x_1 - 3x_3 = 7$ and $x_2 + 2x_3 = 4$

A52 $x_1 - 2x_2 + x_3 = 1$ and $3x_1 + 4x_2 - x_3 = 5$

A53 $x_1 - 2x_2 + x_3 = 0$ and $3x_1 + 4x_2 - x_3 = 0$

For Problems A54–A56, calculate the area of the parallelogram determined by each pair of vectors.

A54 $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\-1\end{bmatrix}$ **A55** $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}$ **A56** $\begin{bmatrix}-3\\1\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}$

(Hint: For A56, think of the vectors as $\begin{bmatrix}-3\\1\\0\end{bmatrix}, \begin{bmatrix}4\\3\\0\end{bmatrix} \in \mathbb{R}^3$.)

A57 What does it mean, geometrically, if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$?

A58 Show that $(\vec{u} - \vec{v}) \times (\vec{u} + \vec{v}) = 2(\vec{u} \times \vec{v})$.

Homework Problems

For Problems B1–B6, calculate the length of the vector.

B1 $\begin{bmatrix}1\\-4\end{bmatrix}$ **B2** $\begin{bmatrix}2\\3\end{bmatrix}$ **B3** $\begin{bmatrix}0\\0\\0\end{bmatrix}$

B4 $\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}$ **B5** $\begin{bmatrix}-1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$ **B6** $\begin{bmatrix}2/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$

For Problems B7–B13, calculate the distance from P to Q.

B7 $P(3,1)$, $Q(-2,2)$ **B8** $P(2,5)$, $Q(3,9)$

B9 $P(1,0)$, $Q(-3,5)$ **B10** $P(3,-1,3)$, $Q(6,-2,4)$

B11 $P(7,3,-5)$, $Q(9,-1,-3)$

B12 $P(4,0,2)$, $Q(5,-2,-1)$

B13 $P(1,-2,1)$, $Q(3,5,-1)$

For Problems B14–B19, determine whether the pair of vectors is orthogonal.

B14 $\begin{bmatrix}9\\-3\end{bmatrix}, \begin{bmatrix}2\\-6\end{bmatrix}$ **B15** $\begin{bmatrix}4\\6\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}$

B16 $\begin{bmatrix}4\\3\\-5\end{bmatrix}, \begin{bmatrix}-4\\-3\\-5\end{bmatrix}$ **B17** $\begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}-2\\0\\2\end{bmatrix}$

B18 $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-3\\5\\1\end{bmatrix}$ **B19** $\begin{bmatrix}1\\3\\8\end{bmatrix}, \begin{bmatrix}-2\\-2\\1\end{bmatrix}$

For Problems B20–B23, determine all values of k for which the pair of vectors is orthogonal.

B20 $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}k\\2k\end{bmatrix}$ **B21** $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}2k\\k^2\end{bmatrix}$

B22 $\begin{bmatrix}1\\3\\-1\end{bmatrix}, \begin{bmatrix}k\\2k\\2\end{bmatrix}$ **B23** $\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}k\\3k\\-k^2\end{bmatrix}$

For Problems B24–B28, find a scalar equation of the plane that contains the given point with the given normal vector.

B24 $P(3,9,2)$, $\vec{n} = \begin{bmatrix}1\\-1\\5\end{bmatrix}$ **B25** $P(4,3,1)$, $\vec{n} = \begin{bmatrix}3\\3\\-4\end{bmatrix}$

B26 $P(0,2,1)$, $\vec{n} = \begin{bmatrix}0\\-4\\-2\end{bmatrix}$ **B27** $P(1,3,1)$, $\vec{n} = \begin{bmatrix}1\\3\\1\end{bmatrix}$

B28 $P(0,0,0)$, $\vec{n} = \begin{bmatrix}5\\-6\\3\end{bmatrix}$

For Problems B29–B34, calculate the cross product.

B29 $\begin{bmatrix}2\\-1\\1\end{bmatrix} \times \begin{bmatrix}-6\\3\\-6\end{bmatrix}$ **B30** $\begin{bmatrix}3\\1\\-2\end{bmatrix} \times \begin{bmatrix}-1\\3\\-1\end{bmatrix}$

B31 $\begin{bmatrix}1\\2\\1\end{bmatrix} \times \begin{bmatrix}2\\4\\2\end{bmatrix}$ **B32** $\begin{bmatrix}1\\3\\-2\end{bmatrix} \times \begin{bmatrix}5\\-1\\1\end{bmatrix}$

B33 $\begin{bmatrix}5\\-1\\1\end{bmatrix} \times \begin{bmatrix}1\\3\\-2\end{bmatrix}$ **B34** $\begin{bmatrix}2\\4\\6\end{bmatrix} \times \begin{bmatrix}4\\2\\2\end{bmatrix}$

B35 Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. Check by calculation that the following general properties hold.

(a) $\vec{u} \times \vec{u} = \vec{0}$
(b) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
(c) $\vec{u} \times 2\vec{w} = 2(\vec{u} \times \vec{w})$
(d) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
(e) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times \vec{v})$
(f) $\vec{u} \cdot (\vec{v} \times \vec{w}) = -\vec{v} \cdot (\vec{u} \times \vec{w})$

For Problems B36–B41, determine a scalar equation of the plane with the given vector equation.

B36 $\vec{x} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad s, t \in \mathbb{R}$

B37 $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad s, t \in \mathbb{R}$

B38 $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$

B39 $\vec{x} = s\begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad s, t \in \mathbb{R}$

B40 $\vec{x} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad s, t \in \mathbb{R}$

B41 $\vec{x} = s\begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad s, t \in \mathbb{R}$

For Problems B42–B47, determine a vector equation of the plane with the given scalar equation.

B42 $-x_1 + 5x_2 + 2x_3 = 2$

B43 $x_2 + x_3 = 1$

B44 $2x_1 + 2x_2 + 4x_3 = 0$

B45 $x_1 + x_2 - x_3 = 6$

B46 $3x_1 + 5x_2 - x_3 = 0$

B47 $-2x_1 - 2x_2 + 3x_3 = 4$

For Problems B48–B53, determine a scalar equation of the plane that contains the given points.

B48 $P(6, 3, 2), Q(4, 3, 3), R(9, 2, 6)$

B49 $P(4, 3, -2), Q(2, 1, 5), R(3, 1, 1)$

B50 $P(0, 2, 1), Q(4, 1, 0), R(2, 2, 0)$

B51 $P(3, 1, 2), Q(-2, -6, -1), R(1, 1, 1)$

B52 $P(9, 2, -1), Q(8, 3, 3), R(7, 3, 1)$

B53 $P(-3, 0, 3), Q(5, 5, 1), R(-2, -2, 0)$

For Problems B54–B59, find a scalar equation of the plane through the given point and parallel to the given plane.

B54 $P(2, 4, -3), 4x_1 + x_2 + 2x_3 = 2$

B55 $P(1, -2, 6), -x_1 + 2x_2 - 3x_3 = 3$

B56 $P(3, 1, 2), 2x_1 + 3x_3 = 1$

B57 $P(1, 1, 0), -x_1 - 5x_2 + 3x_3 = 5$

B58 $P(0, 0, 0), 2x_1 + 3x_2 - 4x_3 = 1$

B59 $P(-5, 2, 8), 4x_1 + 2x_2 + 2x_3 = 5$

For Problems B60–B64, determine a vector equation of the line of intersection of the given planes.

B60 $x_1 + 2x_2 + x_3 = 1$ and $2x_1 - 3x_2 + x_3 = 4$

B61 $-x_1 + 2x_2 - 2x_3 = 1$ and $x_1 + 2x_2 - x_3 = 2$

B62 $x_1 - 2x_2 + x_3 = 1$ and $3x_1 + x_2 - x_3 = 4$

B63 $2x_1 + 3x_2 + x_3 = 5$ and $4x_1 - 2x_2 + x_3 = 6$

B64 $5x_1 + 2x_2 - x_3 = 0$ and $4x_1 + x_2 - 3x_3 = 2$

For Problems B65–B70, calculate the area of the parallelogram determined by the given vectors.

B65 $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

B66 $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

B67 $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

B68 $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \end{bmatrix}$

B69 $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

B70 $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

(Hint: For Problems B54–B59 use the same method as for Problem A56.)

Conceptual Problems

C1 (a) Using geometrical arguments in $\mathbb{R}^3$, what can you say about the vectors $\vec{p}$, $\vec{n}$, and $\vec{d}$ if the line with vector equation $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$ and the plane with scalar equation $\vec{n} \cdot \vec{x} = k$ have no point of intersection?

(b) Confirm your answer in part (a) by determining when it is possible to find a value of the parameter t that gives a point of intersection.

C2 Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$. Prove the following properties of the dot product.

(a) $\vec{x} \cdot \vec{x} \geq 0$

(b) $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$

(c) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

(d) $\vec{x} \cdot (s\vec{y} + t\vec{z}) = s(\vec{x} \cdot \vec{y}) + t(\vec{x} \cdot \vec{z})$

C3 Prove that the zero vector $\vec{0}$ in $\mathbb{R}^2$ is orthogonal to every vector $\vec{x} \in \mathbb{R}^2$ in two ways.

(a) By directly calculating $\vec{x} \cdot \vec{0}$.

(b) By using property (4) in Theorem 1.3.2.

C4 Determine an equation of the set of points in $\mathbb{R}^3$ that are equidistant from points P and Q. Explain why the set is a plane, and determine its normal vector.

C5 Find a scalar equation of the plane such that each point of the plane is equidistant from the points $P(2, 2, 5)$ and $Q(-3, 4, 1)$ in two ways.

(a) Write and simplify the equation $\|\vec{PX}\| = \|\vec{QX}\|$.

(b) Determine a point on the plane and the normal vector by geometrical arguments.

C6 Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$. Consider the following statement: "If $\vec{x} \cdot \vec{y} = \vec{x} \cdot \vec{z}$, then $\vec{y} = \vec{z}$."

(a) If the statement is true, prove it. If the statement is false, provide a counterexample.

(b) If we specify $\vec{x} \neq \vec{0}$, does that change the result?

C7 Show that if X is a point in $\mathbb{R}^3$ on the line through P and Q, then $\vec{x} \times (\vec{q} - \vec{p}) = \vec{p} \times \vec{q}$, where $\vec{x} = \overrightarrow{OX}$, $\vec{p} = \overrightarrow{OP}$, and $\vec{q} = \overrightarrow{OQ}$.

C8 (a) Let $\vec{n}$ be a vector in $\mathbb{R}^3$ such that $\|\vec{n}\| = 1$ (called a **unit vector**). Let α be the angle between $\vec{n}$ and the x_1-axis; let β be the angle between $\vec{n}$ and the x_2-axis; and let γ be the angle between $\vec{n}$ and the x_3-axis. Explain why

$$\vec{n} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$$

(Hint: Take the dot product of $\vec{n}$ with the standard basis vectors.)

Because of this equation, the components n_1, n_2, n_3 are sometimes called the **direction cosines**.

(b) Explain why $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

(c) Give a two-dimensional version of the direction cosines, and explain the connection to the identity $\cos^2\theta + \sin^2\theta = 1$.

C9 Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Consider the following statement: "If $\vec{u} \neq \vec{0}$, and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$."
If the statement is true, prove it. If it is false, give a counterexample.

C10 Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Explain why $\vec{u} \times (\vec{v} \times \vec{w})$ must be a vector that satisfies the vector equation $\vec{x} = s\vec{v} + t\vec{w}$.

C11 Give an example of distinct vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ such that

(a) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$

(b) $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$

C12 Prove that if $\vec{x}$ and $\vec{y}$ are non-zero orthogonal vectors in $\mathbb{R}^2$, then $\{\vec{x}, \vec{y}\}$ is linearly independent. (Hint: Use the definition of linear independence, and take the dot product of both sides with respect to $\vec{x}$.)

C13 A set $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is said to be an **orthogonal basis** for $\mathbb{R}^2$ if $\mathcal{B}$ is a basis for $\mathbb{R}^2$, and $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. Show that the coordinates of any vector $\vec{x} \in \mathbb{R}^2$ with respect to $\mathcal{B}$ are $c_1 = \dfrac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}$ and $c_2 = \dfrac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}$.

1.4 Vectors in $\mathbb{R}^n$

We now extend the ideas from Sections 1.1 and 1.2 to n-dimensional Euclidean space.

Applications of Vectors in $\mathbb{R}^n$

Students sometimes do not see the point in discussing n-dimensional space because it does not seem to correspond to any physical realistic geometry. It is important to realize that a vector in $\mathbb{R}^n$ does not have to refer to an object in n-dimensional space. Here are just a few uses of vectors in $\mathbb{R}^n$.

String Theory Some scientists working in string theory work with vectors in six-, ten-, or eleven-dimensional space.

Position of a Rigid Object To discuss the position of a particle, an engineer needs to specify its position (3 variables) and the direction it is pointing (3 more variables); the engineer therefore uses a vector with 6 components.

Economic Models An economist seeking to model the Canadian economy uses many variables. One standard model has more than 1500 variables.

Population Age Distribution A biologist may use a vector with n components where the i-th component of the vector is the number of individuals in a given age class.

Analog Signal Sampling To convert an analog signal into digital form, the average intensity level of the signal over n equally spaced time intervals is recorded into a vector with n components.

Addition and Scalar Multiplication of Vectors in $\mathbb{R}^n$

Definition
$\mathbb{R}^n$

$\mathbb{R}^n$ is the set of all vectors of the form $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, where $x_i \in \mathbb{R}$. Mathematically,

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ are two vectors in $\mathbb{R}^n$ such that $x_i = y_i$ for $1 \leq i \leq n$, then we say that $\vec{x}$ and $\vec{y}$ are **equal** and write $\vec{x} = \vec{y}$.

As we have seen already in $\mathbb{R}^2$ and $\mathbb{R}^3$, we write vectors in $\mathbb{R}^n$ as columns. These are sometimes called **column vectors.**

Definition
Addition and Scalar Multiplication in $\mathbb{R}^n$

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$, then we define **addition** of vectors by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

We define **scalar multiplication** of a vector $\vec{x}$ by a scalar $t \in \mathbb{R}$ by

$$t\vec{x} = t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix}$$

As in $\mathbb{R}^2$ and $\mathbb{R}^3$, we call a sum of scalar multiples of vectors in $\mathbb{R}^n$ a **linear combination**, and by $\vec{x} - \vec{y}$ we mean $\vec{x} + (-1)\vec{y}$.

EXAMPLE 1.4.1

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 4 \end{bmatrix} \in \mathbb{R}^4$. Calculate the linear combination $2\vec{u} - 3\vec{v}$.

Solution: We get

$$2\vec{u} - 3\vec{v} = 2\begin{bmatrix} 1 \\ 2 \\ -5 \\ 1 \end{bmatrix} + (-3)\begin{bmatrix} 2 \\ 1 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -10 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \\ 15 \\ -12 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 5 \\ -10 \end{bmatrix}$$

Of course, we get exactly the same properties that we saw in Theorem 1.1.1.

Theorem 1.4.1

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

- V1 $\vec{x} + \vec{y} \in \mathbb{R}^n$ (closed under addition)
- V2 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (addition is commutative)
- V3 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ (addition is associative)
- V4 There exists a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{R}^n$ (zero vector)
- V5 For each $\vec{x} \in \mathbb{R}^n$ there exists a vector $-\vec{x} \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverses)
- V6 $t\vec{x} \in \mathbb{R}^n$ (closed under scalar multiplication)
- V7 $s(t\vec{x}) = (st)\vec{x}$ (scalar multiplication is associative)
- V8 $(s + t)\vec{x} = s\vec{x} + t\vec{x}$ (a distributive law)
- V9 $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$ (another distributive law)
- V10 $1\vec{x} = \vec{x}$ (scalar multiplicative identity)

Proof: We will prove property V2 and leave the other proofs to the reader.

For V2,

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ \vdots \\ y_n + x_n \end{bmatrix} = \vec{y} + \vec{x}$$

■

EXERCISE 1.4.1 Prove properties V5 and V7 from Theorem 1.4.1.

Observe that properties V2, V3, V7, V8, V9, and V10 from Theorem 1.4.1 refer only to the operations of addition and scalar multiplication, while the other properties, V1, V4, V5, and V6, are about the relationship between the operations and the set $\mathbb{R}^n$. These facts should be clear in the proof of the theorem. Moreover, we see that the zero vector of $\mathbb{R}^n$ is the vector $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, and the additive inverse of $\vec{x}$ is $-\vec{x} = (-1)\vec{x}$.

Subspaces

Properties V1 and V6 from Theorem 1.4.1 show that $\mathbb{R}^n$ is **closed under linear combinations**. That is, if $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ is also a vector in $\mathbb{R}^n$ for any $c_1, \dots, c_k \in \mathbb{R}$. As previously mentioned, not all subsets of $\mathbb{R}^n$ are closed under linear combinations.

EXAMPLE 1.4.2 Consider the line L in $\mathbb{R}^2$ defined by

$$L = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

Show that L is not closed under linear combinations.

Solution: If we take two vectors in the line, say

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

We get $\vec{x} + \vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. However, we see that there is no value of $t \in \mathbb{R}$ such that

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Consequently, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is not a vector on the line. So, this set is not closed under linear combinations.

We are most interested in non-empty subsets of $\mathbb{R}^n$ that are closed under linear combinations.

Definition
Subspace

A non-empty subset S of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if for all vectors $\vec{x}, \vec{y} \in S$ and $s, t \in \mathbb{R}$ we have

$$s\vec{x} + t\vec{y} \in S$$

The definition requires that a subspace S be non-empty. In particular, it follows from the definition that if we pick any two vectors $\vec{x}, \vec{y} \in S$ and take $s = t = 0$, then $\vec{0} = 0\vec{x} + 0\vec{y} \in S$. Hence, every subspace S of $\mathbb{R}^n$ contains the zero vector. This fact provides an easy method for disqualifying any subsets that do not contain the zero vector as subspaces. For instance, as we saw in Example 1.4.2, a line in $\mathbb{R}^2$ cannot be a subspace if it does not pass through the origin. Thus, when checking to determine if a set S is non-empty, it makes sense to first check if $\vec{0} \in S$.

It is easy to see that the set $\{\vec{0}\}$ consisting of only the zero vector in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$; this is called the **trivial subspace**. Additionally, $\mathbb{R}^n$ is a subspace of itself. We will see throughout the text that other subspaces arise naturally in linear algebra.

EXAMPLE 1.4.3

Show that $T = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 - x_2 + x_3 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

Solution: By definition, T is a subset of $\mathbb{R}^3$ and we have that $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in T$ since taking $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ satisfies $x_1 - x_2 + x_3 = 0$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in T$. Then they must satisfy the condition of the set, so $x_1 - x_2 + x_3 = 0$ and $y_1 - y_2 + y_3 = 0$.

We must show that $s\vec{x} + t\vec{y}$ satisfies the condition on T. We have

$$s\vec{x} + t\vec{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ sx_3 + ty_3 \end{bmatrix}$$

and

$$(sx_1 + ty_1) - (sx_2 + ty_2) + (sx_3 + ty_3) = s(x_1 - x_2 + x_3) + t(y_1 - y_2 + y_3) = s(0) + t(0) = 0$$

Hence, $s\vec{x} + t\vec{y} \in T$.

Therefore, by definition, T is a subspace of $\mathbb{R}^3$.

EXAMPLE 1.4.4

Show that $U = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 x_2 = 0 \right\}$ is not a subspace of $\mathbb{R}^2$.

Solution: To show that U is not a subspace, we just need to give one example showing that U does not satisfy the definition of a subspace.

Observe that $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both in U, but $\vec{x} + \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U$, since $1(1) \neq 0$. Thus, U is not a subspace of $\mathbb{R}^2$.

EXERCISE 1.4.2

Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid 2x_1 = x_2 \right\}$ is a subspace of $\mathbb{R}^2$ and $T = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 + x_2 = 2 \right\}$ is not a subspace of $\mathbb{R}^2$.

EXERCISE 1.4.3

Prove that if P is a plane in $\mathbb{R}^3$ with vector equation

$$\vec{x} = a\vec{v}_1 + b\vec{v}_2, \quad a, b \in \mathbb{R}$$

then P is a subspace of $\mathbb{R}^3$.

Spanning Sets and Linear Independence

It can be shown that the only subspaces of $\mathbb{R}^2$ are $\{\vec{0}\}$, lines through the origin, and $\mathbb{R}^2$ itself. Similarly, the only subspaces of $\mathbb{R}^3$ are $\{\vec{0}\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$ itself. We see that all of these sets can be described as the span of a set of vectors. Thus, we also extend our definition of spanning to $\mathbb{R}^n$.

Definition
Span in $\mathbb{R}^n$

Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. We define the **span** of B, denoted Span B, to be the set of all possible linear combinations of the vectors in B. Mathematically,

$$\operatorname{Span} B = \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

A **vector equation** for Span B is

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k, \quad c_1, \dots, c_k \in \mathbb{R}$$

If $S = \operatorname{Span} B$, then we say that B **spans** S, that B is a **spanning set** for S, and that S is **spanned** by B.

The following theorem says that, like in $\mathbb{R}^2$ and $\mathbb{R}^3$, all spanned sets are subspaces.

Theorem 1.4.2

If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$ and $S = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then S is a subspace of $\mathbb{R}^n$.

Proof: By properties V1 and V6 of Theorem 1.4.1, $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k \in \mathbb{R}^n$, so S is a subset of $\mathbb{R}^n$.

Taking $t_i = 0$ for $1 \leq i \leq k$, we get $\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_k \in S$, so S is non-empty.

Let $\vec{x}, \vec{y} \in S$. Then, for some real numbers c_i and d_i, $1 \leq i \leq k$, $\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ and $\vec{y} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$. It follows that

$$\begin{aligned} s\vec{x} + t\vec{y} &= s(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + t(d_1\vec{v}_1 + \cdots + d_k\vec{v}_k) \\ &= (sc_1 + td_1)\vec{v}_1 + \cdots + (sc_k + td_k)\vec{v}_k \end{aligned}$$

so, $s\vec{x} + t\vec{y} \in S$ since $(sc_i + td_i) \in \mathbb{R}$.

Therefore, S is a subspace of $\mathbb{R}^n$. ■

To simplify spanning sets in $\mathbb{R}^n$, we use the following theorem, which corresponds to what we saw in Section 1.2.

Theorem 1.4.3

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. Some vector $\vec{v}_i$, $1 \leq i \leq k$, can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k$ if and only if

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$$

As before, we define linear independence so that a spanning set is as simple as possible if and only if it is linearly independent.

Definition

Linearly Dependent

Linearly Independent

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. The set $\mathcal{B}$ is said to be **linearly dependent** if there exist real coefficients $t_1, \dots, t_k$ not all zero such that

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$$

The set $\mathcal{B}$ is said to be **linearly independent** if the only solution to

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$$

is $t_1 = t_2 = \cdots = t_k = 0$.

EXAMPLE 1.4.5

Prove that the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is linearly independent.

Solution: Consider the vector equation

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 + t_3 \\ t_3 \\ t_1 - 2t_2 - t_3 \end{bmatrix}$$

Comparing entries gives the system of equations

$$t_1 = 0, \quad t_2 + t_3 = 0, \quad t_3 = 0, \quad t_1 - 2t_2 - t_3 = 0$$

Solving the system gives $t_1 = t_2 = t_3 = 0$. Hence, $\mathcal{B}$ is linearly independent.

CONNECTION

Observe that determining whether a set $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly dependent or linearly independent requires determining solutions of the vector equation

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$$

Similarly, determining whether a vector $\vec{b} \in \mathbb{R}^n$ is in a spanned set $\operatorname{Span}\{\vec{w}_1, \dots, \vec{w}_\ell\}$ in $\mathbb{R}^n$ requires determining if the vector equation

$$s_1\vec{w}_1 + \cdots + s_k\vec{w}_k = \vec{b}$$

has a solution. Both of these vector equations actually represents n equations (one for each entry of the vectors) in the k unknown scalars. In the next chapter, we will look at how to efficiently solve such systems of equations.

We can now extend our geometrical concepts of lines and planes to $\mathbb{R}^n$ for $n > 3$. To match what we did in $\mathbb{R}^2$ and $\mathbb{R}^3$, we make the following definitions.

Definition
Line in $\mathbb{R}^n$

Let $\vec{p}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. We call the set with vector equation

$$\vec{x} = \vec{p} + t_1\vec{v}, \quad t_1 \in \mathbb{R}$$

a **line** in $\mathbb{R}^n$ that passes through $\vec{p}$.

Definition
Plane in $\mathbb{R}^n$

Let $\vec{v}_1, \vec{v}_2, \vec{p} \in \mathbb{R}^n$, with $\{\vec{v}_1, \vec{v}_2\}$ being a linearly independent set. The set with vector equation

$$\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2, \quad t_1, t_2 \in \mathbb{R}$$

is called a **plane** in $\mathbb{R}^n$ that passes through $\vec{p}$.

Definition
Hyperplane in $\mathbb{R}^n$

Let $\vec{v}_1, \dots, \vec{v}_{n-1}, \vec{p} \in \mathbb{R}^n$, with $\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$ being linearly independent. The set with vector equation

$$\vec{x} = \vec{p} + t_1\vec{v}_1 + \cdots + t_{n-1}\vec{v}_{n-1}, \quad t_1, \dots, t_{n-1} \in \mathbb{R}$$

is called a **hyperplane** in $\mathbb{R}^n$ that passes through $\vec{p}$.

EXAMPLE 1.4.6

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$. Since $\mathcal{B}$ is linearly independent, the set $\operatorname{Span}\mathcal{B}$ is a hyperplane in $\mathbb{R}^4$.

Bases of Subspaces of $\mathbb{R}^n$

In Section 1.2, we defined a basis for $\mathbb{R}^2$ to be a set $\mathcal{B}$ such that $\operatorname{Span}\mathcal{B} = \mathbb{R}^2$ and $\mathcal{B}$ is linearly independent. We now generalize the concept of a basis to any subspace of $\mathbb{R}^n$.

Definition
Basis of a Subspace

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set in $\mathbb{R}^n$, and let S be a non-trivial subspace of $\mathbb{R}^n$. If $\mathcal{B}$ is linearly independent and $\operatorname{Span}\mathcal{B} = S$, then the set $\mathcal{B}$ is called a **basis** for S.
A basis for the trivial subspace $\{\vec{0}\}$ is defined to be the empty set.

EXAMPLE 1.4.7

Prove that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the plane P in $\mathbb{R}^3$ with scalar equation $2x_1 + 3x_2 - x_3 = 0$.

Solution: By definition of P, every $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in P$ satisfies $2x_1 + 3x_2 - x_3 = 0$. Solving this for x_3 gives $x_3 = 2x_1 + 3x_2$. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 2c_1 + 3c_2 \end{bmatrix}$$

Solving gives $c_1 = x_1$, $c_2 = x_2$. Thus, $\mathcal{B}$ spans P. Now consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 2c_1 + 3c_2 \end{bmatrix}$$

Comparing entries, we get that $c_1 = c_2 = 0$. Hence, $\mathcal{B}$ is also linearly independent.

Since $\mathcal{B}$ is linearly independent and spans P, it is a basis for P.

We can think of the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ for P in Example 1.4.7 in exactly the same way as for any basis of $\mathbb{R}^2$. That is, the lines Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ and Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ form coordinate axes for the plane as in Figure 1.4.1.

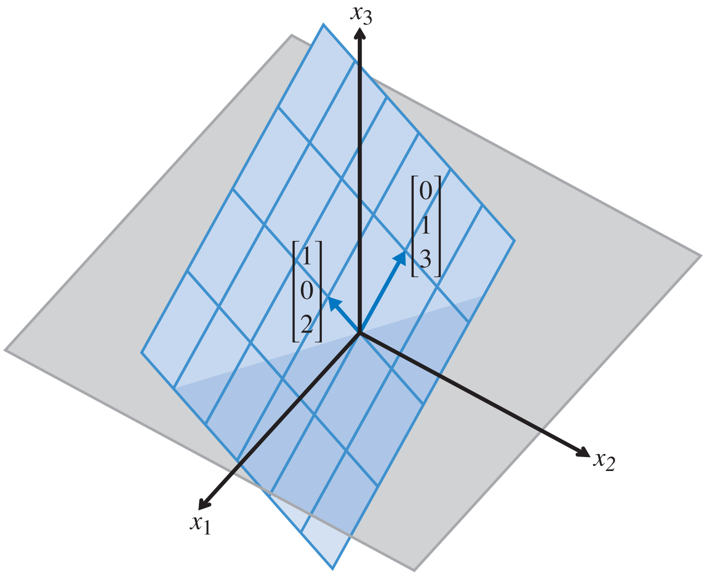


Figure 1.4.1 A basis for P.

The standard basis for $\mathbb{R}^n$ matches what we saw for $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition
Standard Basis for $\mathbb{R}^n$

In $\mathbb{R}^n$, let $\vec{e}_i$ represent the vector whose i-th component is 1 and all other components are 0. The set $\{\vec{e}_1, \dots, \vec{e}_n\}$ is called the **standard basis for** $\mathbb{R}^n$.

EXERCISE 1.4.4

State the standard basis for $\mathbb{R}^4$. Prove that it is linearly independent, and show that it is a spanning set for $\mathbb{R}^4$.

As we saw in Section 1.2, a basis $\mathcal{B}$ defines a coordinate system for the subspace S. In particular, because $\mathcal{B}$ spans S, it means that every vector $\vec{x} \in S$ can be written as a linear combination of the vectors in $\mathcal{B}$. Moreover, since $\mathcal{B}$ is linearly independent, it means that there is only one such linear combination for each $\vec{x} \in S$. We state this as a theorem.

Theorem 1.4.4

If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subspace S of $\mathbb{R}^n$, then for each $\vec{x} \in S$ there exists unique scalars $c_1, \dots, c_k \in \mathbb{R}$ such that

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

PROBLEMS 1.4

Practice Problems

For Problems A1–A4, compute the linear combination.

A1 $\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

A2 $\begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix}$

A3 $\begin{bmatrix} 3 \\ -4 \\ -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$

A4 $2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

For Problems A5–A10, determine whether the set is a subspace of the appropriate $\mathbb{R}^n$.

A5 Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

A6 $\left\{ \vec{x} \in \mathbb{R}^3 \mid x_1^2 - x_2^2 = x_3 \right\}$

A7 $\left\{ \vec{x} \in \mathbb{R}^3 \mid x_1 = x_3 \right\}$

A8 $\left\{ \vec{x} \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\}$

A9 $\left\{ \vec{x} \in \mathbb{R}^3 \mid x_1 x_2 = x_3 \right\}$

A10 $\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

For Problems A11–A16, determine whether the set is a subspace of $\mathbb{R}^4$.

A11 $\left\{ \vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0 \right\}$

A12 $\{\vec{v}_1 \in \mathbb{R}^4\}$, where $\vec{v}_1 \neq \vec{0}$

A13 $\left\{ \vec{x} \in \mathbb{R}^4 \mid x_1 + 2x_3 = 5, x_1 - 3x_4 = 0 \right\}$

A14 $\left\{ \vec{x} \in \mathbb{R}^4 \mid x_1 = x_3 x_4, x_2 - x_4 = 0 \right\}$

A15 $\left\{ \vec{x} \in \mathbb{R}^4 \mid 2x_1 = 3x_4, x_2 - 5x_3 = 0 \right\}$

A16 $\left\{ \vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 = -x_4, x_3 = 2 \right\}$

For Problems A17–A20, show that the set is linearly dependent by writing a non-trivial linear combination of the vectors that equals the zero vector.

A17 $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ -3 \end{bmatrix} \right\}$

A18 $\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ -2 \end{bmatrix} \right\}$

A19 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

A20 $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -8 \\ 3 \end{bmatrix} \right\}$

For Problems A21–A24, determine whether the set is linearly independent. If it is linearly dependent, write one of the vectors in the set as a linear combination of the others.

A21 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 2 \\ -2 \end{bmatrix} \right\}$

A22 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

A23 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

A24 $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

For Problems A25–A27, prove that B is a basis for the plane in $\mathbb{R}^3$ with the given scalar equation.

A25 $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$, $2x_1 + x_2 + x_3 = 0$.

A26 $B = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, $3x_1 + x_2 - 2x_3 = 0$.

A27 $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$, $3x_1 + x_2 - 2x_3 = 0$.

A28 Prove that $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the hyperplane P in $\mathbb{R}^4$ with scalar equation $x_1 + x_2 + x_3 - x_4 = 0$.

For Problems A29–A32, determine whether the set represents a line, a plane, or a hyperplane in $\mathbb{R}^4$. Give a basis for the subspace.

A29 Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

A30 Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

A31 Span $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -2 \\ 0 \end{bmatrix} \right\}$

A32 Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

A33 Let $\vec{p}, \vec{d} \in \mathbb{R}^n$. Prove that $\vec{x} = \vec{p} + t\vec{d}, t \in \mathbb{R}$ is a subspace of $\mathbb{R}^n$ if and only if $\vec{p}$ is a scalar multiple of $\vec{d}$.

A34 Suppose that $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set in $\mathbb{R}^n$. Prove that any non-empty subset of $\mathcal{B}$ is linearly independent.

A35 Let $\vec{v}_1, \dots, \vec{v}_k$ be vectors in $\mathbb{R}^n$.

(a) Prove if $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$, then $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$.

(b) Prove if $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

A36 A factory produces thingamajiggers and whatchamacallits. It takes 3kg of steel, 2L of whipped cream, 4 nails, and 3 calculators to create a thingamajigger, while it takes 1kg of steel, 10L of whipped cream, 2 nails, and 5 calculators to create a whatchamacallit. We can represent the amount of each building material for thingamajiggers and whatchamacallits as the vectors $\vec{t} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 10 \\ 2 \\ 5 \end{bmatrix}$. What does the linear combination $100\vec{t} + 250\vec{w}$ represent in this situation?

Homework Problems

For Problems B1–B4, compute the linear combination.

B1 $\begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \\ 6 \\ 11 \end{bmatrix}$

B2 $\begin{bmatrix} 5 \\ 1 \\ 7 \\ 3 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}$

B3 $\begin{bmatrix} 3 \\ -3 \\ 1 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 6 \\ 1 \\ -2 \end{bmatrix}$

B4 $5\begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \\ 1 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 \\ -7 \\ 6 \\ -7 \\ 0 \end{bmatrix}$

For Problems B5–B14, determine whether the set is a subspace of the appropriate $\mathbb{R}^n$.

B5 $\left\{\vec{x} \in \mathbb{R}^2 \mid x_1 + 3x_2 = 0\right\}$

B6 $\left\{\vec{x} \in \mathbb{R}^2 \mid x_1 + x_2 \le 0\right\}$

B7 $\left\{\vec{x} \in \mathbb{R}^2 \mid x_1^2 = x_2^3\right\}$

B8 $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\right\}$

B9 $\left\{\vec{x} \in \mathbb{R}^3 \mid x_2 = 1\right\}$

B10 $\left\{\vec{x} \in \mathbb{R}^3 \mid 3x_1 + 3x_2 - 2x_3 = 0\right\}$

B11 $\left\{\vec{x} \in \mathbb{R}^3 \mid x_1 - 2x_2 = 3, x_1 + x_2 = 0\right\}$

B12 $\left\{\vec{x} \in \mathbb{R}^3 \mid x_1 - 2x_2 = 0, x_1 + x_2 = 0\right\}$

B13 $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_1\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, t_1, t_2 \in \mathbb{R}$

B14 $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + t_1\begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ -2 \\ -8 \end{bmatrix}, t_1, t_2 \in \mathbb{R}$

For Problems B15–B18, determine whether the set is a subspace of $\mathbb{R}^4$.

B15 $\left\{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + 3x_4 = 0, 3x_2 = 2x_4\right\}$

B16 $\left\{\vec{x} \in \mathbb{R}^4 \mid x_1 + 2x_2 - x_3 = x_4\right\}$

B17 $\left\{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 - 3x_3 = 1, x_1 = x_4\right\}$

B18 $\left\{\vec{x} \in \mathbb{R}^4 \mid x_1 = 2x_3 - x_4, x_1 - 3x_4 = 0\right\}$

For Problems B19–B22, show that the set is linearly dependent by writing a non-trivial linear combination of the vectors that equals the zero vector.

B19 $\left\{\begin{bmatrix} 1 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

B20 $\left\{\begin{bmatrix} 4 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -9 \\ -3 \end{bmatrix}\right\}$

B21 $\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ -1 \\ -1 \end{bmatrix}\right\}$

B22 $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

For Problems B23–B26, determine whether the set is linearly independent. If it is linearly dependent, write one of the vectors in the set as a linear combination of the others.

B23 $\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 0 \end{bmatrix}\right\}$

B24 $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$

B25 $\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 7 \end{bmatrix}\right\}$

B26 $\left\{\begin{bmatrix} -2 \\ 5 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ 3 \end{bmatrix}\right\}$

B27 Prove that $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the plane P in $\mathbb{R}^3$ with scalar equation $x_1 - 3x_2 + x_3 = 0$.

B28 Prove that $B = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the plane P in $\mathbb{R}^3$ with scalar equation $-2x_1 + 3x_2 + 5x_3 = 0$.

B29 Prove that $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$ is a basis for the hyperplane P in $\mathbb{R}^4$ with scalar equation $2x_1 - 3x_3 - x_4 = 0$.

For Problems B30–B35, determine whether the set represents a line, a plane, or a hyperplane in $\mathbb{R}^4$. Give a basis for the subspace.

B30 Span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \\ -2 \end{bmatrix} \right\}$

B31 Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

B32 Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

B33 Span $\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

B34 Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

B35 Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \\ 3 \end{bmatrix} \right\}$

Conceptual Problems

C1 Prove property V8 from Theorem 1.4.1.

C2 Prove property V9 from Theorem 1.4.1.

C3 Prove if $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subspace S of $\mathbb{R}^n$, then for each $\vec{x} \in S$, there exists unique scalars $c_1, \dots, c_k \in \mathbb{R}$ such that $x = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$.

C4 Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. Prove that if $\vec{v}_i = \vec{0}$ for some i, then $\mathcal{B}$ is linearly dependent.

C5 Let U and V be subspaces of $\mathbb{R}^n$.

(a) Prove that the intersection of U and V is a subspace of $\mathbb{R}^n$.

(b) Give an example to show that the union of two subspaces of $\mathbb{R}^n$ does not have to be a subspace of $\mathbb{R}^n$.

(c) Define $U + V = \{\vec{u} + \vec{v} \mid \vec{u} \in U, \vec{v} \in V\}$. Prove that $U + V$ is a subspace of $\mathbb{R}^n$.

C6 Pick vectors $\vec{p}, \vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ in $\mathbb{R}^4$ such that the vector equation $\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3$

(a) is a hyperplane not passing through the origin.

(b) is a plane passing through the origin.

(c) is the point $(1, 3, 1, 1)$.

(d) is a line passing through the origin.

C7 Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, and let s and t be fixed real numbers with $t \neq 0$. Prove that

$$\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}$$

C8 Explain the difference between a subset of $\mathbb{R}^n$ and a subspace of $\mathbb{R}^n$.

For Problems C9–C14, given that $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$, state whether each of the following statements is true or false. If the statement is true, explain briefly. If the statement is false, give a counterexample.

C9 If $\vec{v}_2 = t\vec{v}_1$ for some real number t, then $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

C10 If $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$, then $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

C11 If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then $\vec{v}_1$ can be written as a linear combination of $\vec{v}_2$ and $\vec{v}_3$.

C12 If $\vec{v}_1$ can be written as a linear combination of $\vec{v}_2$ and $\vec{v}_3$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

C13 $\{\vec{v}_1\}$ is not a subspace of $\mathbb{R}^n$.

C14 $\operatorname{Span}\{\vec{v}_1\}$ is a subspace of $\mathbb{R}^n$.

1.5 Dot Products and Projections in $\mathbb{R}^n$

We now extend everything we did with dot products, lengths, and orthogonality in $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$. We will use these concepts to define projections in $\mathbb{R}^n$.

Definition
Dot Product

Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$. We define the **dot product** of $\vec{x}$ and $\vec{y}$ by

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

CONNECTION

The dot product is also sometimes called the **scalar product** or the **standard inner product**. We will look at other inner products in Section 7.4.

EXAMPLE 1.5.1

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -2 \\ 5 \\ 0 \\ -4 \end{bmatrix}$. Calculate $\vec{x} \cdot \vec{y}$.

Solution: We have $\vec{x} \cdot \vec{y} = 1(-2) + 2(5) + (-1)(0) + 3(-4) = -4$.

From this definition, some important properties follow.

Theorem 1.5.1

Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Then,

(1) $\vec{x} \cdot \vec{x} \geq 0$
(2) $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$
(3) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
(4) $\vec{x} \cdot (s\vec{y} + t\vec{z}) = s(\vec{x} \cdot \vec{y}) + t(\vec{x} \cdot \vec{z})$

Proof: We leave the proof of these properties to the reader.

Because of property (1), we can now define the length of a vector in $\mathbb{R}^n$. The word **norm** is often used as a synonym for **length** when we are speaking of vectors.

Definition
Norm

Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. We define the **norm**, or **length**, of $\vec{x}$ by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$$

EXAMPLE 1.5.2

Let $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ -2/3 \end{bmatrix}$. Find $\|\vec{x}\|$ and $\|\vec{y}\|$.

Solution: We have

$$\|\vec{x}\| = \sqrt{2^2 + 1^2 + 3^2 + (-1)^2} = \sqrt{15}$$
$$\|\vec{y}\| = \sqrt{(1/3)^2 + (-2/3)^2 + 0^2 + (-2/3)^2} = \sqrt{1/9 + 4/9 + 0 + 4/9} = 1$$

EXERCISE 1.5.1

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and let $\vec{y} = \frac{1}{\|\vec{x}\|}\vec{x}$. Determine $\|\vec{x}\|$ and $\|\vec{y}\|$.

We now give some important properties of the norm in $\mathbb{R}^n$.

Theorem 1.5.2

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

(1) $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$
(2) $\|t\vec{x}\| = |t|\,\|\vec{x}\|$
(3) $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$, with equality if and only if $\{\vec{x}, \vec{y}\}$ is linearly dependent
(4) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Remark

Property (3) is called the **Cauchy-Schwarz Inequality**.
Property (4) is called the **Triangle Inequality**.

EXERCISE 1.5.2

Prove that the vector $\hat{x} = \frac{1}{\|\vec{x}\|}\vec{x}$ is parallel to $\vec{x}$ and satisfies $\|\hat{x}\| = 1$.

Definition
Unit Vector

A vector $\vec{x} \in \mathbb{R}^n$ such that $\|\vec{x}\| = 1$ is called a **unit vector**.

We will see that unit vectors can be very useful. We often want to find a unit vector that has the same direction as a given vector $\vec{x}$. Using the result in Exercise 1.5.2, we see that we can use the vector

$$\hat{x} = \frac{1}{\|\vec{x}\|}\vec{x}$$

We could now define angles between vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ by matching what we did in $\mathbb{R}^2$ and $\mathbb{R}^3$. However, in linear algebra we are generally interested only in whether two vectors are orthogonal.

Definition
Orthogonal

Two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ are **orthogonal** to each other if and only if $\vec{x} \cdot \vec{y} = 0$.

Notice that this definition implies that $\vec{0}$ is orthogonal to every vector in $\mathbb{R}^n$.

EXAMPLE 1.5.3

Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$. Show that $\vec{v}$ is orthogonal to $\vec{w}$ but $\vec{v}$ is not orthogonal to $\vec{z}$.

Solution: We have $\vec{v} \cdot \vec{w} = 1(2) + 0(3) + 3(0) + (-2)(1) = 0$, so they are orthogonal. $\vec{v} \cdot \vec{z} = 1(-1) + 0(-1) + 3(1) + (-2)(2) = -2$, so they are not orthogonal.

The Scalar Equation of a Hyperplane

We can repeat the argument we used to find a scalar equation of a plane in Section 1.3 to find a scalar equation of a hyperplane in $\mathbb{R}^n$. In particular, if we have a vector $\vec{m}$ that is orthogonal to any directed line segment $\vec{PQ}$ lying in the hyperplane, then for any point $X(x_1, \dots, x_n)$ in the hyperplane, we have

$$0 = \vec{m} \cdot \vec{PX}$$

As before, we can rearrange this as

$$\begin{aligned} 0 &= \vec{m} \cdot (\vec{x} - \vec{p}) \\ 0 &= \vec{m} \cdot \vec{x} - \vec{m} \cdot \vec{p} \\ \vec{m} \cdot \vec{x} &= \vec{m} \cdot \vec{p} \\ m_1 x_1 + \cdots + m_n x_n &= \vec{m} \cdot \vec{p} \end{aligned}$$

Thus, we see that a single scalar equation in $\mathbb{R}^n$ represents a hyperplane in $\mathbb{R}^n$.

EXAMPLE 1.5.4

Find a scalar equation of the hyperplane in $\mathbb{R}^4$ that has normal vector $\vec{m} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ and passes through the point $P(1, 0, 2, -1)$.

Solution: The equation is

$$2x_1 + 3x_2 - 2x_3 + x_4 = 2(1) + 3(0) + (-2)(2) + 1(-1) = -3$$

Projections in $\mathbb{R}^n$

The idea of a projection is one of the most important applications of the dot product. Suppose that we want to know how much of a given vector $\vec{y}$ is in the direction of some other given vector $\vec{x}$ (see Figure 1.5.1). In elementary physics, this is exactly what is required when a force is "resolved" into its components along certain directions (for example, into its vertical and horizontal components). When we define projections, it is helpful to think of examples in two or three dimensions, but the ideas do not really depend on whether the vectors are in $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$.

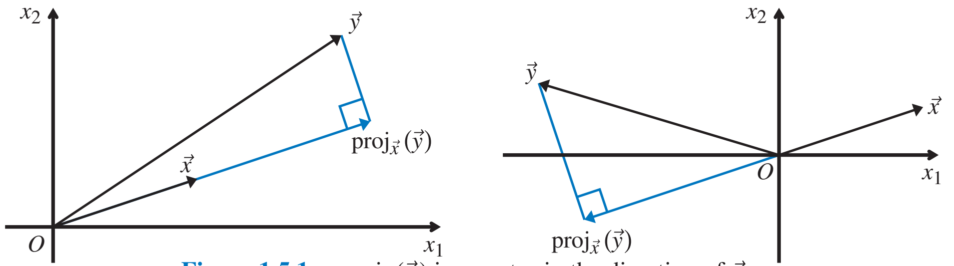


Figure 1.5.1 $\mathrm{proj}_{\vec{x}}(\vec{y})$ is a vector in the direction of $\vec{x}$.

First, let us consider the case where $\vec{x} = \vec{e}_1$ in $\mathbb{R}^2$. How much of an arbitrary vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ points along $\vec{x}$? Clearly, the part of $\vec{y}$ that is in the direction of $\vec{x}$ is $\begin{bmatrix} y_1 \\ 0 \end{bmatrix} = y_1\vec{e}_1 = (\vec{x} \cdot \vec{y})\vec{x}$. This will be called the **projection of $\vec{y}$ onto $\vec{x}$** and is denoted $\mathrm{proj}_{\vec{x}}(\vec{y})$.

Next, consider the case where $\vec{x} \in \mathbb{R}^2$ has arbitrary direction and is a unit vector. First, draw the line through the origin with direction vector $\vec{x}$. Now, draw the line perpendicular to this line that passes through the point (y_1, y_2). This forms a right triangle, as in Figure 1.5.1. The projection of $\vec{y}$ onto $\vec{x}$ is the portion of the triangle that lies on the line with direction $\vec{x}$. Thus, the resulting projection is a scalar multiple of $\vec{x}$, that is $\mathrm{proj}_{\vec{x}}(\vec{y}) = k\vec{x}$. We need to determine the value of k. To do this, let $\vec{z}$ denote the vector from $\mathrm{proj}_{\vec{x}}(\vec{y})$ to $\vec{y}$. Then, by definition, $\vec{z}$ is orthogonal to $\vec{x}$ and we can write

$$\vec{y} = \vec{z} + \mathrm{proj}_{\vec{x}}(\vec{y}) = \vec{z} + k\vec{x}$$

We now employ a very useful and common trick which is to take the dot product of $\vec{y}$ with $\vec{x}$:

$$\vec{x} \cdot \vec{y} = \vec{x} \cdot (\vec{z} + k\vec{x}) = \vec{x} \cdot \vec{z} + \vec{x} \cdot (k\vec{x}) = 0 + k(\vec{x} \cdot \vec{x}) = k\|\vec{x}\|^2 = k$$

since $\vec{x}$ is orthogonal to $\vec{z}$ and is a unit vector. Thus,

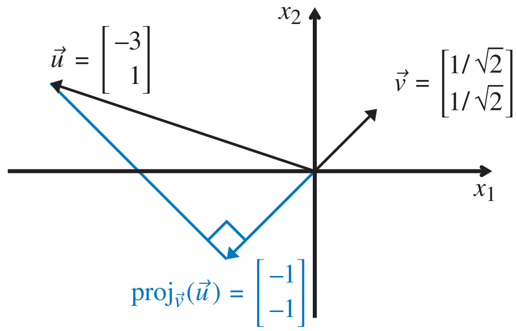
$$\mathrm{proj}_{\vec{x}}(\vec{y}) = (\vec{x} \cdot \vec{y})\vec{x}$$

EXAMPLE 1.5.5

Find the projection of $\vec{u} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ onto the unit vector $\vec{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Solution: We have

$$\begin{aligned} \operatorname{proj}_{\vec{v}}(\vec{u}) &= (\vec{v} \cdot \vec{u})\vec{v} \\ &= \left(\frac{-3}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\ &= \frac{-2}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \end{aligned}$$



If $\vec{x} \in \mathbb{R}^2$ is an arbitrary non-zero vector, then a unit vector in the direction of $\vec{x}$ is $\hat{x} = \frac{\vec{x}}{\|\vec{x}\|}$. Hence, we find that the projection of $\vec{y}$ onto $\vec{x}$ is

$$\operatorname{proj}_{\vec{x}}(\vec{y}) = \operatorname{proj}_{\hat{x}}(\vec{y}) = (\hat{x} \cdot \vec{y})\hat{x} = \left(\frac{\vec{x}}{\|\vec{x}\|} \cdot \vec{y} \right) \frac{\vec{x}}{\|\vec{x}\|} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\vec{x}$$

To match this result, we make the following definition for vectors in $\mathbb{R}^n$.

Definition
Projection

For any vectors $\vec{y}, \vec{x}$ in $\mathbb{R}^n$, with $\vec{x} \neq \vec{0}$, we define the **projection** of $\vec{y}$ onto $\vec{x}$ by

$$\operatorname{proj}_{\vec{x}}(\vec{y}) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\vec{x}$$

EXAMPLE 1.5.6

Let $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$. Determine $\operatorname{proj}_{\vec{v}}(\vec{u})$ and $\operatorname{proj}_{\vec{u}}(\vec{v})$.

Solution: We have

$$\operatorname{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{(4)(-2) + 3(5) + (-1)(3)}{4^2 + 3^2 + (-1)^2}\vec{v} = \frac{4}{26}\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 6/13 \\ -2/13 \end{bmatrix}$$

$$\operatorname{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\vec{u} = \frac{(-2)(4) + 5(3) + 3(-1)}{(-2)^2 + 5^2 + 3^2}\vec{u} = \frac{4}{38}\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -4/19 \\ 10/19 \\ 6/19 \end{bmatrix}$$

Remarks

1. This example illustrates that, in general, $\operatorname{proj}_{\vec{x}}(\vec{y}) \neq \operatorname{proj}_{\vec{y}}(\vec{x})$. Of course, we should not expect equality, because $\operatorname{proj}_{\vec{x}}(\vec{y})$ is in the direction of $\vec{x}$, whereas $\operatorname{proj}_{\vec{y}}(\vec{x})$ is in the direction of $\vec{y}$.

2. Observe that for any $\vec{x} \in \mathbb{R}^n$, we can consider $\operatorname{proj}_{\vec{x}}$ a function whose domain and codomain are $\mathbb{R}^n$. To indicate this, we can write $\operatorname{proj}_{\vec{x}} : \mathbb{R}^n \to \mathbb{R}^n$. Since the output of this function is a vector, we call it a **vector-valued function**.

EXAMPLE 1.5.7

A 30 kg rail cart with a wind sail is on a frictionless track that runs northeast and southwest. A wind applies a force vector of $\vec{F} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$ to the cart. Calculate the acceleration of the cart.

Solution: Since the rail cart is on a track, only the amount of force in the direction of the track will create acceleration in the direction of the track. The track has direction vector $\vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, the force vector in that direction is

$$\text{proj}_{\vec{d}}(\vec{F}) = \frac{\vec{d} \cdot \vec{F}}{\|\vec{d}\|^2}\vec{d} = \frac{150}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 75 \\ 75 \end{bmatrix}$$

Thus, the amount of force in the direction of the track is

$$\left\|\text{proj}_{\vec{d}}(\vec{F})\right\| = \sqrt{(75)^2 + (75)^2} \approx 106N$$

Consequently, the acceleration of the cart along the track will be $a = \dfrac{F}{m} \approx 3.53m/s^2$.

The Perpendicular Part

When you resolve a force in physics, you often not only want the component of the force in the direction of a given vector $\vec{x}$, but also the component of the force perpendicular to $\vec{x}$.

We begin by restating the problem. In $\mathbb{R}^n$, given a non-zero vector $\vec{x}$, express any $\vec{y} \in \mathbb{R}^n$ as the sum of a vector parallel to $\vec{x}$ and a vector orthogonal to $\vec{x}$. That is, write $\vec{y} = \vec{w} + \vec{z}$, where $\vec{w} = c\vec{x}$ for some $c \in \mathbb{R}$ and $\vec{z} \cdot \vec{x} = 0$.

We use the same trick we did $\mathbb{R}^2$. Taking the dot product of $\vec{x}$ and $\vec{y}$ gives

$$\vec{x} \cdot \vec{y} = \vec{x} \cdot (\vec{z} + \vec{w}) = \vec{x} \cdot \vec{z} + \vec{x} \cdot (c\vec{x}) = 0 + c(\vec{x} \cdot \vec{x}) = c\|\vec{x}\|^2$$

Therefore, $c = \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}$, so in fact, $\vec{w} = c\vec{x} = \text{proj}_{\vec{x}}(\vec{y})$, as we might have expected. One bonus of approaching the problem this way is that it is now clear that this is the only way to choose $\vec{w}$ to satisfy the problem.

Next, since $\vec{y} = \text{proj}_{\vec{x}}(\vec{y}) + \vec{z}$, it follows that $\vec{z} = \vec{y} - \text{proj}_{\vec{x}}(\vec{y})$. Is this $\vec{z}$ really orthogonal to $\vec{x}$? To check, calculate

$$\begin{aligned}
\vec{x} \cdot \vec{z} &= \vec{x} \cdot (\vec{y} - \text{proj}_{\vec{x}}(\vec{y})) \\
&= \vec{x} \cdot \vec{y} - \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\vec{x}\right) \cdot \vec{x} \\
&= \vec{x} \cdot \vec{y} - \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\right)(\vec{x} \cdot \vec{x}) \\
&= \vec{x} \cdot \vec{y} - \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\right)\|\vec{x}\|^2 \\
&= \vec{x} \cdot \vec{y} - \vec{x} \cdot \vec{y} = 0
\end{aligned}$$

So, $\vec{z}$ is orthogonal to $\vec{x}$, as required. Since it is often useful to construct a vector $\vec{z}$ in this way, we introduce a name for it.

Definition
Perpendicular of a Projection

For any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, with $\vec{x} \neq \vec{0}$, define the **projection of $\vec{y}$ perpendicular to $\vec{x}$** to be

$$\text{perp}_{\vec{x}}(\vec{y}) = \vec{y} - \text{proj}_{\vec{x}}(\vec{y})$$

Notice that $\text{perp}_{\vec{x}}(\vec{y})$ is again a vector-valued function on $\mathbb{R}^n$. Also observe that $\vec{y} = \text{proj}_{\vec{x}}(\vec{y}) + \text{perp}_{\vec{x}}(\vec{y})$. See Figure 1.5.2.

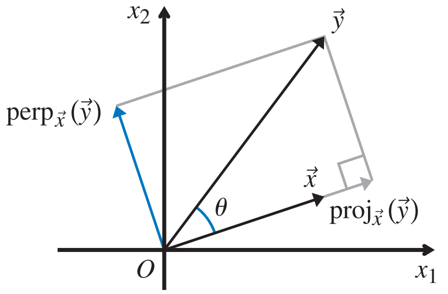


Figure 1.5.2 $\text{perp}_{\vec{x}}(\vec{y})$ is perpendicular to $\vec{x}$, and $\text{proj}_{\vec{x}}(\vec{y}) + \text{perp}_{\vec{x}}(\vec{y}) = \vec{y}$.

EXAMPLE 1.5.8

Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$. Determine $\text{proj}_{\vec{v}}(\vec{u})$ and $\text{perp}_{\vec{v}}(\vec{u})$.

Solution:

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{8}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \\ 4/3 \end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/3 \\ 4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 11/3 \\ -1/3 \end{bmatrix}$$

EXERCISE 1.5.3

Let $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Determine $\text{proj}_{\vec{v}}(\vec{u})$ and $\text{perp}_{\vec{v}}(\vec{u})$.

Two Properties of Projections

Projections will appear several times in this book, and some of their special properties are important. The first is called the **linearity property**, and the second is called the **projection property**. Let $\vec{x} \in \mathbb{R}^n$ with $\vec{x} \neq \vec{0}$, then

(L1) $\text{proj}_{\vec{x}}(s\vec{y} + t\vec{z}) = s\,\text{proj}_{\vec{x}}(\vec{y}) + t\,\text{proj}_{\vec{x}}(\vec{z})$ for all $\vec{y}, \vec{z} \in \mathbb{R}^n$, $s, t \in \mathbb{R}$

(L2) $\text{proj}_{\vec{x}}(\text{proj}_{\vec{x}}(\vec{y})) = \text{proj}_{\vec{x}}(\vec{y})$, for all $\vec{y}$ in $\mathbb{R}^n$

EXERCISE 1.5.4 Verify that properties (L1) and (L2) are true.

It follows that $\operatorname{perp}_{\vec{x}}$ also satisfies the corresponding equations. We shall see that $\operatorname{proj}_{\vec{x}}$ and $\operatorname{perp}_{\vec{x}}$ are just two cases amongst the many functions that satisfy the linearity property.

Applications of Projections

Minimum Distance What is the distance from a point $Q(q_1, q_2)$ to the line with vector equation $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$? In this and similar problems, *distance* always means the minimum distance. Geometrically, we see that the minimum distance is found along a line segment perpendicular to the given line through a point P on the line. A formal proof that minimum distance requires perpendicularity can be given by using the Pythagorean Theorem. (See Problem C9.)

To answer the question, take *any* point on the line $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$. The obvious choice is $P(p_1, p_2)$ corresponding to $\vec{p}$. From Figure 1.5.3, we see that the required distance is the length $\operatorname{perp}_{\vec{d}}\left(\vec{PQ}\right)$.

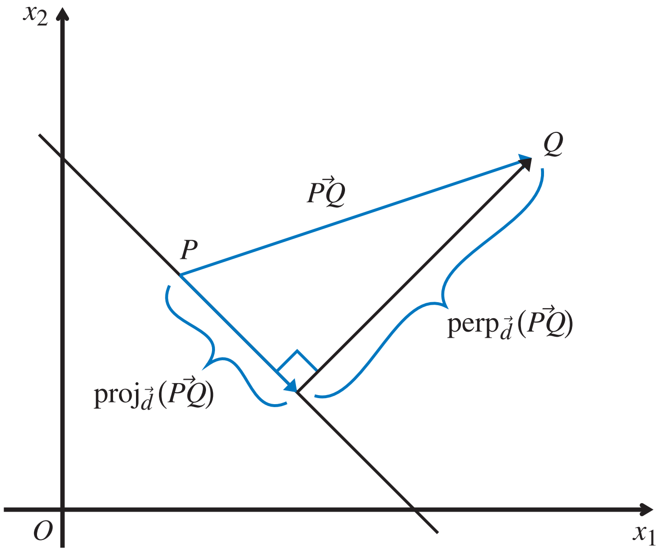


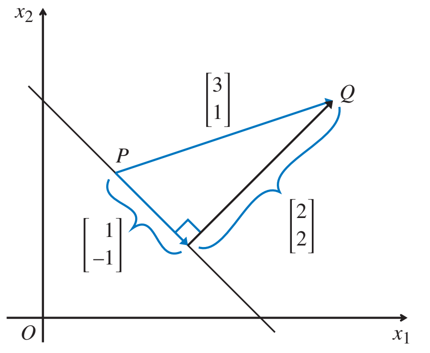
Figure 1.5.3 The distance from Q to the line $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$ is $\left\|\operatorname{perp}_{\vec{d}}\left(\vec{PQ}\right)\right\|$.

EXAMPLE 1.5.9

Find the distance from $Q(4,3)$ to the line $\vec{x} = \begin{bmatrix}1\\2\end{bmatrix} + t\begin{bmatrix}-1\\1\end{bmatrix}$, $t \in \mathbb{R}$.

Solution: We pick the point $P(1,2)$ on the line. Then, $\vec{PQ} = \begin{bmatrix}4-1\\3-2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$. So, the distance is

$$\begin{aligned}
\|\operatorname{perp}_{\vec{d}}(\vec{PQ})\| &= \|\vec{PQ} - \operatorname{proj}_{\vec{d}}(\vec{PQ})\| \\
&= \left\| \begin{bmatrix}3\\1\end{bmatrix} - \left(\frac{-3+1}{1+1}\right)\begin{bmatrix}-1\\1\end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix} \right\| = \left\| \begin{bmatrix}2\\2\end{bmatrix} \right\| = 2\sqrt{2}
\end{aligned}$$



Notice that in this problem and similar problems, we take advantage of the fact that the direction vector $\vec{d}$ can be thought of as "starting" at any point. When $\operatorname{perp}_{\vec{d}}(\vec{AB})$ is calculated, both vectors are "located" at point P. When projections were originally defined, it was implicitly assumed that all vectors were located at the origin. Now, it is apparent that the definitions make sense as long as all vectors in the calculation are located at the same point.

We now look at the similar problem of finding the distance from a point $Q(q_1, q_2, q_3)$ to a plane in $\mathbb{R}^3$ with normal vector $\vec{n}$. If P is any point in the plane, then $\operatorname{proj}_{\vec{n}}(\vec{PQ})$ is the directed line segment from the plane to the point Q that is perpendicular to the plane. Hence, $\|\operatorname{proj}_{\vec{n}}(\vec{PQ})\|$ is the distance from Q to the plane. Moreover, $\operatorname{perp}_{\vec{n}}(\vec{PQ})$ is a directed line segment lying in the plane. In particular, it is the projection of $\vec{PQ}$ onto the plane. See Figure 1.5.4.

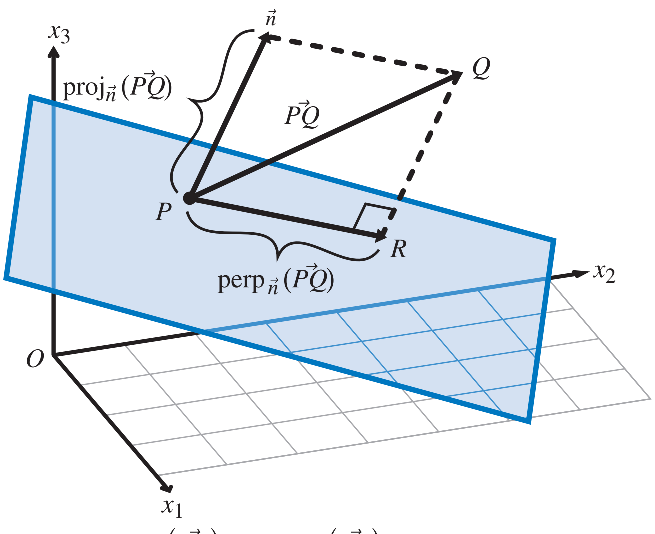


Figure 1.5.4 $\operatorname{proj}_{\vec{n}}(\vec{PQ})$ and $\operatorname{perp}_{\vec{n}}(\vec{PQ})$, where $\vec{n}$ is normal to the plane.

EXAMPLE 1.5.10 What is the distance from $Q(q_1, q_2, q_3)$ to a plane in $\mathbb{R}^3$ with equation $n_1x_1 + n_2x_2 + n_3x_3 = d$?

Solution: Assuming that $n_1 \neq 0$, we pick $P(d/n_1, 0, 0)$ to be our point in the plane. Thus, the distance is

$$\begin{aligned}
\|\operatorname{proj}_{\vec{n}}\left(\vec{PQ}\right)\| &= \left\| \frac{(\vec{q} - \vec{p}) \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} \right\| \\
&= \left| \frac{(\vec{q} - \vec{p}) \cdot \vec{n}}{\|\vec{n}\|^2} \right| \|\vec{n}\| \\
&= \left| \frac{(\vec{q} - \vec{p}) \cdot \vec{n}}{\|\vec{n}\|} \right| \\
&= \left| \frac{(q_1 - d/n_1)n_1 + q_2n_2 + q_3n_3}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \right| \\
&= \left| \frac{q_1n_1 + q_2n_2 + q_3n_3 - d}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \right|
\end{aligned}$$

This is a standard formula for this distance problem. However, the lengths of projections along or perpendicular to a suitable vector can be used for all of these problems. It is better to learn to use this powerful and versatile idea, as illustrated in the problems above, than to memorize complicated formulas.

Finding the Nearest Point In some applications, we need to determine the point in the plane that is nearest to the point Q. Let us call this point R, as in Figure 1.5.4. Then we can determine R by observing that

$$\vec{OR} = \vec{OP} + \vec{PR} = \vec{OP} + \operatorname{perp}_{\vec{n}}\left(\vec{PQ}\right)$$

However, we get an easier calculation if we observe from the figure that

$$\vec{OR} = \vec{OQ} + \vec{QR} = \vec{OQ} + \operatorname{proj}_{\vec{n}}\left(\vec{QP}\right)$$

Notice that we need $\vec{QP}$ here instead of $\vec{PQ}$. Problem C10 asks you to check that these two calculations of $\vec{OR}$ are consistent.

If the plane in this problem passes through the origin, then we may take $P = O$, and the point in the plane that is closest to Q is given by $\operatorname{perp}_{\vec{n}}(\vec{q})$.

EXAMPLE 1.5.11 Find the point on the plane $x_1 - 2x_2 + 2x_3 = 5$ that is closest to the point $Q(2, 1, 1)$.

Solution: We pick $P(1, -1, 1)$ to be the point on the plane. Then $\vec{QP} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$, and we find that the point on the plane closest to Q is

$$\vec{OR} = \vec{q} + \operatorname{proj}_{\vec{n}}\left(\vec{QP}\right) = \vec{q} + \frac{\vec{n} \cdot \vec{QP}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{9}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 1/3 \\ 5/3 \end{bmatrix}$$

The Scalar Triple Product and Volumes in $\mathbb{R}^3$ The three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ may be taken to be the three adjacent edges of a parallelepiped (see Figure 1.5.5). Is there an expression for the volume of the parallelepiped in terms of the three vectors? To obtain such a formula, observe that the parallelogram determined by $\vec{u}$ and $\vec{v}$ can be regarded as the base of the solid of the parallelepiped. This base has area $\|\vec{u} \times \vec{v}\|$. With respect to this base, the altitude of the solid is the length of the amount of $\vec{w}$ in the direction of the normal vector $\vec{n} = \vec{u} \times \vec{v}$ to the base.

$$\text{altitude} = \|\operatorname{proj}_{\vec{n}}(\vec{w})\| = \frac{|\vec{n} \cdot \vec{w}|}{\|\vec{n}\|} = \frac{|(\vec{u} \times \vec{v}) \cdot \vec{w}|}{\|\vec{u} \times \vec{v}\|}$$

To get the volume, multiply this altitude by the area of the base to get

$$\text{volume of the parallelepiped} = \frac{|(\vec{u} \times \vec{v}) \cdot \vec{w}|}{\|\vec{u} \times \vec{v}\|} \times \|\vec{u} \times \vec{v}\| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

The product $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is called the **scalar triple product** of $\vec{w}$, $\vec{u}$, and $\vec{v}$. Notice that the result is a real number (a scalar).

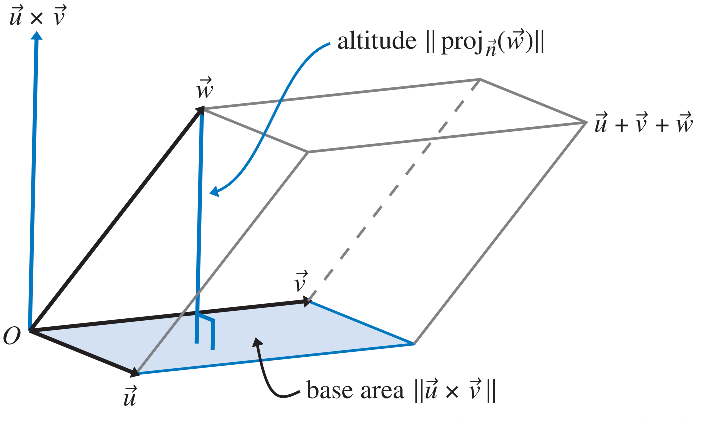


Figure 1.5.5 The parallelepiped with edges $\vec{u}$, $\vec{v}$, $\vec{w}$ has volume $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$.

The sign of the scalar triple product also has an interpretation. Recall that the ordered triple of vectors $\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ is right-handed; we can think of $\vec{u} \times \vec{v}$ as the "upwards" normal vector to the plane with vector equation $\vec{x} = s\vec{u} + t\vec{v}$, $s, t \in \mathbb{R}$. Some other vector $\vec{w}$ is then "upwards," and $\{\vec{u}, \vec{v}, \vec{w}\}$ (in that order) is right-handed, if and only if the scalar triple product is positive. If the triple scalar product is negative, then $\{\vec{u}, \vec{v}, \vec{w}\}$ is a left-handed system.

It is often useful to note that $(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v}$. This is straightforward but tedious to verify.

EXAMPLE 1.5.12

Find the volume of the parallelepiped determined by the vectors $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Solution: The volume V is

$$V = \left| \left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right| = \left| \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right| = 2$$

Best Approximation Finding the point Q on a line L that is closest to a given point P does not have to be interpreted geometrically. For example, it can be used to find the best approximation of a given set of data. For now we look at just one example.

EXAMPLE 1.5.13

Hooke's Law states that the force F applied to a spring is proportional to the distance x that the spring is stretched. That is, $F = kx$ where k is a constant called the *spring constant*. To determine the spring constant for a given spring, some physics students experiment with different forces on the spring. They find that forces of 2 N, 3 N, 5 N, and 6 N stretch the spring by 0.25 m, 0.40 m, 0.55 m, and 0.75 m respectively. Thus, by Hooke's Law, they find that

$$\begin{aligned} 2 &= 0.25k \\ 3 &= 0.40k \\ 5 &= 0.55k \\ 6 &= 0.75k \end{aligned}$$

Not surprisingly, due to measurement error, each equation gives a different value of k. To solve this problem, we interpret it in terms of vectors. Let $\vec{p} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 0.25 \\ 0.40 \\ 0.55 \\ 0.75 \end{bmatrix}$.

We want to find the value of k that makes the vector $k\vec{d}$ closest to the point $P(2, 3, 5, 6)$. Observe that we can interpret $k\vec{d}$ as the line L with vector equation

$$\vec{x} = k\begin{bmatrix} 0.25 \\ 0.40 \\ 0.55 \\ 0.75 \end{bmatrix}, \quad k \in \mathbb{R}$$

The vector on L that is closest to P is the projection of P onto L. Moreover, we know that the coefficient k of the projection is

$$k = \frac{\vec{d} \cdot \vec{p}}{\|\vec{d}\|^2} = \frac{8.95}{1.0875} \approx 8.23$$

Thus, based on the data, the best approximation of k would be $k \approx 8.23$.

CONNECTION

In this section, we have only looked at how to do projections onto lines in $\mathbb{R}^n$ and onto planes in $\mathbb{R}^3$. In Chapter 7 we will extend this to finding projections onto subspaces of $\mathbb{R}^n$. This will also allow us to find more general equations of best fit via the method of least squares.

PROBLEMS 1.5

Practice Problems

For Problems A1–A3, calculate the dot product.

A1 $\begin{bmatrix}5\\3\\-6\\1\end{bmatrix}\cdot\begin{bmatrix}3\\2\\4\\0\end{bmatrix}$ **A2** $\begin{bmatrix}1\\-2\\-2\\4\end{bmatrix}\cdot\begin{bmatrix}2\\1/2\\1/2\\-1\end{bmatrix}$ **A3** $\begin{bmatrix}1\\4\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\-1\\1\end{bmatrix}$

For Problems A4–A6, find the length.

A4 $\left\|\begin{bmatrix}\sqrt{2}\\1\\-\sqrt{2}\\-1\end{bmatrix}\right\|$ **A5** $\left\|\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\right\|$ **A6** $\left\|\begin{bmatrix}1\\2\\-1\\3\end{bmatrix}\right\|$

For Problems A7–A12, find a unit vector in the direction of $\vec{x}$.

A7 $\vec{x}=\begin{bmatrix}1\\2\\5\end{bmatrix}$ **A8** $\vec{x}=\begin{bmatrix}3\\-2\\-1\\1\end{bmatrix}$ **A9** $\vec{x}=\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$

A10 $\vec{x}=\begin{bmatrix}1\\2\\5\\-3\end{bmatrix}$ **A11** $\vec{x}=\begin{bmatrix}1/2\\1/2\\1/2\\1/2\\1/2\end{bmatrix}$ **A12** $\vec{x}=\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}$

For Problems A13 and A14, verify the triangle inequality and the Cauchy-Schwarz inequality for the given vectors.

A13 $\vec{x}=\begin{bmatrix}4\\3\\1\end{bmatrix},\vec{y}=\begin{bmatrix}2\\1\\5\end{bmatrix}$ **A14** $\vec{x}=\begin{bmatrix}1\\-1\\2\end{bmatrix},\vec{y}=\begin{bmatrix}-3\\2\\4\end{bmatrix}$

For Problems A15–A20, determine a scalar equation of the hyperplane that passes through the given point with the given normal vector.

A15 $P(1,1,-1),\vec{n}=\begin{bmatrix}3\\1\\4\end{bmatrix}$ **A16** $P(2,-2,0,1),\vec{n}=\begin{bmatrix}0\\1\\3\\3\end{bmatrix}$

A17 $P(2,1,1,5),\vec{n}=\begin{bmatrix}3\\-2\\-5\\1\end{bmatrix}$ **A18** $P(3,1,0,7),\vec{n}=\begin{bmatrix}2\\-4\\1\\-3\end{bmatrix}$

A19 $P(0,0,0,0),\vec{n}=\begin{bmatrix}1\\-4\\5\\-2\end{bmatrix}$ **A20** $P(1,0,1,2,1),\vec{n}=\begin{bmatrix}0\\1\\2\\1\\1\end{bmatrix}$

For Problems A21–A25, determine a normal vector of the hyperplane.

A21 $2x_1+x_2=3$ in $\mathbb{R}^2$
A22 $3x_1-2x_2+3x_3=7$ in $\mathbb{R}^3$
A23 $-4x_1+3x_2-5x_3-6=0$ in $\mathbb{R}^3$
A24 $x_1-x_2+2x_3-3x_4=5$ in $\mathbb{R}^4$
A25 $x_1+x_2-x_3+2x_4-x_5=0$ in $\mathbb{R}^5$

For Problems A26–A31, determine $\text{proj}_{\vec{v}}(\vec{u})$ and $\text{perp}_{\vec{v}}(\vec{u})$. Check your results by verifying that $\vec{v}\cdot\text{perp}_{\vec{v}}(\vec{u})=0$ and $\text{proj}_{\vec{v}}(\vec{u})+\text{perp}_{\vec{v}}(\vec{u})=\vec{u}$.

A26 $\vec{v}=\begin{bmatrix}0\\1\end{bmatrix},\vec{u}=\begin{bmatrix}3\\-5\end{bmatrix}$ **A27** $\vec{v}=\begin{bmatrix}3/5\\4/5\end{bmatrix},\vec{u}=\begin{bmatrix}-4\\6\end{bmatrix}$

A28 $\vec{v}=\begin{bmatrix}0\\1\\0\end{bmatrix},\vec{u}=\begin{bmatrix}-3\\5\\2\end{bmatrix}$ **A29** $\vec{v}=\begin{bmatrix}1/3\\-2/3\\2/3\end{bmatrix},\vec{u}=\begin{bmatrix}4\\1\\-3\end{bmatrix}$

A30 $\vec{v}=\begin{bmatrix}1\\1\\0\\-2\end{bmatrix},\vec{u}=\begin{bmatrix}-1\\-1\\2\\-1\end{bmatrix}$ **A31** $\vec{v}=\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\vec{u}=\begin{bmatrix}2\\3\\2\\-3\end{bmatrix}$

For Problems A32–A37, determine $\text{proj}_{\vec{v}}(\vec{u})$ and $\text{perp}_{\vec{v}}(\vec{u})$.

A32 $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix},\vec{u}=\begin{bmatrix}3\\-3\end{bmatrix}$ **A33** $\vec{v}=\begin{bmatrix}2\\3\\-2\end{bmatrix},\vec{u}=\begin{bmatrix}4\\-1\\3\end{bmatrix}$

A34 $\vec{v}=\begin{bmatrix}-2\\1\\-1\end{bmatrix},\vec{u}=\begin{bmatrix}5\\-1\\3\end{bmatrix}$ **A35** $\vec{v}=\begin{bmatrix}1\\1\\-2\end{bmatrix},\vec{u}=\begin{bmatrix}4\\1\\-2\end{bmatrix}$

A36 $\vec{v}=\begin{bmatrix}-1\\2\\1\\-3\end{bmatrix},\vec{u}=\begin{bmatrix}2\\-1\\2\\1\end{bmatrix}$ **A37** $\vec{v}=\begin{bmatrix}2\\0\\1\\1\end{bmatrix},\vec{u}=\begin{bmatrix}-1\\2\\-1\\2\end{bmatrix}$

For Problems A38 and A39:

(a) Determine a unit vector in the direction of $\vec{u}$.
(b) Calculate $\text{proj}_{\vec{u}}(\vec{F})$.
(c) Calculate $\text{perp}_{\vec{u}}(\vec{F})$.

A38 $\vec{u}=\begin{bmatrix}2\\6\\3\end{bmatrix},\vec{F}=\begin{bmatrix}10\\18\\-6\end{bmatrix}$ **A39** $\vec{u}=\begin{bmatrix}3\\1\\-2\end{bmatrix},\vec{F}=\begin{bmatrix}3\\11\\2\end{bmatrix}$

For Problems A40–A43, use a projection to find the point on the line that is closest to the given point, and find the distance from the point to the line.

A40 $Q(0,0)$, line $\vec{x} = \begin{bmatrix}1\\4\end{bmatrix} + t\begin{bmatrix}-2\\2\end{bmatrix}, \quad t \in \mathbb{R}$

A41 $Q(2,5)$, line $\vec{x} = \begin{bmatrix}3\\7\end{bmatrix} + t\begin{bmatrix}1\\-4\end{bmatrix}, \quad t \in \mathbb{R}$

A42 $Q(1,0,1)$, line $\vec{x} = \begin{bmatrix}2\\2\\-1\end{bmatrix} + t\begin{bmatrix}1\\-2\\1\end{bmatrix}, \quad t \in \mathbb{R}$

A43 $Q(2,3,2)$, line $\vec{x} = \begin{bmatrix}1\\1\\-1\end{bmatrix} + t\begin{bmatrix}1\\4\\1\end{bmatrix}, \quad t \in \mathbb{R}$

For Problems A44–A50, use a projection to find the distance from the point to the plane in $\mathbb{R}^3$.

A44 $Q(2,3,1)$, plane $3x_1 - x_2 + 4x_3 = 5$

A45 $Q(-2,3,-1)$, plane $2x_1 - 3x_2 - 5x_3 = 5$

A46 $Q(0,2,-1)$, plane $2x_1 - x_3 = 5$

A47 $Q(-1,-1,1)$, plane $2x_1 - x_2 - x_3 = 4$

A48 $Q(1,0,1)$, plane $x_1 + x_2 + 3x_3 = 7$

A49 $Q(0,0,2)$, plane $2x_1 + x_2 - 4x_3 = 5$

A50 $Q(2,-1,2)$, plane $x_1 - x_2 - x_3 = 6$

For Problems A51–A54, use a projection to determine the point in the hyperplane that is closest to the given point.

A51 $Q(1,0,0,1)$, hyperplane $2x_1 - x_2 + x_3 + x_4 = 0$

A52 $Q(1,2,1,3)$, hyperplane $x_1 - 2x_2 + 3x_3 = 1$

A53 $Q(2,4,3,4)$, hyperplane $3x_1 - x_2 + 4x_3 + x_4 = 0$

A54 $Q(-1,3,2,-2)$, hyperplane $x_1 + 2x_2 + x_3 - x_4 = 4$

For Problems A55–A58, find the volume of the parallelepiped determined by the given vectors.

A55 $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

A56 $\begin{bmatrix}4\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\5\\2\end{bmatrix}, \begin{bmatrix}1\\1\\6\end{bmatrix}$

A57 $\begin{bmatrix}-2\\1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\5\end{bmatrix}$

A58 $\begin{bmatrix}1\\5\\-3\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}3\\0\\4\end{bmatrix}$

A59 To determine the spring constant for a given spring, some physics students apply different forces on the spring. They find that forces of 3.0 N, 6.5 N, and 9.0 N stretch the spring by 1.0 cm, 2.0 cm, and 3.0 cm respectively. Approximate the spring constant k using the method outlined in Example 1.5.13.

Homework Problems

For Problems B1–B3, calculate the dot product.

B1 $\begin{bmatrix}1\\2\\-1\\3\end{bmatrix} \cdot \begin{bmatrix}3\\-2\\1\\0\end{bmatrix}$

B2 $\begin{bmatrix}4\\2\\-5\\1\end{bmatrix} \cdot \begin{bmatrix}3\\0\\-2\\1\end{bmatrix}$

B3 $\begin{bmatrix}2\\2\\5\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\-1\\4\end{bmatrix}$

For Problems B4–B6, find the length.

B4 $\left\| \begin{bmatrix}1\\2\\2\\3\end{bmatrix} \right\|$

B5 $\left\| \begin{bmatrix}3\\1\\1\\4\end{bmatrix} \right\|$

B6 $\left\| \begin{bmatrix}2/3\\1/3\\1/3\\2/3\end{bmatrix} \right\|$

For Problems B7 and B8, evaluate the expression.

B7 $\left(2\begin{bmatrix}1\\2\\-1\\3\end{bmatrix}\right) \cdot \left(3\begin{bmatrix}3\\-2\\1\\0\end{bmatrix}\right)$

B8 $\left(\begin{bmatrix}3\\5\\11\\3\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\-2\\1\end{bmatrix}\right)\begin{bmatrix}1\\2\\0\\-1\end{bmatrix}$

For Problems B9–B14, find a unit vector in the direction of $\vec{x}$.

B9 $\vec{x} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$

B10 $\vec{x} = \begin{bmatrix}5\\0\\0\\1\end{bmatrix}$

B11 $\vec{x} = \begin{bmatrix}3\\2\\-1\\0\end{bmatrix}$

B12 $\vec{x} = \begin{bmatrix}-2\\1\\-2\\3\end{bmatrix}$

B13 $\vec{x} = \begin{bmatrix}1/3\\1/2\\1/6\\0\end{bmatrix}$

B14 $\vec{x} = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$

For Problems B15 and B16, verify the triangle inequality and the Cauchy-Schwarz inequality for the given vectors.

B15 $\vec{x} = \begin{bmatrix}-1\\2\\4\end{bmatrix}, \vec{y} = \begin{bmatrix}-3\\1\\5\end{bmatrix}$

B16 $\vec{x} = \begin{bmatrix}3\\1\\-2\end{bmatrix}, \vec{y} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

For Problems B17–B20, determine a scalar equation of the hyperplane that passes through the given point with the given normal vector.

B17 $P(1,2,3,5), \vec{n} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ -1 \end{bmatrix}$ **B18** $P(3,1,4,1), \vec{n} = \begin{bmatrix} 1 \\ 5 \\ 9 \\ 2 \end{bmatrix}$

B19 $P(2,1,2,1), \vec{n} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ **B20** $P(1,2,0,1), \vec{n} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

For Problems B21–B26, determine a normal vector of the hyperplane.

B21 $3x_1 + x_2 = 0$ in $\mathbb{R}^2$

B22 $x_1 + 2x_2 + 7x_3 = 1$ in $\mathbb{R}^3$

B23 $3x_1 - 5x_2 + x_3 - x_4 = 4$ in $\mathbb{R}^4$

B24 $x_1 - 3x_3 + 9x_4 = 15$ in $\mathbb{R}^4$

B25 $2x_1 + x_3 + 3x_5 = 2$ in $\mathbb{R}^5$

B26 $-2x_1 - x_2 - 2x_3 + 2x_4 - 2x_5 = 0$ in $\mathbb{R}^5$

For Problems B27–B34, determine $\text{proj}_{\vec{v}}(\vec{u})$ and $\text{perp}_{\vec{v}}(\vec{u})$. Check your results by verifying that $\vec{v} \cdot \text{perp}_{\vec{v}}(\vec{u}) = 0$ and $\text{proj}_{\vec{v}}(\vec{u}) + \text{perp}_{\vec{v}}(\vec{u}) = \vec{u}$.

B27 $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **B28** $\vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \vec{u} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

B29 $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **B30** $\vec{v} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}, \vec{u} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

B31 $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$ **B32** $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \vec{u} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$

B33 $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 3 \\ 3 \\ -4 \\ 2 \end{bmatrix}$ **B34** $\vec{v} = \begin{bmatrix} 7 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \vec{u} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

For Problems B35–B42, determine $\text{proj}_{\vec{v}}(\vec{u})$ and $\text{perp}_{\vec{v}}(\vec{u})$.

B35 $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **B36** $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

B37 $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **B38** $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \vec{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

B39 $\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \vec{u} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$ **B40** $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

B41 $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$ **B42** $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{u} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$

For Problems B43 and B44:

(a) Determine a unit vector in the direction of $\vec{u}$.

(b) Calculate $\text{proj}_{\vec{u}}(\vec{F})$.

(c) Calculate $\text{perp}_{\vec{u}}(\vec{F})$.

B43 $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \vec{F} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$ **B44** $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \vec{F} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

For Problems B45–B48, use a projection to find the point on the line that is closest to the given point, and find the distance from the point to the line.

B45 $Q(4,-5)$, line $\vec{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad t \in \mathbb{R}$

B46 $Q(0,0,1)$, line $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$

B47 $Q(2,-2,1)$, line $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$

B48 $Q(3,2,0)$, line $\vec{x} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \quad t \in \mathbb{R}$

For Problems B49–B52, use a projection to find the distance from the point to the plane in $\mathbb{R}^3$.

B49 $Q(3,5,2)$, plane $2x_1 - 3x_2 - 5x_3 = 7$

B50 $Q(1,0,1)$, plane $2x_1 + x_2 - 4x_3 = 5$

B51 $Q(2,6,2)$, plane $x_1 - x_2 - x_3 = 6$

B52 $Q(0,0,0)$, plane $x_1 + 2x_2 - x_3 = 4$

For Problems B53–B56, use a projection to determine the point in the hyperplane that is closest to the given point.

B53 $Q(2,1,0,-1)$, hyperplane $2x_1 + 2x_3 + 3x_4 = 0$

B54 $Q(1,3,0,1)$, hyperplane $2x_1 - 2x_2 + x_3 + 3x_4 = 0$

B55 $Q(3,1,2,6)$, hyperplane $3x_1 - x_2 - x_3 + x_4 = 3$

B56 $Q(3,1,3,0)$, hyperplane $2x_1 + x_2 + 4x_3 + 3x_4 = 4$

For Problems B57–B60, find the volume of the parallelepiped determined by the given vectors.

B57 $\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$ **B58** $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$

B59 $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ **B60** $\begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$

Conceptual Problems

C1 Consider the statement "If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ such that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{v} = \vec{w}$."

(a) If the statement is true, prove it. If it is false, provide a counterexample.

(b) If we specify $\vec{u} \neq \vec{0}$, does this change the result?

C2 Prove that, as a consequence of the triangle inequality, $\left| \|\vec{x}\| - \|\vec{y}\| \right| \leq \|\vec{x} - \vec{y}\|$. (Hint: $\|\vec{x}\| = \|\vec{x} - \vec{y} + \vec{y}\|$.)

C3 Let $\vec{v}_1$ and $\vec{v}_2$ be orthogonal vectors in $\mathbb{R}^n$. Prove that $\|\vec{v}_1 + \vec{v}_2\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2$.

C4 Prove that if $\vec{x} \in \mathbb{R}^n$ with $\vec{x} \neq \vec{0}$, then $\dfrac{1}{\|\vec{x}\|}\vec{x}$ is a unit vector.

C5 Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of non-zero vectors in $\mathbb{R}^n$ such that all of the vectors are mutually orthogonal. That is, $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. Prove that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

C6 Let S be any set of vectors in $\mathbb{R}^n$. Let $S^\perp$ be the set of all vectors that are orthogonal to every vector in S. That is,

$$S^\perp = \{\vec{w} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{v} \in S\}$$

Show that $S^\perp$ is a subspace of $\mathbb{R}^n$.

C7 (a) Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, verify that the composite map $C : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $C(\vec{x}) = \text{proj}_{\vec{u}}(\text{proj}_{\vec{v}}\, \vec{x})$ also has the linearity property (L1).

(b) Suppose that $C(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^3$, where C is defined as in part (a). What can you say about $\vec{u}$ and $\vec{v}$? Explain.

C8 By the linearity property (L1), we know that $\text{proj}_{\vec{u}}(-\vec{x}) = -\text{proj}_{\vec{u}}\, \vec{x}$. Check, and explain geometrically, that $\text{proj}_{-\vec{u}}(\vec{x}) = \text{proj}_{\vec{u}}(\vec{x})$.

C9 (a) (Pythagorean Theorem) Use the fact that $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$ to prove that $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ if and only if $\vec{x} \cdot \vec{y} = 0$.

(b) Let ℓ be the line in $\mathbb{R}^n$ with vector equation $\vec{x} = t\vec{d}$ and let P be any point that is not on ℓ. Prove that for any point Q on the line, the smallest value of $\|\vec{p} - \vec{q}\|^2$ is obtained when $\vec{q} = \text{proj}_{\vec{d}}(\vec{p})$ (that is, when $\vec{p} - \vec{q}$ is perpendicular to $\vec{d}$). (Hint: Consider $\|\vec{p} - \vec{q}\| = \|\vec{p} - \text{proj}_{\vec{d}}(\vec{p}) + \text{proj}_{\vec{d}}(\vec{p}) - \vec{q}\|$.)

C10 By using the definition of $\text{perp}_{\vec{n}}$ and the fact that $\vec{PQ} = -\vec{QP}$, show that

$$\vec{OP} + \text{perp}_{\vec{n}}\left(\vec{PQ}\right) = \vec{OQ} + \text{proj}_{\vec{n}}\left(\vec{QP}\right)$$

C11 (a) Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Show that $\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}(\vec{x})) = \vec{0}$.

(b) For any $\vec{u} \in \mathbb{R}^3$, prove algebraically that for any $\vec{x} \in \mathbb{R}^3$, $\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}(\vec{x})) = \vec{0}$.

(c) Explain geometrically why $\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}(\vec{x})) = \vec{0}$ for every $\vec{x} \in \mathbb{R}^3$.

C12 A set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to be an **orthonormal set** if each vector in the set is orthogonal to every other vector in the set, and each vector is a unit vector.

(a) Prove that the standard basis for $\mathbb{R}^3$ is an orthonormal set.

(b) Prove that any orthonormal set is a basis for the set it spans.

CHAPTER REVIEW

Suggestions for Student Review

Organizing your own review is an important step towards mastering new material. It is much more valuable than memorizing someone else's list of key ideas. To retain new concepts as useful tools, you must be able to state definitions and make connections between various ideas and techniques. You should also be able to give (or, even better, create) instructive examples. The suggestions below are not intended to be an exhaustive checklist; instead, they suggest the kinds of activities and questioning that will help you gain a confident grasp of the material.

1 Find some person or persons to talk with about mathematics. There is lots of evidence that this is the best way to learn. Be sure you do your share of asking and answering. Note that a little bit of embarrassment is a small price for learning. Also, be sure to get lots of practice in writing answers independently. Looking for how you will need to apply linear algebra in your chosen field can be extremely helpful and motivating.

2 Draw pictures to illustrate addition, subtraction, and scalar multiplication of vectors. (Section 1.1)

3 Explain how you find a vector equation for a line, and make up examples to show why the vector equation of a line is not unique. (Section 1.1)

4 How do you determine if a vector is on a given line or plane? If you were asked to find 10 points on a given plane, how would you do it? (Sections 1.1, 1.3)

5 What are the differences and similarities between vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and directed line segments? Make sure to discuss the different applications. (Section 1.1)

6 State the formal definition of spanning. Can a line in $\mathbb{R}^3$ that does not pass through the origin have a spanning set? What are the theorems related to spanning, and how do we use them? (Sections 1.2, 1.4)

7 State the formal definition of linear independence. Explain the connection between the formal definition of linear dependence and an intuitive geometric understanding of linear dependence. Why is linear independence important when looking at spanning sets? (Sections 1.2, 1.4)

8 State the formal definition of a basis. What does a basis represent geometrically? What is the difference between a basis and a spanning set? What is the importance of that difference? (Sections 1.2, 1.4)

9 State the relation (or relations) between the length and angles in $\mathbb{R}^3$ and the dot product in $\mathbb{R}^3$. Use examples to illustrate. (Sections 1.3, 1.5)

10 What are the properties of dot products and lengths? What are some applications of dot products? (Sections 1.3, 1.5)

11 State the important algebraic and geometric properties of the cross product. What are some applications of the cross product? (Section 1.3)

12 State the definition of a subspace of $\mathbb{R}^n$. Give examples of subspaces in $\mathbb{R}^3$ that are lines, planes, and all of $\mathbb{R}^3$. Show that there is only one subspace in $\mathbb{R}^3$ that does not have infinitely many vectors in it. (Section 1.4)

13 Show that the subspace spanned by three vectors in $\mathbb{R}^3$ can either be a point, a line, a plane, or all of $\mathbb{R}^3$, by giving examples. Explain how this relates to the concept of linear independence. (Section 1.4)

14 Let $\{\vec{v}_1, \vec{v}_2\}$ be a linearly independent spanning set for a subspace S of $\mathbb{R}^3$. Explain how you could construct other spanning sets and other linearly independent spanning sets for S. (Section 1.4)

15 Explain how the projection onto a vector $\vec{v}$ is defined in terms of the dot product. Illustrate with a picture. Define the part of a vector $\vec{x}$ perpendicular to $\vec{v}$ and verify (in the general case) that it is perpendicular to $\vec{v}$. (Section 1.5)

16 Explain with a picture how projections help us to solve the minimum distance problem. (Section 1.5)

17 Discuss the role of a normal vector to a plane or hyperplane in determining a scalar equation of the plane or hyperplane. Explain how you can get from a scalar equation of a plane to a vector equation for the plane and from a vector equation of the plane to the scalar equation. (Sections 1.3, 1.5)

Chapter Quiz

Note: Your instructor may have different ideas of an appropriate level of difficulty for a test on this material.

E1 Calculate $\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and illustrate with a sketch.

E2 Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$. Determine a unit vector that is orthogonal to both $\vec{x}$ and $\vec{y}$.

E3 Let $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ be vectors in $\mathbb{R}^4$. Calculate $\text{proj}_{\vec{u}}(\vec{v})$ and $\text{perp}_{\vec{u}}(\vec{v})$.

E4 Determine a vector equation of the line passing through points $P(-2, 1, -4)$ and $Q(5, -2, 1)$.

E5 Determine a vector equation of the plane in $\mathbb{R}^3$ that satisfies $x_1 - 2x_3 = 3$.

E6 Determine a scalar equation of the plane that contains the points $P(1, -1, 0)$, $Q(3, 1, -2)$, and $R(-4, 1, 6)$.

E7 Describe Span $\left\{ \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 7 \end{bmatrix} \right\}$ geometrically and give a basis for the spanned set.

E8 Determine whether the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent or linearly dependent.

E9 Let $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

(a) Prove that B is a basis for $\mathbb{R}^2$.

(b) Find the coordinates of $\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ with respect to the basis B.

(c) Find the coordinates of $\vec{y} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ with respect to the basis B.

E10 Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_2 = 3 - 5x_1 \right\}$. Determine whether S is a subspace of $\mathbb{R}^2$.

E11 Prove that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid a_1x_2 + a_2x_2 + a_3x_3 = d \right\}$ is a subspace of $\mathbb{R}^3$ for any real numbers a_1, a_2, a_3 if and only if $d = 0$.

E12 Prove that $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis of the plane P in $\mathbb{R}^3$ with scalar equation $x_1 - 3x_2 + x_3 = 0$.

E13 Find the point on the line $\vec{x} = t\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, t \in \mathbb{R}$ that is closest to the point $P(2, 3, 4)$. Illustrate your method of calculation with a sketch.

E14 Find the point on the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ that is closest to the point $P(3, -2, 0, 2)$, and determine the distance from the point to the plane.

E15 Prove that the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ has the same volume as the parallelepiped determined by $(\vec{u} + k\vec{v})$, $\vec{v}$, and $\vec{w}$.

For Problems E16–E22, determine whether the statement is true, and if so, explain briefly. If false, give a counterexample. Each statement is to be interpreted in $\mathbb{R}^3$.

E16 Any three distinct points lie in exactly one plane.

E17 The subspace spanned by a single non-zero vector is a line passing through the origin.

E18 If $\vec{x} = s\vec{v}_1 + t\vec{v}_2$, $s, t \in \mathbb{R}$ is a plane, then $\{\vec{v}_1, \vec{v}_2\}$ is a basis for the plane.

E19 The dot product of a vector with itself cannot be zero.

E20 For any vectors $\vec{x}$ and $\vec{y}$, $\text{proj}_{\vec{x}}(\vec{y}) = \text{proj}_{\vec{y}}(\vec{x})$.

E21 For any vectors $\vec{x}$ and $\vec{y}$, the set $\{\text{proj}_{\vec{x}}(\vec{y}), \text{perp}_{\vec{x}}(\vec{y})\}$ is linearly independent.

E22 The area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is the same as the area of the parallelogram determined by $\vec{u}$ and $(\vec{v} + 3\vec{u})$.

Further Problems

These problems are intended to be a little more challenging than the problems at the end of each section. Some explore topics beyond the material discussed in the text, and some preview topics that will appear later in the text.

F1 Consider the statement "If $\vec{u} \neq \vec{0}$, and both $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$." Either prove the statement or give a counterexample.

F2 Suppose that $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in $\mathbb{R}^3$. Prove that for every $\vec{x} \in \mathbb{R}^3$,

$$\text{perp}_{\vec{u}\times\vec{v}}(\vec{x}) = \text{proj}_{\vec{u}}(\vec{x}) + \text{proj}_{\vec{v}}(\vec{x})$$

F3 In Section 1.3 Problem C10, you were asked to show that $\vec{u} \times (\vec{v} \times \vec{w}) = s\vec{v} + t\vec{w}$ for some $s, t \in \mathbb{R}$.

(a) By direct calculation, prove that

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

(b) Prove that

$$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}$$

F4 What condition is required on the values of $a, b, c, d \in \mathbb{R}$ so that $\mathcal{B} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

F5 Let $\{\vec{v}_1, \vec{v}_2\}$ be a basis for a plane P in $\mathbb{R}^3$.

(a) Find a vector $\vec{w}$ such that $\vec{w}$ is orthogonal to $\vec{v}_1$ and $\{\vec{v}_1, \vec{w}\}$ is also a basis for P. (The set $\{\vec{v}_1, \vec{w}\}$ is called an **orthogonal basis** for P.)

(b) Find an orthogonal basis for $\mathbb{R}^3$ that includes the vectors $\vec{v}_1$ and $\vec{w}$. That is, find another vector $\vec{y} \in \mathbb{R}^3$ such that $B = \{\vec{v}_1, \vec{w}, \vec{y}\}$ is a basis for $\mathbb{R}^3$.

(c) Find a formula for the coordinates of any vector $\vec{x} \in \mathbb{R}^3$ with respect to the basis B. These are called the coordinates with respect to an orthogonal basis.

F6 Let $\mathbb{U}, \mathbb{W}$ be subspaces of $\mathbb{R}^n$ such that $\mathbb{U} \cap \mathbb{W} = \{\vec{0}\}$. Define a subset of $\mathbb{R}^n$ by

$$\mathbb{U} \oplus \mathbb{W} = \{\vec{u} + \vec{w} \mid \vec{u} \in \mathbb{U} \text{ and } \vec{w} \in \mathbb{W}\}$$

(a) Prove that $\mathbb{U} \oplus \mathbb{W}$ is a subspace of $\mathbb{R}^n$.

(b) Prove that if $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a basis for $\mathbb{U}$ and $\{\vec{w}_1, \dots, \vec{w}_\ell\}$ is a basis for $\mathbb{W}$, then $\{\vec{u}_1, \dots, \vec{u}_k, \vec{w}_1, \dots, \vec{w}_\ell\}$ is a basis for $\mathbb{U} \oplus \mathbb{W}$.

F7 Prove that

(a) $\vec{u} \cdot \vec{v} = \frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2$

(b) $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

(c) Interpret (a) and (b) in terms of a parallelogram determined by vectors $\vec{u}$ and $\vec{v}$.

F8 Show that if P, Q, and R are collinear points and $\vec{OP} = \vec{p}$, $\vec{OQ} = \vec{q}$, and $\vec{OR} = \vec{r}$, then

$$(\vec{p} \times \vec{q}) + (\vec{q} \times \vec{r}) + (\vec{r} \times \vec{p}) = \vec{0}$$

F9 In $\mathbb{R}^2$, two lines fail to have a point of intersection only if they are parallel. However, in $\mathbb{R}^3$, a pair of lines can fail to have a point of intersection even if they are not parallel. Two such lines in $\mathbb{R}^3$ are called **skew**.

(a) Observe that if two lines are skew, then they do not lie in a common plane. Show that two skew lines do lie in parallel planes.

(b) Find the distance between the skew lines

$$\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, s \in \mathbb{R} \text{ and } \vec{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad t \in \mathbb{R}$$

CHAPTER 2

Systems of Linear Equations

CHAPTER OUTLINE

In Chapter 1 there were several times that we needed to find a vector $\vec{x}$ in $\mathbb{R}^n$ that simultaneously satisfied several linear equations. For example, when determining whether a set was linearly independent, and when deriving the formula for the cross product. In such cases, we used a **system of linear equations**. *Such systems arise frequently in almost every conceivable area where mathematics is applied: in analyzing stresses in complicated structures; in allocating resources or managing inventory; in determining appropriate controls to guide aircraft or robots; and as a fundamental tool in the numerical analysis of the flow of fluids or heat.*

You have previously learned how to solve small systems of linear equations with substitution and elimination. However, in real-life problems, it is possible for a system of linear equations to have thousands of equations and thousands of variables. Hence, we want to develop some theory that will allow us to solve such problems quickly. In this chapter, we will see that substitution and elimination can be represented by row reduction of a matrix to its reduced row echelon form. This is a fundamental procedure in linear algebra. Obtaining and interpreting the reduced row echelon form of a matrix will play an important role in almost everything we do in the rest of this book.

2.1 Systems of Linear Equations and Elimination

Definition
Linear Equation

A **linear equation** in n variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \tag{2.1}$$

The numbers $a_1, \dots, a_n$ are called the **coefficients** of the equation, and b is usually referred to as "the right-hand side" or "the constant term." The x_i are the unknowns or variables to be solved for.

CONNECTION

1. In Chapters 1 – 8 the coefficients and constant term will be real numbers. In Chapter 9, we will look at linear equations where the coefficients and constant term are a little more 'complex'.

2. From our work in Section 1.5, we know that a linear equation in n variables with real coefficients and constant term geometrically represents a hyperplane in $\mathbb{R}^n$.

EXAMPLE 2.1.1

The equation $x_1 + 2x_2 = 4$ is linear.

The equation $x_1 + \sqrt{3}x_2 - 1 = \pi x_3$ is also linear as it can be written in the form $x_1 + \sqrt{3}x_2 - \pi x_3 = 1$.

The equations $x_1^2 - x_2 = 1$ and $x_1 x_2 = 0$ are both not linear.

Definition
System of Linear Equations

A set of m linear equations in the same variables $x_1, \ldots, x_n$ is called a **system of** m **linear equations in** n **variables**.

A general **system of** m **linear equations in** n **variables** is written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Note that for each coefficient a_{ij}, the first index i indicates in which equation the coefficient appears. The second index j indicates which variable the coefficient multiplies. That is, a_{ij} is the coefficient of x_j in the i-th equation. The index i on the constant term b_i indicates which equation the constant appears in.

Definition
Solution of a System

Solution Set

A vector $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{R}^n$ is called a **solution** of a system of m linear equations in n variables if all m equations are satisfied when we set $x_i = s_i$ for $1 \leq i \leq n$. The set of all solutions of a system of linear equations is called the **solution set** of the system.

Definition
Consistent

Inconsistent

If a system of linear equations has at least one solution, then it is said to be **consistent**. Otherwise, it is said to be **inconsistent**.

Observe that geometrically a system of m linear equations in n variables represents m hyperplanes in $\mathbb{R}^n$. A solution of the system is a vector in $\mathbb{R}^n$ which lies on all m hyperplanes. The system is inconsistent if all m hyperplanes do not share a point of intersection.

EXAMPLE 2.1.2

The system of 2 equations in 3 variables

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 3x_1 + 6x_2 - 3x_3 &= 2 \end{aligned}$$

does not have any solutions since the two corresponding planes are parallel. Hence, the system is inconsistent.

EXAMPLE 2.1.3

The system of 3 linear equations in 2 variables

$$\begin{aligned} x_1 - x_2 &= -5 \\ 2x_1 + 3x_2 &= 0 \\ -x_1 + 4x_2 &= 11 \end{aligned}$$

is consistent since if we take $x_1 = -3$ and $x_2 = 2$, then we get

$$\begin{aligned} -3 - 2 &= -5 \\ 2(-3) + 3(2) &= 0 \\ -(-3) + 4(2) &= 11 \end{aligned}$$

It can be shown this is the only solution, so the solution set is $\left\{\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$.

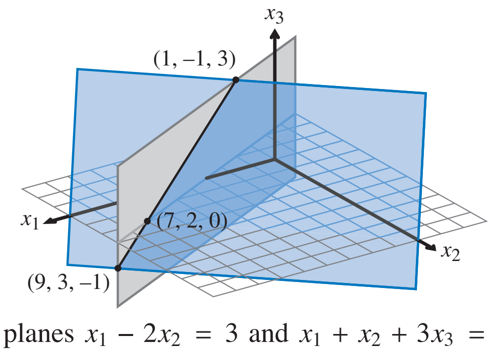
EXERCISE 2.1.1

Sketch the lines $x_1 - x_2 = -5$, $2x_1 + 3x_2 = 0$, and $-x_1 + 4x_2 = 11$ on a single graph to verify the result of Example 2.1.3 geometrically.

EXAMPLE 2.1.4

It can be shown that the solution set of the system of 2 equations in 3 variables

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ x_1 + x_2 + 3x_3 &= 9 \end{aligned}$$



has vector equation

$$\vec{x} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Geometrically, this means that the planes $x_1 - 2x_2 = 3$ and $x_1 + x_2 + 3x_3 = 9$ intersect in the line with the given vector equation.

Remark

It may look like the first equation in Example 2.1.4 has only 2 variables. However, we always assume that all the equations have the same variables. Thus, we interpret the first equation as

$$x_1 - 2x_2 + 0x_3 = 0$$

EXERCISE 2.1.2 Verify that every vector on the line with vector equation $\vec{x} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $t \in \mathbb{R}$ is indeed a solution of the system of linear equations in Example 2.1.4.

To illustrate the possibilities, consider a system of three linear equations in three unknowns. Each equation represents a plane in $\mathbb{R}^3$ which we will label as P_1, P_2, P_3. A solution of the system determines a point of intersection of the three planes. Figure 2.1.1 illustrates an inconsistent system: there is no point common to all three planes. Figure 2.1.2 illustrates a unique solution: all three planes intersect in exactly one point. Figure 2.1.3 demonstrates a case where there are infinitely many solutions.

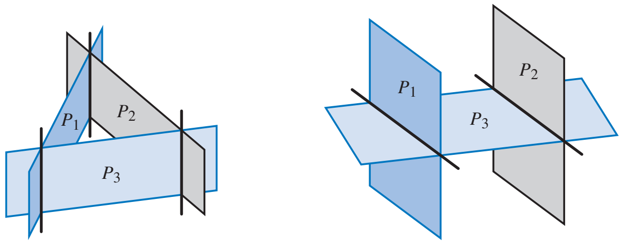


Figure 2.1.1 Two cases where three planes have no common point of intersection: the corresponding system is inconsistent.

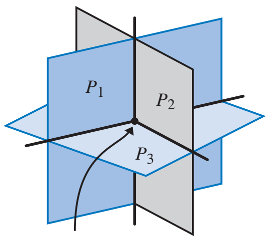


Figure 2.1.2 Three planes with one intersection point: the corresponding system of equations has a unique solution.

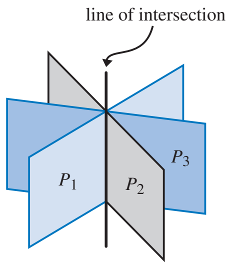


Figure 2.1.3 Three planes that meet in a common line: the corresponding system has infinitely many solutions.

EXERCISE 2.1.3 For each of the four pictures above, create a system of 3 linear equations in 3 unknowns which will be graphically represented by the picture.

These are, in fact, the only three possibilities for any system of linear equations. That is, every system of linear equations is either inconsistent, consistent with a unique solution, or consistent with infinitely many solutions. The following theorem shows that if a system of linear equations has two solutions, then it must have infinitely many solutions. In particular, every vector lying on the line that passes through the two vectors is also a solution of the system.

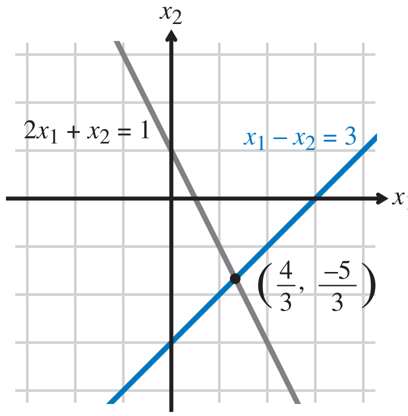
Theorem 2.1.1 If a system of linear equations has two distinct solutions $\vec{s}$ and $\vec{t}$, then $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

You are asked to prove this theorem in Problem C4.

EXAMPLE 2.1.5 We have now seen that the system of linear equations

$$\begin{aligned} x_1 - x_2 &= 3 \\ 2x_1 + x_2 &= 1 \end{aligned}$$

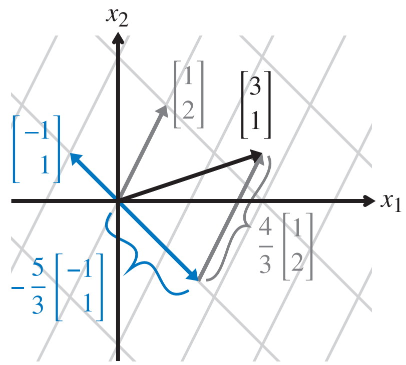
can be interpreted geometrically as a pair of lines in $\mathbb{R}^2$ and its solution $(4/3, -5/3)$ viewed as the point of intersection.



However, we do have an alternate view. From Example 1.2.2 on page 19, we see that this system also represents trying to write the vector $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and a linear combination of $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. That is, the solution $(4/3, -5/3)$ tells us that

$$\frac{4}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

This is represented in the diagram.



Solving Systems of Linear Equations

We now want to establish a standard procedure for determining the solution set of any system of linear equations. To do this, we begin by carefully analyzing how we solve a small system of equations using substitution and elimination.

EXAMPLE 2.1.6 Find all solutions of the system of linear equations

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ x_1 + 3x_2 - x_3 &= 7 \\ 2x_1 + x_2 - 5x_3 &= 7 \end{aligned}$$

Solution: To solve this system by elimination, we begin by eliminating x_1 from all equations except the first one.

EXAMPLE 2.1.6
(continued)

Add (−1) times the first equation to the second equation. The first and third equations are unchanged, so the system is now

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 5x_3 &= 7 \end{aligned}$$

Note two important things about this step. First, if x_1, x_2, x_3 satisfy the original system, then they certainly satisfy the revised system after the step. This follows from the rule of arithmetic that if $P = Q$ and $R = S$, then $P+R = Q+S$. So, when we add a multiple of one equation to another equation and both are satisfied, the resulting equation is satisfied. Thus, the revised system has the same solution set as the original system.

Add (−2) times the first equation to the third equation.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ 2x_2 + x_3 &= 3 \\ -x_2 - x_3 &= -1 \end{aligned}$$

Again, this step does not change the solution set. Now that x_1 has been eliminated from all equations except the first one, we leave the first equation alone and turn our attention to x_2.

Although we will not modify or use the first equation in the next several steps, we keep writing the entire system after each step. This is important because it leads to a good general procedure for dealing with large systems.

It is convenient, but not necessary, to work with an equation in which x_2 has the coefficient 1. We could multiply the second equation by $1/2$. However, to avoid fractions, we instead just swap the order of the equations.

Interchange the second and third equations. This step definitely does not change the solution set.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ -x_2 - x_3 &= -1 \\ 2x_2 + x_3 &= 3 \end{aligned}$$

Multiply the second equation by (−1). This does not change the solution set.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ x_2 + x_3 &= 1 \\ 2x_2 + x_3 &= 3 \end{aligned}$$

Add (−2) times the second equation to the third equation. This does not change the solution set.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ x_2 + x_3 &= 1 \\ -x_3 &= 1 \end{aligned}$$

In the third equation, all variables except x_3 have been eliminated; by elimination, we have solved for x_3. Using similar steps, we could continue and eliminate x_3 from the second and first equations and x_2 from the first equation. However, it is often a much simpler task to complete the solution process by **back-substitution**.

EXAMPLE 2.1.6
(continued)

We have $x_3 = -1$. Substitute this value into the second equation to find that

$$x_2 = 1 - x_3 = 1 - (-1) = 2$$

Next, substitute both these values into the first equation to obtain

$$x_1 = 4 - x_2 + 2x_3 = 4 - 2 + 2(-1) = 0$$

Thus, the only solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

But, all of operations did not change the solution set, so this solution is also the unique solution of the original system.

Observe that we can easily check that $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ satisfies the original system of equations:

$$\begin{aligned} 0 + 2 - 2(-1) &= 4 \\ 0 + 3(2) - (-1) &= 7 \\ 2(0) + 2 - 5(-1) &= 7 \end{aligned}$$

It is important to observe the form of the equations in our final system. The first variable with a non-zero coefficient in each equation, called a **leading variable**, does not appear in any equation below it. Also, the leading variable in the second equation is to the right of the leading variable in the first, and the leading variable in the third is to the right of the leading variable in the second.

The system solved in Example 2.1.6 is a simple one. However, the solution procedure introduces all the steps that are needed in the process of elimination.

Types of Steps in Elimination

(1) Multiply one equation by a non-zero constant.

(2) Interchange two equations.

(3) Add a multiple of one equation to another equation.

Warning! Do *not* combine steps of type (1) and type (3) into one step of the form "Add a multiple of one equation to a multiple of another equation." Although such a combination would not lead to errors in this chapter, it can lead to errors when we apply these ideas in Chapter 5.

Observe that after each step we actually have a new system of linear equations to solve. The idea is that each new system has the same solution set as the original and is easier to solve. Moreover, observe that each elimination step is reversible, so that we can always return to the original system from the new system. Hence, we make the following definition.

Definition
Equivalent

Two systems of linear equations that have the same solution set are said to be **equivalent**.

EXERCISE 2.1.4

Solve the system below by using substitution and elimination. Clearly show/explain all of your steps used in solving the system, and describe your general procedure.

$$\begin{aligned} 2x_1 + 3x_2 &= 11 \\ 3x_1 + 6x_2 &= 7 \end{aligned}$$

EXAMPLE 2.1.7

Determine the solution set of the system of 2 linear equations in 4 variables

$$\begin{aligned} x_1 + 2x_3 + x_4 &= 14 \\ x_1 + 3x_3 + 3x_4 &= 19 \end{aligned}$$

Solution: First, notice that neither equation contains x_2. This may seem peculiar, but it happens in some applications that one of the variables of interest does not appear in any of the linear equations. If it truly is one of the variables of the problem, ignoring it is incorrect. We rewrite the equations to make it explicit:

$$\begin{aligned} x_1 + 0x_2 + 2x_3 + x_4 &= 14 \\ x_1 + 0x_2 + 3x_3 + 3x_4 &= 19 \end{aligned}$$

As in Example 2.1.6, we want our leading variable in the first equation to be to the left of the leading variable in the second equation, and we want the leading variable to be eliminated from the second equation. Thus, we use a type (3) step to eliminate x_1 from the second equation.
Add (-1) times the first equation to the second equation:

$$\begin{aligned} x_1 + 0x_2 + 2x_3 + x_4 &= 14 \\ x_3 + 2x_4 &= 5 \end{aligned}$$

Observe that x_2 is not shown in the second equation because the leading variable must have a non-zero coefficient. Moreover, we have already finished our elimination procedure as we have our desired form. The solution can now be completed by back-substitution.

Note that the equations do not completely determine both x_3 and x_4: one of them can be chosen arbitrarily, and the equations can still be satisfied. For consistency, we always choose the variables that do not appear as a leading variable in any equation to be the ones that will be chosen arbitrarily. We will call these **free variables**.

Thus, in the revised system, we see that neither x_2 nor x_4 appears as a leading variable in any equation. Therefore, x_2 and x_4 are the free variables and may be chosen arbitrarily (for example, $x_4 = t \in \mathbb{R}$ and $x_2 = s \in \mathbb{R}$). Then the second equation can be solved for the leading variable x_3:

$$x_3 = 5 - 2x_4 = 5 - 2t$$

Now, solve the first equation for its leading variable x_1:

$$x_1 = 14 - 2x_3 - x_4 = 14 - 2(5 - 2t) - t = 4 + 3t$$

EXAMPLE 2.1.7
(continued)

Thus, the solution set of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4+3t \\ s \\ 5-2t \\ t \end{bmatrix}, \quad s,t \in \mathbb{R}$$

In this case, there are infinitely many solutions because for each value of s and for each value of t that we choose, we get a different solution. We say that this equation is the **general solution** of the system, and we call s and t the **parameters** of the general solution. For many purposes, it is useful to recognize that this solution can be split into a constant part, a part in t, and a part in s:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

This will be the standard format for displaying general solutions. It is acceptable to leave x_2 in the place of s and x_4 in the place of t, but then you *must* say $x_2, x_4 \in \mathbb{R}$. Observe that one immediate advantage of this form is that we can instantly see the geometric interpretation of the solution. The intersection of the two hyperplanes $x_1 + 2x_3 + x_4 = 14$ and $x_1 + 3x_3 + 3x_4 = 19$ in $\mathbb{R}^4$ is the plane in $\mathbb{R}^4$ that passes through $P(4, 0, 5, 0)$ with vector equation

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

The solution procedure we have introduced is known as **Gaussian elimination with back-substitution**. A slight variation of this procedure is introduced in the next section.

EXERCISE 2.1.5

Find the general solution to the system of linear equations

$$\begin{aligned} 2x_1 + 4x_2 + 0x_3 &= 12 \\ x_1 + 2x_2 - x_3 &= 4 \end{aligned}$$

Use the general solution to find three different solutions of the system.

CONNECTION

In Chapter 1 (see page 37) when we were using a scalar equation of a plane to find a vector equation of the plane, we were really just solving a system of 1 linear equation in 3 variables. Observe that the procedure we used there is exactly the same as what we are now doing.

The Matrix Representation of a System of Linear Equations

After you have solved a few systems of equations using elimination, you may realize that you could write the solution faster if you could omit the letters x_1, x_2, and so on—as long as you could keep the coefficients lined up properly. To do this, we write out the coefficients in a rectangular array called a **matrix**.

Definition
Augmented Matrix
Coefficient Matrix

A general linear system of m equations in n unknowns can be represented by the matrix

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{array}\right]$$

where the coefficient a_{ij} appears in the i-th row and j-th column of the coefficient matrix. This is called the **augmented matrix** of the system. It is augmented because it includes as its last column the right-hand side of the equations. The matrix without this last column is called the **coefficient matrix** of the system:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

For convenience, we sometimes denote the augmented matrix of a system with coefficient matrix A and right-hand side $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ by $\left[A \mid \vec{b}\right]$. In Chapter 3, we will develop an even better way of representing a system of linear equations.

EXAMPLE 2.1.8 Write the coefficient matrix and augmented matrix for the following system:

$$\begin{aligned} 3x_1 + 8x_2 - 18x_3 + x_4 &= 35 \\ x_1 + 2x_2 - 4x_3 &= 11 \\ x_1 + 3x_2 - 7x_3 + x_4 &= 10 \end{aligned}$$

Solution: The coefficient matrix is formed by writing the coefficients of each equation as the rows of the matrix. Thus, we get the matrix

$$A = \begin{bmatrix} 3 & 8 & -18 & 1 \\ 1 & 2 & -4 & 0 \\ 1 & 3 & -7 & 1 \end{bmatrix}$$

EXAMPLE 2.1.8
(continued)

For the augmented matrix, we just add the right-hand side as the last column. We get

$$\left[\begin{array}{cccc|c} 3 & 8 & -18 & 1 & 35 \\ 1 & 2 & -4 & 0 & 11 \\ 1 & 3 & -7 & 1 & 10 \end{array}\right]$$

EXAMPLE 2.1.9

Write the system of linear equations that has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Solution: The rows of the matrix tell us the coefficients and constant terms of each equation. We get the system

$$\begin{aligned} x_1 \qquad + 2x_3 &= 3 \\ -x_2 + x_3 &= 1 \\ x_3 &= -2 \end{aligned}$$

Remark

Another way to view the coefficient matrix is to see that the j-th column of the coefficient matrix is the vector containing all the coefficients of x_j. We will use this interpretation throughout the text.

Since each row in the augmented matrix corresponds to an equation in the system of linear equations, performing operations on the equations of the system corresponds to performing the same operations on the rows of the matrix. Thus, the steps in elimination correspond to the following elementary row operations.

Definition
Elementary Row Operations

The three **elementary row operations (EROs)** are:
(1) Multiply one row by a non-zero constant.
(2) Interchange two rows.
(3) Add a multiple of one row to another row.

As with the steps in elimination, we do not combine operations of type (1) and type (3) into one operation.

The process of performing elementary row operations on a matrix to bring it into some simpler form is called **row reduction**.

Recall that if a system of equations is obtained from another system by one or more of the elimination steps, the systems are said to be equivalent. For matrices, if the matrix M is row reduced into a matrix N by a sequence of elementary row operations, then we say that M is **row equivalent** to N and we write $M \sim N$. Note that it would be incorrect to use $=$ or $\Rightarrow$ instead of $\sim$.

Just as elimination steps are reversible, so are elementary row operations. It follows that if M is row equivalent to N, then N is row equivalent to M, so we may say that M and N are row equivalent. It also follows that if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

We often use short hand notation to indicate which operations we are using:

cR_i indicates multiplying the i-th row by $c \neq 0$.

$R_i \updownarrow R_j$ indicates swapping the i-th row and the j-th row.

$R_i + cR_j$ indicates adding c times the j-th row to the i-th row.

As one becomes confident with row reducing, one may omit these indicators. However, including them can make checking the steps easier, is required for one concept in Chapter 5, and instructors may require them in work submitted for grading.

Theorem 2.1.2 If the augmented matrices $\left[A_1 \mid \vec{b}_1\right]$ and $\left[A \mid \vec{b}\right]$ are row equivalent, then the systems of linear equations associated with each augmented matrix are equivalent.

EXAMPLE 2.1.10

Rewrite the elimination steps for the system in Example 2.1.6 in matrix form.

Solution: The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 1 & 3 & -1 & 7 \\ 2 & 1 & -5 & 7 \end{array}\right]$$

The first step in the elimination was to add (-1) times the first equation to the second. Here we add (-1) multiplied by the first row to the second. We write

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 1 & 3 & -1 & 7 \\ 2 & 1 & -5 & 7 \end{array}\right] R_2 - 1R_1 \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -5 & 7 \end{array}\right]$$

The remaining steps are

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -5 & 7 \end{array}\right] R_3 - 2R_1 \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right] R_2 \updownarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & -1 & -1 & -1 \\ 0 & 2 & 1 & 3 \end{array}\right] (-1)R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \end{array}\right] R_3 - 2R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

All the elementary row operations corresponding to the elimination in Example 2.1.6 have been performed. Observe that the last matrix is the augmented matrix for the final system of linear equations that we obtained in Example 2.1.6.

EXAMPLE 2.1.11

Write the matrix representation of the elimination in Example 2.1.7.

Solution: We have

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 14 \\ 1 & 0 & 3 & 3 & 19 \end{array}\right] R_2 + (-1)R_1 \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 14 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

EXERCISE 2.1.6

Write out the matrix representation of the elimination used in Exercise 2.1.5.

In the next example, we will solve a system of linear equations using Gaussian elimination with back-substitution entirely in matrix form.

EXAMPLE 2.1.12 Find the general solution of the system

$$\begin{aligned} 3x_1 + 8x_2 - 18x_3 + x_4 &= 35 \\ x_1 + 2x_2 - 4x_3 \phantom{{}+x_4} &= 11 \\ x_1 + 3x_2 - 7x_3 + x_4 &= 10 \end{aligned}$$

Solution: Write the augmented matrix of the system and row reduce:

$$\left[\begin{array}{cccc|c} 3 & 8 & -18 & 1 & 35 \\ 1 & 2 & -4 & 0 & 11 \\ 1 & 3 & -7 & 1 & 10 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 3 & 8 & -18 & 1 & 35 \\ 1 & 3 & -7 & 1 & 10 \end{array}\right] R_2 - 3R_1 \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 2 & -6 & 1 & 2 \\ 1 & 3 & -7 & 1 & 10 \end{array}\right] R_3 - 1R_1 \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 2 & -6 & 1 & 2 \\ 0 & 1 & -3 & 1 & -1 \end{array}\right] R_2 \updownarrow R_3 \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 2 & -6 & 1 & 2 \end{array}\right] R_3 - 2R_2 \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{array}\right]$$

To find the general solution, we now interpret the final matrix as the augmented matrix of the equivalent system. We get the system

$$\begin{aligned} x_1 + 2x_2 - 4x_3 \phantom{{}+x_4} &= 11 \\ x_2 - 3x_3 + x_4 &= -1 \\ -x_4 &= 4 \end{aligned}$$

Since x_3 is a free variable, we let $x_3 = t \in \mathbb{R}$. Then we use back-substitution to get

$$\begin{aligned} x_4 &= -4 \\ x_2 &= -1 + 3x_3 - x_4 = 3 + 3t \\ x_1 &= 11 - 2x_2 + 4x_3 = 5 - 2t \end{aligned}$$

Thus, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5-2t \\ 3+3t \\ t \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -4 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

Check this solution by substituting these values for x_1, x_2, x_3, x_4 into the original equations.

Observe that there are many different ways that we could choose to row reduce the augmented matrix in any of these examples. For instance, in Example 2.1.12 we could interchange row 1 and row 3 instead of interchanging row 1 and row 2. Alternatively, we could use the elementary row operations $R_2 - \frac{1}{3}R_1$ and $R_3 - \frac{1}{3}R_1$ to eliminate the non-zero entries beneath the first leading variable. It is natural to ask if there is a way of determining which elementary row operations will work the best. Unfortunately, there is no such algorithm for doing these by hand. However, we will give a basic algorithm for row reducing a matrix into the "proper" form. We start by defining this form.

Row Echelon Form

Based on how we used elimination to solve the system of equations, we define the following form of a matrix.

Definition
Row Echelon Form (REF)

A matrix is in **row echelon form (REF)** if

(1) A zero row (all entries in the row are zero) must appear below all rows that contain a non-zero entry.
(2) When two non-zero rows are compared, the first non-zero entry, called the leading entry, in the upper row is to the left of the leading entry in the lower row.

If a matrix A is row equivalent to a matrix R in row echelon form, then we say that R is a row echelon form of A.

Remark

It follows from these properties that all entries in a column beneath a leading entry must be 0. For otherwise, (1) or (2) would be violated.

EXAMPLE 2.1.13 Determine which of the following matrices are in row echelon form. For each matrix that is not in row echelon form, explain why it is not in row echelon form.

(a) $\begin{bmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 2 & 3 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 2 & -1 & 1 \\ 1 & 3 & 1 & 4 & -2 \end{bmatrix}$

Solution: The matrices in (a) and (b) are both in REF. The matrix in (c) is not in REF since the leading entry in the second row is to the right of the leading entry in the third row. The matrix in (d) is not in REF since there is a non-zero entry beneath the leading entry in the first row.

Algorithm 2.1.1

Gaussian Elimination

Any matrix can be row reduced to row echelon form by using the following steps:

1. Working from the left, find the first column of the matrix that contains some non-zero entry. Interchange rows (if necessary) so that the top entry in the column is non-zero. Of course, the column may contain multiple non-zero entries. You can use any of these non-zero entries, but some choices will make your calculations considerably easier than others; see the Remarks on page 95. We will call this entry a **pivot**.
2. Use elementary row operations of type (3) to make all entries beneath the pivot into zeros.
3. Repeat Steps 1 and 2 on the submatrix consisting of all rows below the row with the most recently obtained pivot until you reach the bottom row or there are no remaining non-zero rows.

EXAMPLE 2.1.14

Use Gaussian Elimination to bring the matrix $A = \begin{bmatrix} 2 & 0 & 4 & 4 \\ -1 & 0 & -1 & 0 \\ -3 & 0 & 0 & 6 \end{bmatrix}$ into REF.

Solution: We begin by considering the first column. The top entry of the first column is non-zero, so this becomes our pivot.

$$\begin{bmatrix} 2 & 0 & 4 & 4 \\ -1 & 0 & -1 & 0 \\ -3 & 0 & 0 & 6 \end{bmatrix}$$

We now put zeros beneath the pivot using $R_2 + \frac{1}{2}R_1$ and $R_3 + \frac{3}{2}R_1$.

$$\begin{bmatrix} 2 & 0 & 4 & 4 \\ -1 & 0 & -1 & 0 \\ -3 & 0 & 0 & 6 \end{bmatrix} \begin{matrix} \\ R_2 + \frac{1}{2}R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 2 & 0 & 4 & 4 \\ 0 & 0 & 1 & 2 \\ -3 & 0 & 0 & 6 \end{bmatrix} \begin{matrix} \\ \\ R_3 + \frac{3}{2}R_1 \end{matrix} \sim \begin{bmatrix} 2 & 0 & 4 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}$$

We now temporarily ignore the first row and repeat the procedure.

$$\begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{4} & \cancel{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}$$

The first two columns of the submatrix contain only zeros, so we move to the third column. Its top entry is non-zero, so that becomes our pivot. We use $R_3 - 6R_2$ to put a zero beneath it.

$$\begin{bmatrix} 2 & 0 & 4 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 6R_2 \end{matrix} \sim \begin{bmatrix} 2 & 0 & 4 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we only have zero rows beneath the most recent pivot, the matrix is now in row echelon form.

EXAMPLE 2.1.15

Use Gaussian Elimination to row reduce the augmented matrix of the following system to row echelon form. Determine all solutions of the system.

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_1 + x_2 \quad &= 1 \\ x_1 + 2x_2 + x_3 &= -2 \end{aligned}$$

Solution: We first write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & -2 \end{array}\right]$$

We begin by considering the first column. To make the top entry in the first column non-zero, we exchange row 1 and row 2

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & -2 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & -2 \end{array}\right]$$

We now have our first pivot. The second row already has a zero beneath this pivot, so we just use $R_3 - R_1$ so that all entries beneath the pivot are 0.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & -2 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

We now temporarily ignore the first row and repeat the procedure.

$$\left[\begin{array}{ccc|c} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

We see that the first column of the submatrix contains all zeros, so we move to the next column. The entry in the top of the next column is non-zero, so this becomes our next pivot. We use $R_3 - R_2$ to place a zero beneath it.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

Since we have arrived at the bottom row, the matrix is in row echelon form.

Observe that when we write the system of linear equations represented by this augmented matrix, we get

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_2 + x_3 &= 2 \\ 0 &= -5 \end{aligned}$$

Clearly, the last equation is impossible. This means we cannot find values of x_1, x_2, x_3 that satisfy all three equations. Consequently this system, and hence the original system, has no solution.

Remarks

1. Although the previous algorithm will always work, it is not necessarily the fastest or easiest method to use for any particular matrix. In principle, it does not matter which non-zero entry is chosen as the pivot in the procedure just described. In practice, it can have considerable impact on the amount of work required and on the accuracy of the result. The ability to row reduce a general matrix to REF by hand quickly and efficiently comes only with a considerable amount of practice. Note that for hand calculations on simple integer examples, it is sensible to go to some trouble to postpone fractions because avoiding fractions may reduce both the effort required and the chance of making errors.

2. Every matrix, except the matrix containing all zeros, has infinitely many row echelon forms that are all row equivalent. However, it can be shown that any two row echelon forms for the same matrix A must agree on the position of the leading entries. (This fact may seem obvious, but it is not easy to prove. It follows from Problem F2 in the Chapter 4 Further Problems.)

Row echelon form allows us to answer questions about consistency and uniqueness. In particular, we have the following theorem.

Theorem 2.1.3

Suppose that the augmented matrix $\left[A \mid \vec{b}\right]$ of a system of linear equations is row equivalent to $\left[R \mid \vec{c}\right]$, which is in row echelon form.

(1) The given system is inconsistent if and only if some row of $\left[R \mid \vec{c}\right]$ is of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$, with $c \neq 0$.

(2) If the system is consistent, there are two possibilities. Either the number of pivots in R is equal to the number of variables in the system and the system has a unique solution, or the number of pivots is less than the number of variables and the system has infinitely many solutions.

Proof: (1) If $\left[R \mid \vec{c}\right]$ contains a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$, where $c \neq 0$, then this corresponds to the equation $0 = c$, which clearly has no solution. Hence, the system is inconsistent. On the other hand, if it contains no such row, then each row must either be of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 0 \end{array}\right]$, which corresponds to an equation satisfied by any values of $x_1, \dots, x_n$, or else contains a pivot. We may ignore the rows that consist entirely of zeros, leaving only rows with pivots. In the latter case, the corresponding system can be solved by assigning arbitrary values to the free variables and then determining the remaining variables by back-substitution. Thus, if there is no row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$, the system is consistent.

(2) Now consider the case of a consistent system. The number of leading variables cannot be greater than the number of columns in the coefficient matrix; if it is equal, then each variable is a leading variable and thus is determined uniquely by the system corresponding to $\left[R \mid \vec{c}\right]$. If some variables are not leading variables, then they are free variables, and they may be chosen arbitrarily. Hence, there are infinitely many solutions. ■

Remark

As we will see later in the text, sometimes we are only interested in whether a system is consistent or inconsistent or in how many solutions a system has. We may not necessarily be interested in finding a particular solution. In these cases, Theorem 2.1.3, or the System-Rank Theorem in Section 2.2, can be very useful.

Some Shortcuts and Some Bad Moves

When carrying out elementary row operations, you may get weary of rewriting the matrix every time. Fortunately, we can combine some elementary row operations in one rewriting. For example,

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 1 & 3 & -1 & 7 \\ 2 & 1 & -5 & 7 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

Choosing one particular row (in this case, the first row) and adding multiples of it to several other rows is perfectly acceptable. There are other elementary row operations that can be combined, but these should not be used until one is extremely comfortable with row reducing. This is because some combinations of steps do cause errors. For example,

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{array}{c} R_1 - R_2 \\ R_2 - R_1 \end{array} \sim \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (WRONG!)$$

This is nonsense because the final matrix should have a leading 1 in the first column. By performing one elementary row operation, we change one row; thereafter, we must use that row in its new changed form. Thus, when performing multiple elementary row operations in one step, make sure that you are not modifying a row that you are using in another elementary row operation.

EXERCISE 2.1.7

Use Gaussian Elimination to bring the matrix $A = \begin{bmatrix} 0 & 2 & 1 & -1 \\ 1 & 2 & -1 & 1 \\ -1 & 4 & 4 & 1 \end{bmatrix}$ into a row echelon form.

Applications

To illustrate the application of systems of equations we give a couple simple examples. In Section 2.4 we discuss more applications from physics/engineering.

EXAMPLE 2.1.16 A boy has a jar full of coins. Altogether there are 180 nickels, dimes, and quarters. The number of dimes is one-half of the total number of nickels and quarters. The value of the coins is \$16.00. How many of each kind of coin does he have?

Solution: Let n be the number of nickels, d the number of dimes, and q the number of quarters. Then

$$n + d + q = 180$$

The second piece of information we are given is that

$$d = \frac{1}{2}(n + q)$$

We rewrite this into standard form for a linear equation:

$$n - 2d + q = 0$$

Finally, we have the value of the coins, in cents:

$$5n + 10d + 25q = 1600$$

Thus, n, d, and q satisfy the system of linear equations:

$$\begin{aligned} n \quad + d \quad + q &= 180 \\ n \quad - 2d \quad + q &= 0 \\ 5n + 10d + 25q &= 1600 \end{aligned}$$

Write the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 1 & -2 & 1 & 0 \\ 5 & 10 & 25 & 1600 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 5R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & -3 & 0 & -180 \\ 0 & 5 & 20 & 700 \end{array}\right] \begin{array}{c} \\ (-1/3)R_2 \\ (1/5)R_3 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & 1 & 0 & 60 \\ 0 & 1 & 4 & 140 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 4 & 80 \end{array}\right]$$

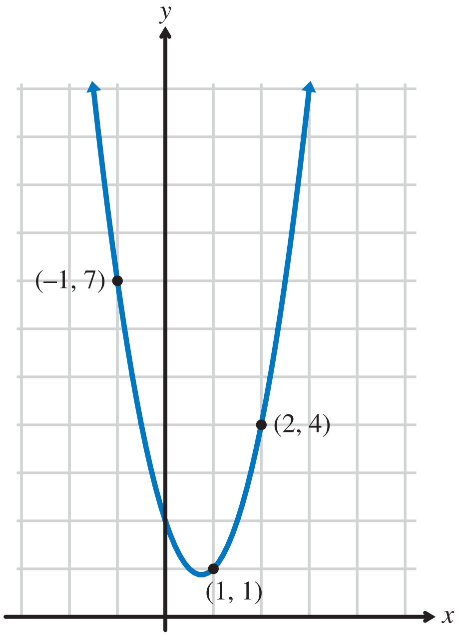
According to Theorem 2.1.3, the system is consistent with a unique solution. In particular, writing the final augmented matrix as a system of equations, we get

$$\begin{aligned} n + d + q &= 180 \\ d &= 60 \\ 4q &= 80 \end{aligned}$$

So, by back-substitution, we get $q = 20$, $d = 60$, $n = 180 - d - q = 100$. Hence, the boy has 100 nickels, 60 dimes, and 20 quarters.

EXAMPLE 2.1.17

Determine the parabola $y = a + bx + cx^2$ that passes through the points $(1, 1)$, $(-1, 7)$, and $(2, 4)$.



Solution: To create a system of linear equations, we substitute the x and y values of each point into the general equation of the parabola $y = a + bx + cx^2$. We get

$$\begin{aligned} 1 &= a + b(1) + c(1)^2 = a + b + c \\ 7 &= a + b(-1) + c(-1)^2 = a - b + c \\ 4 &= a + b(2) + c(2)^2 = a + 2b + 4c \end{aligned}$$

As usual, we write the corresponding augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 7 \\ 1 & 2 & 4 & 4 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 6 \\ 0 & 1 & 3 & 3 \end{array}\right] -(1/2)R_2 \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 3 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

By back-substitution we get $c = 2$, $b = -3$, and $a = 1 - b - c = 1 - (-3) - 2 = 2$. Hence, the parabola is

$$y = 2 - 3x + 2x^2$$

It is easy to verify the answer by checking to ensure that all three points are on this parabola.

A Remark on Computer Calculations

In computer calculations, the choice of the pivot may affect accuracy. The problem is that real numbers are represented to only a finite number of digits in a computer, so inevitably some round-off or truncation errors can occur. When you are doing a large number of arithmetic operations, these errors can accumulate, and they can be particularly serious if at some stage you subtract two nearly equal numbers. The following example gives some idea of the difficulties that might be encountered.

The system

$$\begin{aligned} 0.1000x_1 + 0.9990x_2 &= 1.000 \\ 0.1000x_1 + 1.000x_2 &= 1.006 \end{aligned}$$

is easily found to have solution $x_2 = 6.000$, $x_1 = -49.94$. Notice that the coefficients were given to four digits. Suppose all entries are rounded to three digits. The system becomes

$$\begin{aligned} 0.100x_1 + 0.999x_2 &= 1.00 \\ 0.100x_1 + 1.00x_2 &= 1.01 \end{aligned}$$

The solution is now $x_2 = 10$, $x_1 = -89.9$. Notice that, despite the fact that there was only a small change in one term on the right-hand side, the resulting solution is not close to the solution of the original problem. Geometrically, this can be understood by observing that the solution is the intersection point of two nearly parallel lines; therefore, a small displacement of one line causes a major shift of the intersection point. Difficulties of this kind may arise in higher-dimensional systems of equations in real applications.

Carefully choosing pivots in computer programs can reduce the error caused by these sorts of problems. However, some matrices are **ill conditioned**; even with high-precision calculations, the solutions produced by computers with such matrices may be unreliable. In applications, the entries in the matrices may be experimentally determined, and small errors in the entries may result in large errors in calculated solutions, no matter how much precision is used in computation. To understand this problem better, you need to know something about sources of error in numerical computation—and more linear algebra. We shall not discuss it further in this book, but you should be aware of the difficulty if you use computers to solve systems of linear equations.

PROBLEMS 2.1

Practice Problems

For Problems A1–A4, solve the system using back-substitution. Write the general solution in standard form.

A1 $\begin{aligned} x_1 - 3x_2 &= 5 \\ x_2 &= 4 \end{aligned}$

A2 $\begin{aligned} x_1 + 2x_2 - x_3 &= 7 \\ x_3 &= 6 \end{aligned}$

A3 $\begin{aligned} x_1 + 3x_2 - 2x_3 &= 4 \\ x_2 + 5x_3 &= 2 \\ x_3 &= 2 \end{aligned}$

A4 $\begin{aligned} x_1 - 2x_2 + x_3 + 4x_4 &= 7 \\ x_2 \qquad - x_4 &= -3 \\ x_3 + x_4 &= 2 \end{aligned}$

For Problems A5–A8, determine whether the matrix is in REF. If not, explain why it is not.

A5 $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

A6 $B = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A7 $C = \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$

A8 $D = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

For Problems A9–A14, row reduce the matrix to REF. Show your steps.

A9 $\begin{bmatrix} 4 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

A10 $\begin{bmatrix} 2 & -2 & 5 & 8 \\ 1 & -1 & 2 & 3 \\ -1 & 1 & 0 & 2 \end{bmatrix}$

A11 $\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ 5 & 0 & 0 \\ 3 & 4 & 5 \end{bmatrix}$

A12 $\begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 3 & 6 & 13 & 20 \end{bmatrix}$

A13 $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & -1 & -4 & 1 \\ 2 & 1 & 3 & 6 \end{bmatrix}$

A14 $\begin{bmatrix} 3 & 1 & 8 & 2 & 4 \\ 1 & 0 & 3 & 0 & 1 \\ 0 & 2 & -2 & 4 & 3 \\ -4 & 1 & 11 & 3 & 8 \end{bmatrix}$

For Problems A15–A20, the given matrix is an augmented matrix of a system of linear equations. Either show the system is inconsistent or write a vector equation for the solution set.

A15 $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & -5 \end{array}\right]$

A16 $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

A17 $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

A18 $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 1 \\ 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$

A19 $\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

A20 $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

For Problems A21–A27:

(a) Write the augmented matrix.
(b) Row reduce the augmented matrix to REF.
(c) Determine whether the system is consistent or inconsistent. If it is consistent, determine the number of parameters in the general solution.
(d) If the system is consistent, write its general solution in standard form.

A21 $\begin{aligned} 3x_1 - 5x_2 &= 2 \\ x_1 + 2x_2 &= 4 \end{aligned}$

A22 $\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 - 3x_2 + 2x_3 &= 6 \end{aligned}$

A23 $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 8 \\ x_1 + 3x_2 - 5x_3 &= 11 \\ 2x_1 + 5x_2 - 8x_3 &= 19 \end{aligned}$

A24 $\begin{aligned} -3x_1 + 6x_2 + 16x_3 &= 36 \\ x_1 - 2x_2 - 5x_3 &= -11 \\ 2x_1 - 3x_2 - 8x_3 &= -17 \end{aligned}$

A25 $\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + 5x_2 + x_3 &= 10 \\ 4x_1 + 9x_2 - x_3 &= 19 \end{aligned}$

A26 $\begin{aligned} x_1 + 2x_2 - 3x_3 \qquad &= -5 \\ 2x_1 + 4x_2 - 6x_3 + x_4 &= -8 \\ 6x_1 + 13x_2 - 17x_3 + 4x_4 &= -21 \end{aligned}$

A27 $\begin{aligned} 2x_2 - 2x_3 \qquad + x_5 &= 2 \\ x_1 + 2x_2 - 3x_3 + x_4 + 4x_5 &= 1 \\ 2x_1 + 4x_2 - 5x_3 + 3x_4 + 8x_5 &= 3 \\ 2x_1 + 5x_2 - 7x_3 + 3x_4 + 10x_5 &= 5 \end{aligned}$

For Problems A28–A31, find the parabola $y = a + bx + cx^2$ that passes through the given three points.

A28 $(1, 3), (2, 5), (4, 15)$

A29 $(0, 2), (1, -1), (2, -10)$

A30 $(-2, 9), (-1, 2), (2, 17)$

A31 $(2, 7), (-1, 1), (0, -3)$

For Problems A32–A33, given that the matrix is an augmented matrix of a system of linear equations, determine the values of a, b, c, d for which the system is consistent. If it is consistent, determine whether it has a unique solution.

A32 $\left[\begin{array}{ccc|c} 2 & 4 & -3 & 6 \\ 0 & b & 7 & 2 \\ 0 & 0 & a & a \end{array}\right]$

A33 $\left[\begin{array}{cccc|c} 1 & -1 & 4 & -2 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & d & 5 & 7 \\ 0 & 0 & 0 & cd & c \end{array}\right]$

A34 A fruit seller has apples, bananas, and oranges. Altogether he has 1500 pieces of fruit. On average, each apple weighs 120 grams, each banana weighs 140 grams, and each orange weighs 160 grams. He can sell apples for 25 cents each, bananas for 20 cents each, and oranges for 30 cents each. If the fruit weighs 208 kilograms, and the total selling price is \$380, how many of each kind of fruit does the fruit seller have?

A35 A student is taking courses in algebra, calculus, and physics at a university where grades are given in percentages. To determine her standing for a physics prize, a weighted average is calculated based on 50% of the student's physics grades, 30% of her calculus grade, and 20% of her algebra grade; the weighted average is 84. For an applied mathematics prize, a weighted average based on one-third of each of the three grades is calculated to be 83. For a pure mathematics prize, her average based on 50% of her calculus grade and 50% of her algebra grade is 82.5. What are her grades in the individual courses?

Homework Problems

For Problems B1–B8, solve the system using back-substitution. Write the general solution in standard form.

B1 $\begin{aligned} 2x_1 + 5x_2 &= 6 \\ x_2 &= 2 \end{aligned}$

B2 $\begin{aligned} x_1 - 2x_2 &= -6 \\ x_2 &= 7 \end{aligned}$

B3 $\begin{aligned} x_1 - 4x_2 + x_3 &= 1 \\ x_3 &= -2 \end{aligned}$

B4 $\begin{aligned} 3x_1 - 3x_2 + 2x_3 &= -6 \\ x_2 &= 1 \end{aligned}$

B5 $\begin{aligned} x_1 - 3x_2 + 3x_3 &= 1 \\ 2x_2 + x_3 &= 0 \\ x_3 &= 8 \end{aligned}$

B6 $\begin{aligned} 6x_1 + x_2 - x_3 &= 12 \\ 3x_2 - 6x_3 &= 9 \\ x_3 &= 2 \end{aligned}$

B7 $\begin{aligned} x_1 + \quad +7x_3 &= 4 \\ 4x_2 - 2x_3 &= 2 \\ x_3 &= -3 \end{aligned}$

B8 $\begin{aligned} x_1 + 5x_2 - 2x_3 + x_4 &= 3 \\ x_3 - 2x_4 &= 2 \end{aligned}$

For Problems B9–B12, determine whether the matrix is in REF. If not, explain why it is not.

B9 $A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

B10 $B = \begin{bmatrix} 1 & -4 & 12 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

B11 $C = \begin{bmatrix} 3 & 3 & 1/4 & 0 \\ 0 & 0 & 4 & 2 \\ 5 & -2 & 3 & 4 \end{bmatrix}$

B12 $D = \begin{bmatrix} 0 & 8 & 1 & -3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

For Problems B13–B24, row reduce the matrix to REF. Show your steps.

B13 $\begin{bmatrix} 3 & 6 \\ -2 & 1 \end{bmatrix}$

B14 $\begin{bmatrix} 8 & -6 \\ -12 & 9 \end{bmatrix}$

B15 $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 4 \\ -1 & 0 & 4 \end{bmatrix}$

B16 $\begin{bmatrix} 0 & 2 & 3 \\ 5 & 6 & 9 \\ 1 & 1 & 2 \end{bmatrix}$

B17 $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -3 \\ 5 & 6 & 7 \end{bmatrix}$

B18 $\begin{bmatrix} 2 & 0 & 4 \\ 2 & 8 & -2 \\ -6 & -9 & -6 \end{bmatrix}$

B19 $\begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 3 & -2 & 5 \\ 2 & 4 & 8 & 5 \end{bmatrix}$

B20 $\begin{bmatrix} 1 & 3 & -2 & 1 \\ -3 & 1 & 3 & 1 \\ -3 & 11 & 0 & 5 \end{bmatrix}$

B21 $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & -6 \\ 1 & 2 & -8 \\ 5 & 2 & 0 \end{bmatrix}$

B22 $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 1 \\ -3 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

B23 $\begin{bmatrix} 3 & 2 & 1 & 3 \\ -9 & -4 & -1 & -9 \\ 12 & 0 & 1 & 7 \\ 6 & 2 & 5 & 3 \end{bmatrix}$

B24 $\begin{bmatrix} 0 & 1 & 3 & -5 \\ 3 & 2 & 7 & -6 \\ 1 & 1 & -2 & 7 \\ 0 & -5 & -5 & 5 \end{bmatrix}$

For Problems B25–B30, the given matrix is an augmented matrix of a system of linear equations. Either show the system is inconsistent or write a vector equation for the solution set.

B25 $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ **B26** $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 3 & 6 \end{array}\right]$

B27 $\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ **B28** $\left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & -3 \end{array}\right]$

B29 $\left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ **B30** $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 4 & 2 \end{array}\right]$

For Problems B31–B37:

(a) Write the augmented matrix.
(b) Row reduce the augmented matrix to REF.
(c) Determine whether the system is consistent or inconsistent. If it is consistent, determine the number of parameters in the general solution.
(d) If the system is consistent, write its general solution in standard form.

B31
$$\begin{aligned} 6x_1 + 3x_2 &= 9 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

B32
$$\begin{aligned} 2x_2 + 4x_3 &= 4 \\ x_1 + 5x_2 + 4x_3 &= 8 \end{aligned}$$

B33
$$\begin{aligned} 5x_1 - 2x_2 - x_3 &= 0 \\ -4x_1 + x_2 - x_3 &= 7 \\ x_1 + x_2 + 4x_3 &= 9 \end{aligned}$$

B34
$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 2 \\ 4x_1 + 5x_2 + 12x_3 &= 1 \\ -2x_1 + 7x_2 + 7x_3 &= -4 \end{aligned}$$

B35
$$\begin{aligned} -x_1 - 2x_2 + x_3 &= 17 \\ x_1 + 2x_2 + 5x_3 &= 1 \\ x_1 + 2x_2 + 9x_3 &= 13 \end{aligned}$$

B36
$$\begin{aligned} x_1 + 4x_2 + 6x_3 + 9x_4 &= 1 \\ 2x_1 + 3x_2 + 7x_3 + 3x_4 &= 2 \\ -2x_1 + x_2 - 3x_3 + 9x_4 &= 1 \end{aligned}$$

B37
$$\begin{aligned} x_1 + 4x_2 + 6x_3 + 9x_4 &= 0 \\ 2x_1 + 3x_2 + 7x_3 + 3x_4 &= 5 \\ -2x_1 + x_2 - 3x_3 + 9x_4 &= -9 \end{aligned}$$

For Problems B38–B41, find the parabola $y = a + bx + cx^2$ that passes through the given three points.

B38 $(-1, 8), (1, -2), (2, 14)$ **B39** $(-1, 9), (0, 1), (1, -3)$

B40 $(-2, -5), (1, 10), (2, 17)$ **B41** $(1, -4), (2, 3), (3, 16)$

For Problems B42 and B43, given that the matrix is an augmented matrix of a system of linear equations, determine the values of a, b, c, d for which the system is consistent. If it is consistent, determine whether it has a unique solution.

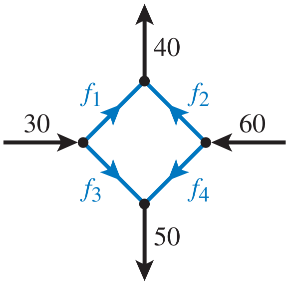
B42 $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & a & 1 & b \\ 0 & 0 & b & 1 \end{array}\right]$ **B43** $\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & c & 0 & d \\ 0 & d & 1 & c \end{array}\right]$

B44 A bookkeeper is trying to determine the prices that a manufacturer was charging. He examines old sales slips that show the number of various items shipped and the total price. He finds that 20 armchairs, 10 sofa beds, and 8 double beds cost \$15200; 15 armchairs, 12 sofa beds, and 10 double beds cost \$15700; and 12 armchairs, 20 sofa beds, and 10 double beds cost \$19600. Determine the cost for each item or explain why the sales slips must be in error.

B45 Steady flow through a network can be described by a system of linear equations. Such networks are used to model, for example, traffic along roads, water through pipes, electricity through a circuit, blood through arteries, or fluxes in a metabolic network. We assume the network in the diagram is in equilibrium; the flow into each node equals the flow out. For instance, at the top node, we get

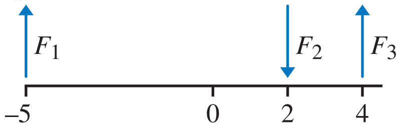
$$\begin{aligned} \text{flow in} &= \text{flow out} \\ f_1 + f_2 &= 40 \end{aligned}$$

(a) Finish creating the corresponding system of linear equations by writing the equation for the other three nodes.
(b) Solve the system to determine how the flows are f_1, f_2, f_3, f_4 when in equilibrium.
(c) What is the physical interpretation of the negative flow?



B46 Students at Linear University write a linear algebra examination. An average mark is computed for 100 students in business, an average is computed for 300 students in liberal arts, and an average is computed for 200 students in science. The average of these three averages is 85%. However, the overall average for the 600 students is 86%. Also, the average for the 300 students in business and science is 4 marks higher than the average for the students in liberal arts. Determine the average for each group of students by solving a system of linear equations.

B47 (Requires knowledge of forces and moments.) A rod 10 m long is pivoted at its centre; it swings in the horizontal plane. Forces of magnitude F_1, F_2, F_3 are applied perpendicular to the rod in the directions indicated by the arrows in the diagram below; F_1 is applied to the left end of the rod, F_2 is applied at a point 2 m to the right of centre, and F_3 at a point 4 m to the right of centre. The total force on the pivot is zero, the moment about the centre is zero, and the sum of the magnitudes of forces is 80 newtons. Write a system of three equations for F_1, F_2, and F_3; write the corresponding augmented matrix; and use the standard procedure to find F_1, F_2, and F_3.



Conceptual Problems

C1 Consider the linear system in x, y, z, and w:

$$\begin{aligned} x + y + w &= b \\ 2x + 3y + z + 5w &= 6 \\ z + w &= 4 \\ 2y + 2z + aw &= 1 \end{aligned}$$

For what values of a and b is the system

(a) Inconsistent?

(b) Consistent with a unique solution?

(c) Consistent with infinitely many solutions?

C2 Recall that two planes $\vec{n} \cdot \vec{x} = c$ and $\vec{m} \cdot \vec{x} = d$ in $\mathbb{R}^3$ are parallel if and only if the normal vector $\vec{m}$ is a non-zero multiple of the normal vector $\vec{n}$. Row reduce a suitable augmented matrix to explain why two parallel planes must either coincide or else have no points in common.

C3 Consider the system

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0 \end{aligned}$$

(a) Explain why the system is consistent.

(b) Prove that if $ad - bc \neq 0$, then the system has a unique solution.

(c) Prove that if the system has a unique solution, then $ad - bc \neq 0$.

C4 Prove Theorem 2.1.1 by

(a) first proving that $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a solution for each $c \in \mathbb{R}$,

(b) and then proving that if $c_1 \neq c_2$, then

$$\vec{s} + c_1(\vec{s} - \vec{t}) \neq \vec{s} + c_2(\vec{s} - \vec{t})$$

2.2 Reduced Row Echelon Form, Rank, and Homogeneous Systems

The standard basic procedure for determining the solution of a system of linear equations is elimination with back-substitution, as described in Section 2.1. In some situations and applications, however, it is advantageous to carry the elimination steps (elementary row operations) as far as possible to avoid the need for back-substitution.

To see what further elementary row operations might be worthwhile, recall that the Gaussian elimination procedure proceeds by selecting a pivot and using elementary row operations to create zeros beneath the pivot. The only further elimination steps that simplify the system are steps that create zeros above the pivot.

EXAMPLE 2.2.1

In Example 2.1.10 on page 90 we row reduced the augmented matrix for the original system to a row equivalent matrix in row echelon form. That is, we found that

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 1 & 3 & -1 & 7 \\ 2 & 1 & -5 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

Instead of using back-substitution to solve the system as we did in Example 2.1.10, we instead perform the following elementary row operations:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} \\ \\ (-1)R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} \\ R_2 - R_3 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 + 2R_3 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This is the augmented matrix for the system $x_1 = 0$, $x_2 = 2$, and $x_3 = -1$, which gives us the solution we found in Example 2.1.10.

This system has been solved by **complete elimination**. The leading variable in the j-th equation has been eliminated from every other equation. This procedure is often called **Gauss-Jordan elimination** to distinguish it from Gaussian elimination with back-substitution. Observe that the elementary row operations used in Example 2.2.1 are exactly the same as the operations performed in the back-substitution in Example 2.1.10.

A matrix corresponding to a system on which Gauss-Jordan elimination has been carried out is in a special kind of row echelon form.

Definition
Reduced Row Echelon Form (RREF)

A matrix R is said to be in **reduced row echelon form (RREF)** if

(1) It is in row echelon form.
(2) All leading entries are 1, called a **leading one**.
(3) In a column with a leading one, all the other entries are zeros.

If A is row equivalent to a matrix R in RREF, then we say that R is the **reduced row echelon form** of A.

EXAMPLE 2.2.2

Determine which of the following matrices are in RREF.

(a) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution: The matrices in (a) and (b) are in RREF. The matrix in (c) is not in RREF as there is a non-zero entry above the leading one in the third column. The matrix (d) is not in RREF since the leading one in the second row is to the left of the leading one in the row above it.

EXERCISE 2.2.1

Determine which of the following matrices are in RREF.

(a) $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

As in the case of row echelon form, it is easy to see that every matrix is row equivalent to a matrix in reduced row echelon form via Gauss-Jordan elimination. However, in this case we get a stronger result.

Theorem 2.2.1 For any given matrix A there is a unique matrix in reduced row echelon form that is row equivalent to A.

You are asked to prove that there is only one matrix in reduced row echelon form that is row equivalent to A in Problem F2 in the Chapter 4 Further Problems.

EXAMPLE 2.2.3

Obtain the matrix in reduced row echelon form that is row equivalent to the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & -2 & 2 \\ 3 & 3 & 5 & 0 & 2 \end{bmatrix}$$

Solution: We begin by using Gaussian elimination to put the matrix into REF

$$\begin{bmatrix} 1 & 1 & 2 & -2 & 2 \\ 3 & 3 & 5 & 0 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & -2 & 2 \\ 0 & 0 & -1 & 6 & -4 \end{bmatrix}$$

We also need to put zeros above pivots and to ensure that each pivot is a 1.

$$\begin{bmatrix} 1 & 1 & 2 & -2 & 2 \\ 0 & 0 & -1 & 6 & -4 \end{bmatrix} \begin{matrix} R_1 + 2R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & 10 & -6 \\ 0 & 0 & -1 & 6 & -4 \end{bmatrix} \begin{matrix} \\ (-1)R_2 \end{matrix} \sim$$
$$\begin{bmatrix} 1 & 1 & 0 & 10 & -6 \\ 0 & 0 & 1 & -6 & 4 \end{bmatrix}$$

This final matrix is in reduced row echelon form.

When row reducing to reduced row echelon form by hand, it seems more natural not to obtain a row echelon form first. Instead, you might first turn any pivot into a leading one and then obtain zeros below and above it moving on to the next leading one. However, for programming a computer to row reduce a matrix, this is a poor strategy because it requires more multiplications and additions than the previous strategy. See Problem F2 at the end of the chapter.

EXAMPLE 2.2.4

Solve the following system of equations by row reducing the augmented matrix to RREF.

$$\begin{aligned} x_1 + x_2 &= -7 \\ 2x_1 + 4x_2 + x_3 &= -16 \\ x_1 + 2x_2 + x_3 &= 9 \end{aligned}$$

Solution: Our first pivot is already a leading one, so we place zeros beneath it.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 2 & 4 & 1 & -16 \\ 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 1 & 16 \end{array}\right]$$

To make our next pivot a leading one, rather than introducing fractions, we use $R_2 - R_3$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 1 & 16 \end{array}\right] R_2 - R_3 \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 0 & -18 \\ 0 & 1 & 1 & 16 \end{array}\right]$$

We now need to get zeros above and below this new leading one.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 0 & -18 \\ 0 & 1 & 1 & 16 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 34 \end{array}\right]$$

The matrix is now in reduced row echelon form. The reduced row echelon form corresponds to the system $x_1 = 11$, $x_2 = -18$, $x_3 = 34$. Hence, the solution is $\vec{x} = \begin{bmatrix} 11 \\ -18 \\ 34 \end{bmatrix}$.

EXERCISE 2.2.2

Row reduce $A = \begin{bmatrix} 0 & -2 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ into reduced row echelon form.

Remark

In general, reducing an augmented matrix to reduced row echelon form to solve a system is not more efficient than the method used in Section 2.1. As previously mentioned, both methods are essentially equivalent for solving small systems by hand.

Rank of a Matrix

Theorem 2.1.3 shows that the number of pivots in a row echelon form of the coefficient matrix of a system of linear equations determines whether the system is consistent or inconsistent. It also determines how many solutions (one or infinitely many) the system has if it is consistent. Thus, we make the following definition.

Definition
Rank

The **rank** of a matrix A is the number of leading ones in its reduced row echelon form and is denoted by $\text{rank}(A)$.

The rank of A is also equal to the number of leading entries in any row echelon form of A. However, since the row echelon form is not unique, it is more tiresome to give clear arguments in terms of row echelon form. In Section 3.4 we shall see a more conceptual way of describing rank.

EXAMPLE 2.2.5

In Example 2.2.3 we saw that the RREF of $A = \begin{bmatrix} 1 & 1 & 2 & -2 & 2 \\ 3 & 3 & 5 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 0 & 10 & -6 \\ 0 & 0 & 1 & -6 & 4 \end{bmatrix}$. Thus, $\text{rank}(A) = 2$.

EXAMPLE 2.2.6

In Example 2.2.4 we saw that the RREF of $B = \begin{bmatrix} 1 & 1 & 0 & -7 \\ 2 & 4 & 1 & -16 \\ 1 & 2 & 1 & 9 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 34 \end{bmatrix}$. Thus, $\text{rank}(B) = 3$.

EXAMPLE 2.2.7

The RREF of $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, $\text{rank}(C) = 1$.

EXERCISE 2.2.3

Determine the rank of each of the following matrices:

(a) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Theorem 2.2.2 System-Rank Theorem

Let $\left[A \mid \vec{b}\right]$ be a system of m linear equations in n variables.

(1) The system is consistent if and only if the rank of the coefficient matrix A is equal to the rank of the augmented matrix $\left[A \mid \vec{b}\right]$.

(2) If the system $\left[A \mid \vec{b}\right]$ is consistent, then the number of parameters in the general solution is the number of variables minus the rank of the matrix A:

$$\#\text{ of parameters} = n - \operatorname{rank}(A)$$

(3) $\operatorname{rank}(A) = m$ if and only if the system $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

Proof: (1): The rank of A is less than the rank of the augmented matrix if and only if there is a row in the RREF of the augmented matrix of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$. This is true if and only if the system is inconsistent.

(2): If the system is consistent, then the free variables are the variables that are not leading variables of any equation in a row echelon form of the matrix. Thus, by definition, there are $n - \operatorname{rank}(A)$ free variables and hence $n - \operatorname{rank}(A)$ parameters in the general solution.

The proof of (3) is left as Problem A40. ■

For property (1) of the System-Rank Theorem, it is important to realize that we consider the rank of the entire matrix, even the augmented part.

EXAMPLE 2.2.8 Consider the system of linear equations

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= -2 \end{aligned}$$

from Example 2.1.15. If we row reduce the augmented matrix to RREF, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, the rank of the augmented matrix is 3. If we just row reduce the coefficient matrix to RREF, then we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank of the coefficient matrix is 2. So, by the System-Rank Theorem (1), the system in inconsistent.

Homogeneous Linear Equations

Frequently, systems of linear equations appear where all of the terms on the right-hand side are zero.

Definition
Homogeneous

A linear equation is **homogeneous** if the right-hand side is zero. A system of linear equations is **homogeneous** if all of the equations of the system are homogeneous.

Since a homogeneous system is a special case of the systems already discussed, no new tools or techniques are needed to solve them. However, we normally work only with the coefficient matrix of a homogeneous system since the last column of the augmented matrix consists entirely of zeros.

EXAMPLE 2.2.9

Find the general solution of the homogeneous system

$$\begin{aligned} 2x_1 + x_2 \phantom{{}- x_3} &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$

Solution: We row reduce the coefficient matrix of the system to RREF:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_2 - R_1 \sim$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{aligned} x_1 \phantom{{}+x_2} + x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

Hence, x_3 is a free variable, so we let $x_3 = t \in \mathbb{R}$. Then $x_1 = -x_3 = -t$, $x_2 = 2x_3 = 2t$, and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Observe that every homogeneous system is consistent as the zero vector $\vec{0}$ will certainly be a solution. We call $\vec{0}$ the **trivial solution**. Thus, as we will see frequently throughout the text, when dealing with homogeneous systems, we are often mostly interested in how many parameters are in the general solution. Of course, for this we can apply the System-Rank Theorem.

EXAMPLE 2.2.10

Determine the number of parameters in the general solution of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 \quad + x_4 &= 0 \\ 3x_1 + 7x_2 + 7x_3 + 3x_4 &= 0 \\ 2x_1 + 5x_2 + 5x_3 + 2x_4 &= 0 \end{aligned}$$

Solution: We row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 7 & 7 & 3 \\ 2 & 5 & 5 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the coefficient matrix is 2 and the number of variables is 4. Thus, by the System-Rank Theorem (2), there are $4 - 2 = 2$ parameters in the general solution.

EXERCISE 2.2.4

Write the general solution of the system in Example 2.2.10.

Example 2.2.9 can be interpreted geometrically. It shows that the intersection of the three planes is the line that passes through the origin with vector equation

$$\vec{x} = t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Thus, from our work in Chapter 1, we know that the solution set is a subspace of $\mathbb{R}^3$. Similarly, Example 2.2.10 shows that the three hyperplanes in $\mathbb{R}^5$ intersect in a plane that passes through the origin in $\mathbb{R}^5$. Hence, the solution set of that system is a subspace of $\mathbb{R}^5$. Of course, we can prove this idea in general.

Theorem 2.2.3

The solution set of a homogeneous system of m linear equations in n variables is a subspace of $\mathbb{R}^n$.

You are asked to prove Theorem 2.2.3 in Problem C2.

This allows us to make the following definition.

Definition
Solution Space

The solution set of a homogeneous system is called the **solution space** of the system.

PROBLEMS 2.2

Practice Problems

For Problems A1–A7, determine whether the matrix is in RREF.

A1 $\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & 1 \end{bmatrix}$ **A2** $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}$ **A3** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A4 $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **A5** $\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A6 $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & -2 & 1 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ **A7** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

For Problems A8–A16, determine the RREF and the rank of the matrix.

A8 $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$ **A9** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ **A10** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$

A11 $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ **A12** $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

A13 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ **A14** $\begin{bmatrix} 2 & -1 & 2 & 8 \\ 1 & -1 & 0 & 2 \\ 3 & -2 & 3 & 13 \end{bmatrix}$

A15 $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ **A16** $\begin{bmatrix} 0 & 1 & 0 & 2 & 5 \\ 3 & 1 & 8 & 5 & 3 \\ 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 6 & 7 & 1 \end{bmatrix}$

For Problems A17–A22, the given matrix is the coefficient matrix of a homogeneous system already in RREF. Determine the number of parameters in the general solution and write out the general solution in standard form.

A17 $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **A18** $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A19 $\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **A20** $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

A21 $\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **A22** $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

For Problems A23–A29, solve the system of linear equations by row reducing the corresponding augmented matrix to RREF. Compare your steps with your solutions from Section 2.1 Problems A21–A27.

A23 $\begin{aligned} 3x_1 - 5x_2 &= 2 \\ x_1 + 2x_2 &= 4 \end{aligned}$ **A24** $\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 - 3x_2 + 2x_3 &= 6 \end{aligned}$

A25 $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 8 \\ x_1 + 3x_2 - 5x_3 &= 11 \\ 2x_1 + 5x_2 - 8x_3 &= 19 \end{aligned}$ **A26** $\begin{aligned} -3x_1 + 6x_2 + 16x_3 &= 36 \\ x_1 - 2x_2 - 5x_3 &= -11 \\ 2x_1 - 3x_2 - 8x_3 &= -17 \end{aligned}$

A27
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + 5x_2 + x_3 &= 10 \\ 4x_1 + 9x_2 - x_3 &= 19 \end{aligned}$$

A28
$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= -5 \\ 2x_1 + 4x_2 - 6x_3 + x_4 &= -8 \\ 6x_1 + 13x_2 - 17x_3 + 4x_4 &= -21 \end{aligned}$$

A29
$$\begin{aligned} 2x_2 - 2x_3 + x_5 &= 2 \\ x_1 + 2x_2 - 3x_3 + x_4 + 4x_5 &= 1 \\ 2x_1 + 4x_2 - 5x_3 + 3x_4 + 8x_5 &= 3 \\ 2x_1 + 5x_2 - 7x_3 + 3x_4 + 10x_5 &= 5 \end{aligned}$$

For Problems A30–A33, write the coefficient matrix of the system of linear equations. Determine the rank of the coefficient matrix and write out the general solution in standard form.

A30 $\begin{aligned} 2x_2 - 5x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \\ x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$ **A31** $\begin{aligned} 3x_1 + x_2 - 9x_3 &= 0 \\ x_1 + x_2 - 5x_3 &= 0 \\ 2x_1 + x_2 - 7x_3 &= 0 \end{aligned}$

A32
$$\begin{aligned} x_1 - x_2 + 2x_3 - 3x_4 &= 0 \\ 3x_1 - 3x_2 + 8x_3 - 5x_4 &= 0 \\ 2x_1 - 2x_2 + 5x_3 - 4x_4 &= 0 \\ 3x_1 - 3x_2 + 7x_3 - 7x_4 &= 0 \end{aligned}$$

A33
$$\begin{aligned} x_2 + 2x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + 5x_3 + 3x_4 - x_5 &= 0 \\ 2x_1 + x_2 + 5x_3 + x_4 - 3x_5 &= 0 \\ x_1 + x_2 + 4x_3 + 2x_4 - 2x_5 &= 0 \end{aligned}$$

For Problems A34–A39, solve the system $\left[A \mid \vec{b}\right]$ by row reducing the augmented matrix to RREF. Then, without any further operations, find the general solution to the homogeneous $\left[A \mid \vec{0}\right]$.

A34 $A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

A35 $A = \begin{bmatrix} 1 & 7 & 5 \\ 1 & 0 & 5 \\ -1 & 2 & -5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$

A36 $A = \begin{bmatrix} 0 & -1 & 5 & -2 \\ -1 & -1 & -4 & -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

A37 $A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 4 & 3 & 2 & -4 \\ -1 & -4 & -3 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$

A38 $A = \begin{bmatrix} 1 & -1 & 4 & -1 \\ -1 & -2 & 5 & -2 \\ -4 & -1 & 2 & 2 \\ 5 & 4 & 1 & 8 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 5 \\ -4 \\ 5 \end{bmatrix}$

A39 $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 4 \\ 4 & 4 & 6 & -8 & 4 \\ 1 & 1 & 4 & -2 & 1 \\ 3 & 3 & 2 & -4 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ -4 \\ -6 \\ 6 \end{bmatrix}$

A40 In this problem, we will look at how to prove System-Rank Theorem (3). For each direction of the 'if and only if', we first look at an example to help us figure out how to do the general proof.

(a) Row reduce the coefficient matrix of the system

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= b_1 \\ x_1 + 2x_2 + x_3 - 2x_4 &= b_2 \\ x_1 + 4x_2 + 2x_3 - 7x_4 &= b_3 \end{aligned}$$

and explain how this proves that the system is consistent for all $\vec{b} \in \mathbb{R}^3$.

(b) Prove if $\operatorname{rank}(A) = m$, then $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

(c) Find a vector $\vec{b} \in \mathbb{R}^3$ such that the following system is inconsistent. To find such a vector $\vec{b}$, think about how to work backwards from the RREF of the coefficient matrix A.

$$\begin{aligned} x_1 + x_2 &= b_1 \\ 2x_1 + 2x_2 &= b_2 \\ 2x_1 + 3x_2 &= b_3 \end{aligned}$$

(d) Prove if $\operatorname{rank}(A) < m$, then there exists $\vec{b} \in \mathbb{R}^m$ such that $\left[A \mid \vec{b}\right]$ is inconsistent.

Homework Problems

For Problems B1–B9, determine whether the matrix is in RREF.

B1 $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **B2** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **B3** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

B4 $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ **B5** $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

B6 $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **B7** $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

B8 $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **B9** $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

For Problems B10–B18, determine the RREF and the rank of matrix.

B10 $\begin{bmatrix} 1 & -2 \\ -3 & 6 \\ 4 & -8 \end{bmatrix}$ **B11** $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 0 & -2 & 8 \end{bmatrix}$ **B12** $\begin{bmatrix} -1 & 1 & 2 \\ 4 & -1 & -3 \\ -3 & -3 & 1 \end{bmatrix}$

B13 $\begin{bmatrix} 2 & 1 & 3 & -1 \\ -4 & 3 & -11 & 3 \\ 6 & 8 & 4 & -3 \end{bmatrix}$ **B14** $\begin{bmatrix} 6 & 9 & 1 & -1 \\ 2 & 3 & -1 & 1 \\ 4 & 6 & 2 & -2 \end{bmatrix}$

B15 $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ **B16** $\begin{bmatrix} 0 & 1 & -2 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 1 & 6 & 6 \end{bmatrix}$

B17 $\begin{bmatrix} 1 & -2 & 1 & 1 \\ -2 & 5 & 0 & -5 \\ 6 & -7 & 6 & -9 \\ 3 & -4 & 2 & 2 \end{bmatrix}$ **B18** $\begin{bmatrix} 1 & -2 & 1 & 5 & 2 \\ 0 & 0 & 3 & 6 & 1 \\ -2 & 4 & 5 & 4 & 5 \\ -2 & 4 & 7 & 8 & 1 \end{bmatrix}$

For Problems B19–B24, the given matrix is the coefficient matrix of a homogeneous system already in RREF. Determine the number of parameters in the general solution and write out the general solution in standard form.

B19 $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

B20 $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B21 $\begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B22 $\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B23 $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

B24 $\begin{bmatrix} 1 & 0 & 3 & 0 & 8 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

For Problems B25–B31, solve the system of linear equations by row reducing the corresponding augmented matrix to RREF.

B25
$$\begin{aligned} 3x_1 + 5x_2 &= 4 \\ 2x_1 + 5x_2 &= -4 \end{aligned}$$

B26
$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + 6x_3 &= 2 \end{aligned}$$

B27
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 2 \\ x_1 + 3x_2 + 5x_3 &= 1 \\ x_1 + x_2 + 4x_3 &= 6 \end{aligned}$$

B28
$$\begin{aligned} 2x_1 + 5x_2 + 5x_3 &= 0 \\ 4x_1 + 7x_2 + x_3 &= 3 \\ -4x_1 - 6x_2 + 2x_3 &= 5 \end{aligned}$$

B29
$$\begin{aligned} x_2 + 5x_3 - 4x_4 &= -2 \\ x_1 + 2x_2 + 7x_3 - 3x_4 &= -1 \\ 5x_1 + 4x_2 + 5x_3 + 9x_4 &= 7 \end{aligned}$$

B30
$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 + 4x_4 &= 1 \\ 6x_1 + 8x_2 + 3x_3 + 3x_4 &= 1 \\ -6x_1 - 8x_2 - x_3 + 7x_4 &= 1 \end{aligned}$$

B31
$$\begin{aligned} 3x_1 + 4x_2 - 5x_3 + 8x_4 + x_5 &= 7 \\ 5x_1 - 2x_2 + 9x_3 + 6x_4 + x_5 &= 5 \\ 2x_1 + 4x_2 - 6x_3 + 7x_4 + x_5 &= 6 \\ 5x_1 + x_2 + 3x_3 + 5x_4 - x_5 &= 1 \end{aligned}$$

For Problems B32–B35, write the coefficient matrix of the system of linear equations. Determine the rank of the coefficient matrix and write out the general solution in standard form.

B32
$$\begin{aligned} 2x_2 + x_3 &= 0 \\ 5x_1 + 6x_2 + x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \end{aligned}$$

B33
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ x_1 + 3x_2 + 3x_3 &= 0 \end{aligned}$$

B34
$$\begin{aligned} 3x_1 + 4x_2 - 2x_3 + 7x_4 &= 0 \\ 9x_1 + 5x_2 + x_3 \phantom{{}+7x_4} &= 0 \\ -3x_1 + x_2 - 3x_3 + 8x_4 &= 0 \\ 3x_1 + x_2 + x_3 - 2x_4 &= 0 \end{aligned}$$

B35
$$\begin{aligned} x_1 - 3x_2 + x_3 - 3x_4 - x_5 &= 0 \\ -x_1 + 4x_2 \phantom{{}+x_3} + x_4 + x_5 &= 0 \\ 2x_1 - 13x_2 - 5x_3 + 8x_4 - x_5 &= 0 \\ 3x_1 - 12x_2 + x_3 - 7x_4 - 3x_5 &= 0 \end{aligned}$$

For Problems B36–B41, solve the system $\left[A \mid \vec{b}\right]$ by row reducing the augmented matrix to RREF. Then, without any further operations, find the general solution to the homogeneous $\left[A \mid \vec{0}\right]$.

B36 $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 4 & 2 \\ 2 & 5 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -8 \\ 6 \\ 11 \end{bmatrix}$

B37 $A = \begin{bmatrix} 4 & 6 & 3 \\ 6 & 9 & 5 \\ -4 & -6 & 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 3 \end{bmatrix}$

B38 $A = \begin{bmatrix} 5 & 3 & 12 & 13 \\ 2 & 1 & 5 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 14 \\ 5 \end{bmatrix}$

B39 $A = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 4 & 3 & 6 & -4 \\ -1 & -4 & 5 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

B40 $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 10 & 6 & -4 \\ 1 & 6 & 4 & -3 \\ 1 & 8 & 3 & -8 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 4 \\ 4 \\ -7 \end{bmatrix}$

B41 $A = \begin{bmatrix} 1 & 2 & 5 & 1 & -3 \\ 0 & 2 & 2 & 2 & -4 \\ -1 & -4 & -7 & -1 & 9 \\ 0 & 2 & 2 & 3 & -3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 7 \\ 4 \\ -9 \\ 5 \end{bmatrix}$

Conceptual Problems

C1 We want to find a vector $\vec{x} \neq \vec{0}$ in $\mathbb{R}^3$ that is simultaneously orthogonal to given vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.

(a) Write equations that must be satisfied by $\vec{x}$.

(b) What condition must be satisfied by the rank of the matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ if there are to be non-trivial solutions? Explain.

C2 Prove the solution set of a homogeneous system of m linear equations in n variables is a subspace of $\mathbb{R}^n$.

C3 (a) Suppose that $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ is the coefficient matrix of a homogeneous system $\left[A \mid \vec{0}\right]$. Find the general solution of the system and indicate why it describes a line through the origin.

(b) Suppose that a matrix A with two rows and three columns is the coefficient matrix of a homogeneous system. If A has rank 2, then explain why the solution set of the homogeneous system is a line through the origin. What could you say if $\text{rank}(A) = 1$?

(c) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three vectors in $\mathbb{R}^4$. Write conditions on a vector $\vec{x} \in \mathbb{R}^4$ such that $\vec{x}$ is orthogonal to $\vec{u}$, $\vec{v}$, and $\vec{w}$. (This should lead to a homogeneous system with coefficient matrix C, whose rows are $\vec{u}$, $\vec{v}$, and $\vec{w}$.) What does the rank of C tell us about the set of vectors $\vec{x}$ that are orthogonal to $\vec{u}$, $\vec{v}$, and $\vec{w}$?

For Problems C4–C6, what can you say about the consistency of the system of m linear equations in n variables and the number of parameters in the general solution?

C4 $m = 5, n = 7$, the rank of the coefficient matrix is 4.

C5 $m = 3, n = 6$, the rank of the coefficient matrix is 3.

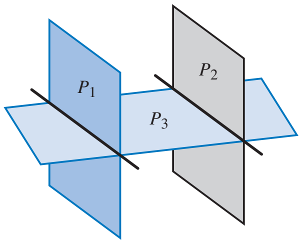
C6 $m = 5, n = 4$, the rank of the augmented matrix is 4.

C7 A system of linear equations has augmented matrix $\left[\begin{array}{ccc|c} 1 & a & b & 1 \\ 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \end{array}\right]$. For which values of a and b is the system consistent? Are there values for which there is a unique solution? Determine the general solution.

C8 Consider three planes P_1, P_2, and P_3, with respective equations

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$,
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$, and
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

The intersections of these planes is illustrated below. Assume that P_1 and P_2 are parallel.



What is the rank of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$?

C9 Prove that if two planes in $\mathbb{R}^3$ intersect, then they intersect either in a line or a plane.

For Problems C10–C12, determine whether the statement is true or false. Justify your answer.

C10 The solution set of a system of linear equations is a subspace of $\mathbb{R}^n$.

C11 If A is the coefficient matrix of a system of m linear equations in n variables where $m < n$, then rank $A = m$.

C12 If a system of m linear equations in n variables $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^m$ with a unique solution, then $m = n$.

2.3 Application to Spanning and Linear Independence

As discussed at the beginning of this chapter, solving systems of linear equations will play an important role in much of what we do in the rest of the text. Here we will show how to use the methods described in this chapter to solve some of the problems we encountered in Chapter 1.

Spanning Problems

Recall that a vector $\vec{v} \in \mathbb{R}^n$ is in the span of a set $\{\vec{v}_1, \dots, \vec{v}_k\}$ of vectors in $\mathbb{R}^n$ if and only if there exists scalars $t_1, \dots, t_k \in \mathbb{R}$ such that

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{v}$$

This vector equation actually represents n equations (one for each component of the vectors) in the k unknowns $t_1, \dots, t_k$. Thus, it is easy to establish whether a vector is in the span of a set; we just need to determine whether the corresponding system of linear equations is consistent or not.

EXAMPLE 2.3.1

Determine whether the vector $\vec{v} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$.

Solution: Consider the vector equation

$$t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Simplifying the left-hand side using vector operations, we get

$$\begin{bmatrix} t_1 + t_2 + 2t_3 \\ t_1 - t_2 + t_3 \\ t_1 + 5t_2 + 4t_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Comparing corresponding entries gives the system of linear equations

$$\begin{aligned} t_1 + t_2 + 2t_3 &= -2 \\ t_1 - t_2 + t_3 &= -3 \\ t_1 + 5t_2 + 4t_3 &= 1 \end{aligned}$$

We row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 1 & -1 & 1 & -3 \\ 1 & 5 & 4 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & -2 & -1 & -1 \\ 0 & 4 & 2 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

By Theorem 2.1.3, the system is inconsistent. Hence, $\vec{v}$ is not in the spanned set.

EXAMPLE 2.3.2

Write $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: We need to find scalars $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Simplifying the left-hand side using vector operations, we get

$$\begin{bmatrix} t_1 - 2t_2 + t_3 \\ 2t_1 + t_2 + t_3 \\ t_1 + t_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

This gives the system of linear equations:

$$\begin{aligned} t_1 - 2t_2 + t_3 &= -1 \\ 2t_1 + t_2 + t_3 &= 1 \\ t_1 + t_3 &= -1 \end{aligned}$$

Row reducing the augmented matrix to RREF gives

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The solution is $t_1 = 2$, $t_2 = 0$, and $t_3 = -3$. This tells us that

$$2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

EXERCISE 2.3.1

Determine whether $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix} \right\}$.

EXAMPLE 2.3.3

Consider the subspace of $\mathbb{R}^4$ defined by $S = \text{Span}\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\3\\1\end{bmatrix}, \begin{bmatrix}3\\5\\5\\3\end{bmatrix}\right\}$. Find a homogeneous system of linear equations that defines S.

Solution: A vector $\vec{x} \in \mathbb{R}^4$ is in this set if and only if for some $t_1, t_2, t_3 \in \mathbb{R}$,

$$t_1\begin{bmatrix}1\\2\\1\\1\end{bmatrix} + t_2\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + t_3\begin{bmatrix}3\\5\\5\\3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

Simplifying the left-hand side gives us a system of equations with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1 \\ 2 & 1 & 5 & x_2 \\ 1 & 3 & 5 & x_3 \\ 1 & 1 & 3 & x_4 \end{array}\right]$$

Row reducing this matrix to row echelon form gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1 \\ 2 & 1 & 5 & x_2 \\ 1 & 3 & 5 & x_3 \\ 1 & 1 & 3 & x_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1 \\ 0 & 1 & 1 & 2x_1 - x_2 \\ 0 & 0 & 0 & -5x_1 + 2x_2 + x_3 \\ 0 & 0 & 0 & -x_1 + x_4 \end{array}\right]$$

The system is consistent if and only if $-5x_1 + 2x_2 + x_3 = 0$ and $-x_1 + x_4 = 0$. Thus, this homogeneous system of linear equations defines S.

EXAMPLE 2.3.4

Show that $\text{Span}\left\{\begin{bmatrix}-3\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\} = \mathbb{R}^3$.

Solution: Denote the vectors by $\vec{v}_1, \vec{v}_2, \vec{v}_3$. To show that every vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, we only need to show that the system

$$t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{b}$$

is consistent for all $\vec{b} \in \mathbb{R}^3$. By the System-Rank Theorem (3), we only need to show that the rank of the coefficient matrix equals the number of rows.

Row reducing the coefficient matrix to RREF gives

$$\begin{bmatrix}-3 & 1 & 2\\ 1 & 3 & -1\\ -2 & -3 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Hence, the rank of the matrix is 3, which equals the number of rows, as required.

We generalize the method used in Example 2.3.4 to get the following important results.

Theorem 2.3.1 A set of k vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ spans $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of the system $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{b}$ is n.

Proof: If $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$, then every $\vec{b} \in \mathbb{R}^n$ can be written as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$. That is, the system of linear equations

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{b}$$

has a solution for every $\vec{b} \in \mathbb{R}^n$. By the System-Rank Theorem (3), this means that the rank of the coefficient matrix of the system equals n (the number of equations).
On the other hand, if the rank of the coefficient matrix of the system is n, then by the System-Rank Theorem (3) the system is consistent for all $\vec{b} \in \mathbb{R}^n$. Therefore, every $\vec{b} \in \mathbb{R}^n$ can be written as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$. Consequently, $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$. ■

Theorem 2.3.2 Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$. If $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$, then $k \geq n$.

Proof: By Theorem 2.3.1, if $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$, then the rank of the coefficient matrix is n. If the matrix has n leading ones, then it must have least n columns to contain the leading ones. Hence, the number of columns, k, must be greater than or equal to n. ■

Linear Independence Problems

Recall that a set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is said to be linearly independent if and only if the only solution to the vector equation

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$$

is the solution $t_i = 0$ for $1 \leq i \leq k$. By the System-Rank Theorem (2), this is true if and only if the rank of the coefficient matrix of the corresponding homogeneous system is equal to the number of variables k. In particular, if the corresponding homogeneous system has no parameters, then the trivial solution is the only solution.

EXAMPLE 2.3.5

Determine whether the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \right\}$ is linearly independent in $\mathbb{R}^3$.

Solution: Consider

$$t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

EXAMPLE 2.3.5
(continued)

Simplifying as above, this gives the homogeneous system with coefficient matrix

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & -1 & 1 & -3 \\ 1 & 5 & 4 & 3 \end{bmatrix}$$

Notice that we do not need to row reduce this matrix. By the System-Rank Theorem (2), the number of parameters in the general solution equals the number of variables minus the rank of the matrix. There are 4 variables, but the maximum the rank can be is 3 since there are only 3 rows. Hence, the number of parameters is at least 1, so the system has infinitely many solutions. Therefore, the set is linearly dependent.

EXAMPLE 2.3.6

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent in $\mathbb{R}^3$.

Solution: Consider $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$. As above, we find that the coefficient matrix of the corresponding system is $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Using the same elementary row operations as in Example 2.3.2, we get

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the set is linearly independent since the system has a unique solution.

EXERCISE 2.3.2

Determine whether the set $\left\{ \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ is linearly independent or dependent.

We generalize the method used in Examples 2.3.5 and 2.3.6 to prove some important results.

Theorem 2.3.3 A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly independent if and only if the rank of the coefficient matrix of the homogeneous system $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$ is k.

Proof: If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, then the system of linear equations

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

has a unique solution. Thus, the rank of the coefficient matrix equals the number of unknowns k by the System-Rank Theorem (2).

On the other hand, if the rank of the coefficient matrix equals k, then the homogeneous system has $k - k = 0$ parameters. Therefore, it has the unique solution $t_1 = \cdots = t_k = 0$, and so the set is linearly independent. ■

Theorem 2.3.4 If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $k \leq n$.

Proof: By Theorem 2.3.3, if $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, then the rank of the coefficient matrix is k. Hence, there must be at least k rows in the matrix to contain the leading ones. Therefore, the number of rows n must be greater than or equal to k. ■

Bases and Dimension of Subspaces

Recall from Section 1.4 that we defined a basis $\mathcal{B}$ of a subspace S of $\mathbb{R}^n$ to be a linearly independent set that spans S. Thus, with our previous tools, we can now easily identify a basis for a subspace. In particular, to show that a set $\mathcal{B}$ of vectors in $\mathbb{R}^n$ is a basis for a subspace S, we just need to show that Span $\mathcal{B} = S$ and $\mathcal{B}$ is linearly independent. We demonstrate this with a couple of examples.

EXAMPLE 2.3.7 Prove that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Solution: Consider

$$t_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix} + t_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \vec{v}$$

We find that the coefficient matrix of the corresponding system is

$$\begin{bmatrix} 1 & 5 & -2 \\ 1 & -2 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

By Theorem 2.3.1, $\mathcal{B}$ spans $\mathbb{R}^3$ if and only if the rank of this matrix equals the number of rows. Moreover, by Theorem 2.3.3, $\mathcal{B}$ is linearly independent if and only if the rank of this matrix equals the number of columns. Hence, we just need to show that the rank of this is matrix is 3. Row reducing the matrix to RREF we get

$$\begin{bmatrix} 1 & 5 & -2 \\ 1 & -2 & 3 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the rank is 3. So, $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

Remark

In Example 2.3.7 we benefited from Theorem 2.3.1 and Theorem 2.3.3 since we were finding a basis for all of $\mathbb{R}^3$. However, this is not always going to be the case. It is important that you do not always just memorize short cuts. It is always necessary to ensure that you have understood the complete concept so that you can solve a variety of problems.

EXAMPLE 2.3.8

Show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the plane $-3x_1 + 2x_2 + x_3 = 0$.

Solution: To prove that $\mathcal{B}$ is a basis for the plane, we must show that $\mathcal{B}$ is linearly independent and spans the plane.

We first observe that $\mathcal{B}$ is clearly linearly independent since neither vector is a scalar multiple of the other.

For spanning, first observe that any vector $\vec{x}$ in the plane must satisfy the condition of the plane. Hence, every vector in the plane has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$$

since $x_3 = 3x_1 - 2x_2$. Therefore, we now just need to show that the equation

$$t_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$$

is always consistent. Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 2 & 1 & x_2 \\ -1 & 1 & 3x_1 - 2x_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 1 & 2x_1 - x_2 \\ 0 & 0 & 0 \end{array}\right]$$

The system is consistent for all $x_1, x_2 \in \mathbb{R}$ and hence $\mathcal{B}$ also spans the plane.

Using the method in Example 2.3.7 we get the following useful theorem.

Theorem 2.3.5 A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{0}$ is n.

Theorem 2.3.5 gives us a condition to test whether a set of n vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$. Moreover, Theorem 2.3.1 and Theorem 2.3.3 give us the following theorem.

Theorem 2.3.6 A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^n$ is linearly independent if and only if it spans $\mathbb{R}^n$.

We now want to prove that every basis of a subspace S of $\mathbb{R}^n$ must contain the same number of vectors.

Theorem 2.3.7

Suppose that $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_\ell\}$ is a basis for a non-trivial subspace $\mathbb{S}$ of $\mathbb{R}^n$ and that $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a set in $\mathbb{S}$. If $k > \ell$, then $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly dependent.

Proof: Since each $\vec{u}_i$, $1 \leq i \leq k$, is a vector in $\mathbb{S}$ and $\mathcal{B}$ is a basis for $\mathbb{S}$, by Theorem 1.4.4 each $\vec{u}_i$ can be written as a unique linear combination of the vectors in $\mathcal{B}$. We get

$$\begin{aligned}
\vec{u}_1 &= a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \cdots + a_{\ell 1}\vec{v}_\ell \\
\vec{u}_2 &= a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \cdots + a_{\ell 2}\vec{v}_\ell \\
&\vdots \\
\vec{u}_k &= a_{1k}\vec{v}_1 + a_{2k}\vec{v}_2 + \cdots + a_{\ell k}\vec{v}_\ell
\end{aligned}$$

Consider the equation

$$\begin{aligned}
\vec{0} &= t_1\vec{u}_1 + \cdots + t_k\vec{u}_k \qquad (2.2)\\
&= t_1(a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \cdots + a_{\ell 1}\vec{v}_\ell) + \cdots + t_k(a_{1k}\vec{v}_1 + a_{2k}\vec{v}_2 + \cdots + a_{\ell k}\vec{v}_\ell) \\
&= (a_{11}t_1 + \cdots + a_{1k}t_k)\vec{v}_1 + \cdots + (a_{\ell 1}t_1 + \cdots + a_{\ell k}t_k)\vec{v}_\ell
\end{aligned}$$

But, $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ is linearly independent, so the only solution to this equation is

$$\begin{aligned}
a_{11}t_1 + \cdots + a_{1k}t_k &= 0 \\
\vdots \qquad\qquad & \vdots \\
a_{\ell 1}t_1 + \cdots + a_{\ell k}t_k &= 0
\end{aligned}$$

The rank of the coefficient matrix of this homogeneous system is at most ℓ because $\ell < k$. By the System-Rank Theorem (2), the solution space has at least $k - \ell > 0$ parameters. Therefore, there are infinitely many possible $t_1, \dots, t_k$ and so $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly dependent since equation (2.2) has infinitely many solutions. ■

Theorem 2.3.8

If $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ and $\{\vec{u}_1, \dots, \vec{u}_k\}$ are both bases of a subspace $\mathbb{S}$ of $\mathbb{R}^n$, then $k = \ell$.

Proof: Since $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ is a basis for $\mathbb{S}$ and $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent, by Theorem 2.3.7, we must have $k \leq \ell$. Similarly, since $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a basis and $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ is linearly independent, we must have $\ell \leq k$. Therefore, $\ell = k$, as required. ■

This theorem justifies the following definition.

Definition

Dimension

If $\mathbb{S}$ is a non-trivial subspace of $\mathbb{R}^n$ with a basis containing k vectors, then we say that the **dimension** of $\mathbb{S}$ is k and write

$$\dim \mathbb{S} = k$$

Since a basis for the trivial subspace $\{\vec{0}\}$ of $\mathbb{R}^n$ is the empty set, the dimension of the trivial subspace is 0.

EXAMPLE 2.3.9

Find a basis and the dimension of the solution space of the homogeneous system

$$\begin{aligned} 2x_1 + x_2 \quad &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$

Solution: We found in Example 2.2.9 that the general solution of this system is

$$\vec{x} = t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

This shows that a spanning set for the solution space is $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$. Since $\mathcal{B}$ contains one non-zero vector, it is also linearly independent and hence a basis for the solution space. Since the basis contains 1 vector, we have that the dimension of the solution space is 1.

EXAMPLE 2.3.10

Find a basis and the dimension of the solution space of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 + 4x_5 &= 0 \\ 3x_1 + 7x_2 + 7x_3 + 3x_4 + 13x_5 &= 0 \\ 2x_1 + 5x_2 + 5x_3 + 2x_4 + 9x_5 &= 0 \end{aligned}$$

Solution: We found in Exercise 2.2.4 that the general solution of this system is

$$\vec{x} = t_1\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad t_1, t_2, t_3 \in \mathbb{R}$$

This shows that a spanning set for the solution space is

$$\mathcal{B} = \left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

It is not difficult to verify that $\mathcal{B}$ is also linearly independent. Consequently, $\mathcal{B}$ is a basis for the solution space, and hence the dimension of the solution space is 3.

PROBLEMS 2.3

Practice Problems

A1 Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$. For each of the following vectors, either express it as a linear combination of the vectors in B or show that it is not in Span B.

(a) $\begin{bmatrix} -3 \\ 2 \\ 8 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ 4 \\ 6 \\ 7 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

A2 Let $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$. For each of the following vectors, either express it as a linear combination of the vectors in B or show that it is not in Span B.

(a) $\begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} -7 \\ 3 \\ 0 \\ 8 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

For Problems A3–A8, find a homogeneous system that defines the given subspace.

A3 Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

A4 Span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

A5 Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$

A6 Span $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

A7 Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

A8 Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ -3 \end{bmatrix} \right\}$

For Problems A9 and A10, find a basis and the dimension of the solution space of the given homogeneous system.

A9
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 0 \\ 2x_1 + 8x_2 + 6x_3 &= 0 \\ -4x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

A10
$$\begin{aligned} x_2 + 3x_3 + x_4 &= 0 \\ x_1 + x_2 + 5x_3 \qquad &= 0 \\ x_1 + 3x_2 + 11x_3 + 2x_4 &= 0 \end{aligned}$$

For Problems A11–A13, determine whether the given set is a basis for the given plane or hyperplane.

A11 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $x_1 + x_2 - x_3 = 0$

A12 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$ for $2x_1 - 3x_2 + x_3 = 0$

A13 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $x_1 - x_2 + 2x_3 - 2x_4 = 0$

For Problems A14–A17, determine whether the set is linearly independent. If the set is linearly dependent, find all linear combinations of the vectors that equal the zero vector.

A14 $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix} \right\}$

A15 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \\ 3 \end{bmatrix} \right\}$

A16 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

A17 $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

For Problems A18–A19, determine all values of k such that the given set is linearly independent.

A18 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ k \end{bmatrix} \right\}$

A19 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ k \\ 1 \end{bmatrix} \right\}$

For Problems A20–A23, determine whether the given set is a basis for $\mathbb{R}^3$.

A20 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

A21 $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\}$

A22 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} \right\}$

A23 $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Homework Problems

B1 Let $B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} \right\}$. For each of the following vectors, either express it as a linear combination of the vectors in B or show that it is not in Span B.

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 \\ -9 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ -5 \\ -1 \end{bmatrix}$

B2 Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right\}$. For each of the following vectors, either express it as a linear combination of the vectors in B or show that it is not in Span B.

(a) $\begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -5 \\ -4 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 5 \\ 4 \\ 3 \\ 9 \end{bmatrix}$

B3 Let $B = \left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -3 \\ 2 \end{bmatrix} \right\}$. For each of the following vectors, either express it as a linear combination of the vectors in B or show that it is not in Span B.

(a) $\begin{bmatrix} 1 \\ 7 \\ 0 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ -5 \\ 3 \\ -8 \end{bmatrix}$

For Problems B4–B11, find a homogeneous system that defines the given subspace.

B4 Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

B5 Span $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$

B6 Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \right\}$

B7 Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

B8 Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

B9 Span $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\}$

B10 Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -8 \\ 9 \end{bmatrix} \right\}$

B11 Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \\ 5 \end{bmatrix} \right\}$

For Problems B12–B15, find a basis and the dimension of the solution space of the given homogeneous system.

B12
$$\begin{aligned} 4x_1 \qquad - 4x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ 6x_1 + 2x_2 - 5x_3 &= 0 \end{aligned}$$

B13
$$\begin{aligned} 3x_2 + 3x_3 - 2x_4 &= 0 \\ x_1 + 5x_2 + x_3 + 3x_4 &= 0 \\ -x_1 + 2x_2 + 6x_3 + x_4 &= 0 \end{aligned}$$

B14
$$\begin{aligned} x_1 + 2x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 + 4x_2 + 6x_3 + x_4 &= 0 \\ 2x_1 + 4x_2 + 7x_3 + 3x_4 &= 0 \end{aligned}$$

B15
$$\begin{aligned} 3x_1 + 5x_2 + 3x_3 - x_4 &= 0 \\ 5x_1 + 9x_2 + 7x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ 2x_1 + x_2 - 5x_3 - 10x_4 &= 0 \end{aligned}$$

For Problems B16–B21, determine whether the given set is a basis for the given plane or hyperplane.

B16 $\left\{ \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \right\}$ for $x_1 + x_2 - x_3 = 0$

B17 $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix} \right\}$ for $4x_1 + 2x_2 + x_3 = 0$

B18 $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \right\}$ for $x_1 - 5x_2 + 3x_3 = 0$

B19 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ for $x_1 + 5x_2 + 2x_3 = 0$

B20 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ -1 \\ 2 \end{bmatrix} \right\}$ for $3x_1 + x_2 + 2x_3 - 2x_4 = 0$

B21 $\left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ for $x_1 - 2x_2 + 5x_3 + 6x_4 = 0$

For Problems B22–B27, determine whether the set is linearly independent. If the set is linearly dependent, find all linear combinations of the vectors that equal the zero vector.

B22 $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}-1\\4\\2\end{bmatrix}\right\}$

B23 $\left\{\begin{bmatrix}1\\1\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\7\end{bmatrix}\right\}$

B24 $\left\{\begin{bmatrix}5\\3\\1\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}, \begin{bmatrix}-7\\7\\1\end{bmatrix}, \begin{bmatrix}1\\9\\2\end{bmatrix}\right\}$

B25 $\left\{\begin{bmatrix}1\\-1\\3\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\12\\5\end{bmatrix}\right\}$

B26 $\left\{\begin{bmatrix}1\\-1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\2\\2\end{bmatrix}\right\}$

B27 $\left\{\begin{bmatrix}0\\3\\5\\2\end{bmatrix}, \begin{bmatrix}2\\5\\3\\3\end{bmatrix}, \begin{bmatrix}4\\5\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\3\\8\\2\end{bmatrix}\right\}$

For Problems B28 and B29, determine all values of k such that the given set is linearly independent.

B28 $\left\{\begin{bmatrix}-1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\-7\\-5\end{bmatrix}, \begin{bmatrix}k\\-1\\4\end{bmatrix}\right\}$

B29 $\left\{\begin{bmatrix}-3\\3\\2\\1\end{bmatrix}, \begin{bmatrix}k\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\9\\7\\k\end{bmatrix}\right\}$

For Problems B30–B33, determine whether the given set is a basis for $\mathbb{R}^3$.

B30 $\left\{\begin{bmatrix}1\\3\\-4\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}\right\}$

B31 $\left\{\begin{bmatrix}3\\5\\3\end{bmatrix}, \begin{bmatrix}2\\5\\-2\end{bmatrix}, \begin{bmatrix}4\\-2\\-1\end{bmatrix}\right\}$

B32 $\left\{\begin{bmatrix}1\\7\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}, \begin{bmatrix}2\\4\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-5\end{bmatrix}\right\}$

B33 $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\5\\1\end{bmatrix}, \begin{bmatrix}6\\7\\-4\end{bmatrix}\right\}$

Conceptual Problems

C1 Let $B = \{\vec{e}_1, \dots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Prove that Span $B = \mathbb{R}^n$ and that B is linearly independent.

C2 Find a basis for the hyperplane in $\mathbb{R}^4$ defined by

$$x_1 + ax_2 + bx_3 + cx_4 = 0$$

C3 Let $B = \{\vec{v}_1, \vec{v}_2\}$ be a basis for a subspace S in $\mathbb{R}^4$ and let $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in S$.

(a) Prove that $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly dependent.

(b) Find vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in S$ such that $\text{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = S$.

(c) Find vectors $\vec{y}_1, \vec{y}_2, \vec{y}_3 \in S$ such that $\text{Span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} \neq S$.

C4 Prove if S is a subspace of $\mathbb{R}^n$ and $\dim S = n$, then $S = \mathbb{R}^n$.

C5 Suppose that S is a subspace of $\mathbb{R}^n$ such that

$$\dim S = k$$

(a) Suppose that $B = \{\vec{w}_1, \dots, \vec{w}_\ell\}$ is a set in S. Prove if $\ell < k$, then B does not span S.

(b) Suppose that $C = \{\vec{u}_1, \dots, \vec{u}_k\}$ spans S. Prove that C is linearly independent.

C6 State, with justification, the dimension of the following subspaces of $\mathbb{R}^5$.

(a) A line through the origin.

(b) A plane through the origin.

(c) A hyperplane through the origin.

2.4 Applications of Systems of Linear Equations

We now look at a few applications of systems of linear equations. The first two of these, Spring-Mass Systems and Electric Circuits, will be referred to again in later chapters to demonstrate how our continued development of linear algebra furthers our ability to work with such applications.

Spring-Mass Systems

Consider the spring-mass system depicted in Figure 2.4.1. We want to determine the equilibrium displacements x_1, x_2 (with positive displacement being to the right) of the masses if constant forces f_1, f_2 (with a positive force being to the right) act on each of the masses m_1, m_2 respectively. As indicated, we are assuming that the springs have the corresponding spring constants k_1, k_2, k_3.

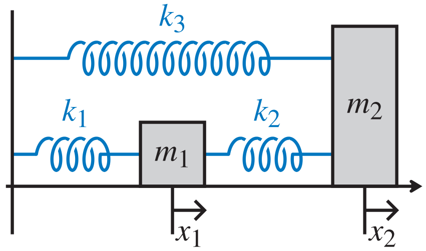


Figure 2.4.1 A spring-mass system.

According to Hooke's Law, we know that the force F required to stretch or compress a spring with spring constant k by a distance x is given by

$$F = kx$$

Using this, and taking into account the direction of force applied to each mass by the spring, we can write an equation for the force acting on each spring.

The mass m_1 is being acted upon by two springs; k_1 and k_2. A displacement of x_1 by mass m_1 will cause the spring k_1 to stretch or compress depending on if x_1 is positive or negative. In either case, by Hooke's Law, the force acting upon mass m_1 by spring k_1 is

$$-k_1x_1$$

We needed to include the negative sign as if x_1 were positive, then the force will need to be negative as the spring will be trying to pull the mass back.

How much spring k_2 will be stretched or compressed depends on displacements x_1 and x_2. In particular, it will be stretched by $x_2 - x_1$. Hence, the force acting on mass m_1 by spring k_2 is

$$k_2(x_2 - x_1)$$

For the system to be in equilibrium, the sum of the forces acting on mass m_1 must equal 0. Hence, we have

$$f_1 - k_1x_1 + k_2(x_2 - x_1) = 0$$

Mass m_2 is being acted upon by springs k_2 and k_3. Using similar analysis, we find that we also have

$$f_2 - k_3x_2 - k_2(x_2 - x_1) = 0$$

Rearranging this gives

$$\begin{aligned} (k_1 + k_2)x_1 - k_2x_2 &= f_1 \\ -k_2x_1 + (k_2 + k_3)x_2 &= f_2 \end{aligned}$$

Therefore, to determine the equilibrium displacements x_1, x_2 we just need to solve this system of linear equations.

EXAMPLE 2.4.1 Consider the spring-mass system in Figure 2.4.1. Assume that spring k_1 has spring constant $k_1 = 2$ N/m, spring k_2 has spring constant $k_2 = 4$ N/m, and spring k_3 has spring constant $k_3 = 3$ N/m. Find the equilibrium displacements x_1, x_2 if forces $f_1 = 10$ N and $f_2 = 5$ N are applied to masses m_1 and m_2 respectively.

Solution: Our work above shows us that we just need to solve the system of linear equations

$$\begin{aligned} (2 + 4)x_1 - 4x_2 &= 10 \\ -4x_1 + (4 + 3)x_2 &= 5 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|c} 6 & -4 & 10 \\ -4 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 45/13 \\ 0 & 1 & 35/13 \end{array}\right]$$

Hence, the equilibrium displacements are $x_1 = 45/13$ m and $x_2 = 35/13$ m.

Resistor Circuits in Electricity

The flow of electrical current in simple electrical circuits is described by simple linear laws. In an electrical circuit, the **current** has a direction and therefore has a sign attached to it; **voltage** is also a signed quantity; **resistance** is a positive scalar. The laws for electrical circuits are discussed next.

Ohm's Law

If an electrical current of magnitude I is flowing through a resistor with resistance R, then the drop in the voltage across the resistor is $V = IR$. The filament in a light bulb and the heating element of an electrical heater are familiar examples of electrical resistors. (See Figure 2.4.2.)

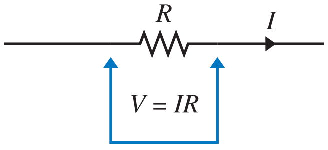


Figure 2.4.2 Ohm's Law: the voltage across the resistor is $V = IR$.

Kirchhoff's Laws

Kirchhoff's Current Law: At a node or junction where several currents enter, the signed sum of the currents entering the node is zero. (See Figure 2.4.3.)

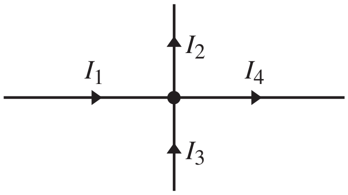


Figure 2.4.3 Kirchhoff's Current Law: $I_1 - I_2 + I_3 - I_4 = 0$.

Kirchhoff's Voltage Law: In a closed loop consisting of only resistors and an **electromotive force** E (for example, E might be due to a battery), the sum of the voltage drops across resistors is equal to E. (See Figure 2.4.4.)

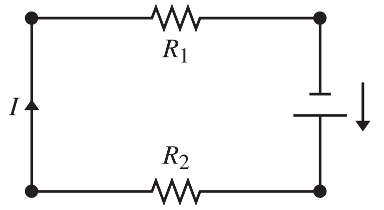


Figure 2.4.4 Kirchhoff's Voltage Law: $E = R_1 I + R_2 I$.

Note that we adopt the convention of drawing an arrow to show the direction of I or of E. These arrows can be assigned arbitrarily, and then the circuit laws will determine whether the quantity has a positive or negative sign. If the quantity has a negative sign, then it means the flow is in the opposite direction of the arrow. It is important to be consistent in using these assigned directions when you use Kirchhoff's Voltage Law.

Sometimes it is necessary to determine the current flowing in each of the loops of a network of loops as shown in Figure 2.4.5. (If the sources of electromotive force are distributed in various places, it will not be sufficient to deal with the problems as a collection of resistors "in parallel and/or in series.") In such problems, it is convenient to introduce the idea of the "current in the loop," which will be denoted i. The true current across any circuit element is given as the algebraic (signed) sum of the "loop currents" flowing through that circuit element. For example, in Figure 2.4.5, the circuit consists of four loops, and a loop current has been indicated in each loop. Across the resistor R_1 in the figure, the true current is simply the loop current i_1; however, across the resistor R_2, the true current (directed from top to bottom) is $i_1 - i_2$. Similarly, across R_4, the true current (from right to left) is $i_1 - i_3$.

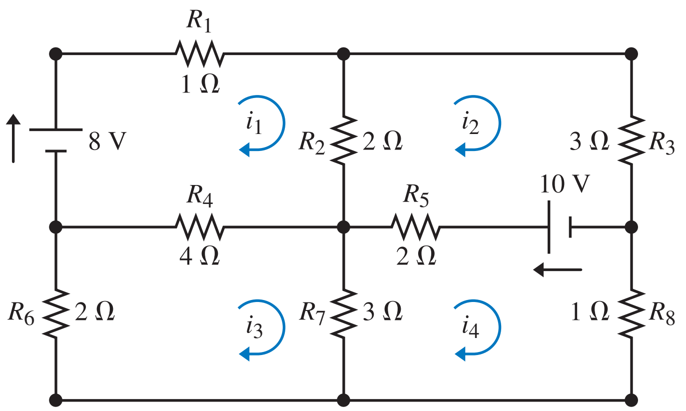


Figure 2.4.5 A resistor circuit.

EXAMPLE 2.4.2

Find all loop currents in the electric circuit in Figure 2.4.5.

Solution: We use Kirchhoff's Voltage Law on each of the four loops:

the top left loop: $i_1 + 2(i_1 - i_2) + 4(i_1 - i_3) = 8$

the top right loop: $3i_2 + 2(i_2 - i_4) + 2(i_2 - i_1) = 10$

the bottom left loop: $2i_3 + 4(i_3 - i_1) + 4(i_3 - i_4) = 0$

the bottom right loop: $2i_4 + 4(i_4 - i_3) + 2(i_4 - i_2) = -10$

Solving this system of 4 equations in the 4 unknowns I_1, I_2, I_3, I_4, we find that the currents are

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \frac{1}{1069} \begin{bmatrix} 2172 \\ 1922 \\ 702 \\ -790 \end{bmatrix}$$

In addition to showing you an application of linear algebra, the point of this example is to show that even for a fairly simple electrical circuit with the most basic elements (resistors), the analysis requires you to be competent in dealing with large systems of linear equations. Systematic, efficient methods of solution are essential.

Obviously, as the number of nodes and loops in the network increases, so does the number of variables and equations. For larger systems, it is important to know whether you have the correct number of equations to determine the unknowns. Thus, the theorems in Sections 2.1 and 2.2, the idea of rank, and the idea of linear independence are all important.

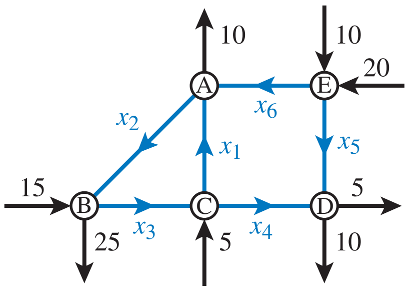
The Moral of This Example Linear algebra is an essential tool for dealing with large systems of linear equations that may arise in dealing with circuits; really interesting examples cannot be given without assuming greater knowledge of electrical circuits and their components.

Water Flow

Kirchhoff's Current Law can be applied to many other situations. Here we will look at water flowing through a system of pipes. However, the same principles and techniques can be applied to a variety of other problems involving networks; for example, in communication networks (including Internet connectivity), transportation networks, and economic networks.

EXAMPLE 2.4.3

The diagram below indicates a system of pipes and the water flow entering or leaving the system. An irrigation engineer may be interested in determining the amount of water flowing through each pipe and the effects of changing the amount of flow in a certain pipe by use of a ball valve.



We set up a system of linear equations by equating the rate in and the rate out at each intersection. We get

Intersection	rate in	=	rate out
A	$x_1 + x_6$	=	$10 + x_2$
B	$x_2 + 15$	=	$25 + x_3$
C	$x_3 + 5$	=	$x_1 + x_4$
D	$x_4 + x_5$	=	$5 + 10$
E	$10 + 20$	=	$x_5 + x_6$

Rearranging this gives the system of linear equations

$$
\begin{aligned}
x_1 - x_2 + x_6 &= 10 \\
x_2 - x_3 &= 10 \\
-x_1 + x_3 - x_4 &= -5 \\
x_4 + x_5 &= 15 \\
-x_5 - x_6 &= -30
\end{aligned}
$$

Row reducing the corresponding augmented matrix gives

$$
\left[\begin{array}{cccccc|c}
1 & -1 & 0 & 0 & 0 & 1 & 10 \\
0 & 1 & -1 & 0 & 0 & 0 & 10 \\
-1 & 0 & 1 & -1 & 0 & 0 & -5 \\
0 & 0 & 0 & 1 & 1 & 0 & 15 \\
0 & 0 & 0 & 0 & -1 & -1 & -30
\end{array}\right] \sim \left[\begin{array}{cccccc|c}
1 & 0 & -1 & 0 & 0 & 1 & 20 \\
0 & 1 & -1 & 0 & 0 & 0 & 10 \\
0 & 0 & 0 & 1 & 0 & -1 & -15 \\
0 & 0 & 0 & 0 & 1 & 1 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

EXAMPLE 2.4.3
(continued)

Hence, the general solution is

$$\begin{aligned} x_1 &= 20 + x_3 - x_6 \\ x_2 &= 10 + x_3 \\ x_4 &= -15 + x_6 \\ x_5 &= 30 - x_6 \end{aligned}$$

with $x_3, x_6 \in \mathbb{R}$.

We do not have enough information to determine the exact flow through all pipes. If we were to able to install water flow meters on pipes x_3 and x_6, then we would be able to deduce the exact amounts flowing through the remaining pipes.

For example, if we find that $x_3 = 20$ L/s and $x_6 = 10$ L/s, then we get that

$$x_1 = 30 \text{ L/s}, x_2 = 30 \text{ L/s}, x_4 = -5 \text{ L/s}, x_5 = 20 \text{ L/s}$$

Observe that in this situation water is flowing along pipe x_4 in the opposite direction that we indicated. If we want to ensure that water is flowing along pipe x_4 to the right, we need to increase the rate of flow in pipe x_6 (by widening the pipe for example) so that the flow rate of pipe x_6 is greater than 15 L/s. Alternately, if we can get the flow rate of pipe x_6 to exactly 15 L/s, then we can safely remove pipe x_4.

Partial Fraction Decomposition

Consider a rational function $\dfrac{N(x)}{D(x)}$, where the degree of the polynomial $N(x)$ is less than the degree of the polynomial $D(x)$. There are several cases in mathematics where it is very useful to rewrite such a rational function as a sum of rational functions whose denominators are linear or irreducible quadratic factors. Most students will first encounter this when learning techniques of integration. They are also used when working with Taylor or Laurent series and when working with the inverse Laplace transform.

For example, given a rational function of the form

$$\frac{cx+d}{(x-a)(x-b)}$$

we want to find constants A and B such that

$$\frac{cx+d}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

Essentially, this procedure, called the **partial fraction decomposition**, is the opposite of adding fractions.

We demonstrate this with a couple of examples.

EXAMPLE 2.4.4

Find constants A and B such that $\dfrac{1}{(x+3)(x-2)} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$.

Solution: If we multiply both sides of the equation by $(x+3)(x-2)$, we get

$$1 = A(x-2) + B(x+3)$$

We rewrite this as

$$0x + 1 = (A+B)x + (-2A+3B)$$

Upon comparing coefficients of like powers of x, we get a system of linear equations in the unknowns A and B

$$\begin{aligned} A + B &= 0 \\ -2A + 3B &= 1 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1/5 \\ 0 & 1 & 1/5 \end{array}\right]$$

Hence,

$$\frac{1}{(x+3)(x-2)} = \frac{-1/5}{x+3} + \frac{1/5}{x-2}$$

It is easy to perform the addition on the right to verify this is correct.

EXAMPLE 2.4.5

Find constants A, B, C, D, E, and F such that

$$\frac{3x^5 - 2x^4 - x + 1}{(x^2+1)^2(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{x^2+x+1}$$

Solution: We first multiply both sides of the equation by $(x^2+1)^2(x^2+x+1)$ to get

$$\begin{aligned} 3x^5 - 2x^4 - x + 1 =& (Ax+B)(x^2+1)(x^2+x+1) + (Cx+D)(x^2+x+1) \\ &+ (Ex+F)(x^2+1)^2 \end{aligned}$$

Expanding the right and collecting coefficients of like powers of x gives

$$\begin{aligned} 3x^5 - 2x^4 - x + 1 =& (A+E)x^5 + (A+B+F)x^4 + (2A+B+C+2E)x^3 \\ &+ (A+2B+C+D+2F)x^2 + (A+B+C+D+E)x + B + D + F \end{aligned}$$

Comparing coefficients of like powers of x gives the system of linear equations

$$\begin{aligned} A + E &= 3 \\ A + B + F &= -2 \\ 2A + B + C + 2E &= 0 \\ A + 2B + C + D + 2F &= 0 \\ A + B + C + D + E &= -1 \\ B + D + F &= 1 \end{aligned}$$

EXAMPLE 2.4.5
(continued)

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 0 & 0 & 1 & -2 \\ 2 & 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Hence,

$$\frac{3x^5 - 2x^4 - x + 1}{(x^2+1)^2(x^2+x+1)} = \frac{-x-7}{x^2+1} + \frac{x+2}{(x^2+1)^2} + \frac{4x+6}{x^2+x+1}$$

Balancing Chemical Equations

When molecules are combined under the correct conditions, a chemical reaction occurs. For example, in Andy Weir's novel *The Martian*, the main character, Mark Watney, uses an iridium catalyst to first convert hydrazine into nitrogen gas and hydrogen gas according to the **chemical equation**

$$N_2H_4 \to N_2 + H_2$$

He then burns the hydrogen gas with the oxygen in the Hab to get the chemical reaction

$$H_2 + O_2 \to H_2O$$

Although these chemical equations indicate how the **reactants** (the molecules on the left of the arrows) are rearranged into the **products** (the molecules on the right of the arrows), we see that the equations are not really complete. In particular, in the first chemical equation, there are four hydrogen atoms on the left side, but only two on the right side. Similarly, in the second chemical equation, there are two oxygen atoms on the left, but only one on the right.

A chemical equation is said to be **balanced** if both sides of the equation have the same number of atoms of each type. So, for example, the chemical equation

$$N_2H_4 \to N_2 + 2H_2$$

is balanced since both the left and right side have two nitrogen atoms and four hydrogen atoms. Similarly, we can see that the chemical equation

$$2H_2 + O_2 \to 2H_2O$$

is balanced. Of course, technically, the chemical equation

$$4H_2 + 2O_2 \to 4H_2O$$

is also balanced, but we always try to use the smallest positive integers that balance the equation.

As usual, we will demonstrate this with a couple of examples.

EXAMPLE 2.4.6

Balance the chemical equation

$$H_3PO_4 + Mg(OH)_2 \rightarrow Mg_3(PO_4)_2 + H_2O$$

Solution: We want to find constants x_1, x_2, x_3, x_4 such that

$$x_1H_3PO_4 + x_2Mg(OH)_2 \rightarrow x_3Mg_3(PO_4)_2 + x_4H_2O$$

is balanced. To turn this into a vector equation, we represent the molecules in the equation with the vectors in $\mathbb{R}^4$:

$$\begin{bmatrix} \text{\# of hydrogen atoms} \\ \text{\# of phosphorus atoms} \\ \text{\# of oxygen atoms} \\ \text{\# of magnesium atoms} \end{bmatrix}$$

We get

$$x_1\begin{bmatrix}3\\1\\4\\0\end{bmatrix} + x_2\begin{bmatrix}2\\0\\2\\1\end{bmatrix} = x_3\begin{bmatrix}0\\2\\8\\3\end{bmatrix} + x_4\begin{bmatrix}2\\0\\1\\0\end{bmatrix}$$

Moving all the terms to the left side and performing the linear combination of vectors, we get the homogeneous system

$$\begin{aligned} 3x_1 + 2x_2 - 2x_4 &= 0 \\ x_1 - 2x_3 &= 0 \\ 4x_1 + 2x_2 - 8x_3 - x_4 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 3 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 4 & 2 & -8 & -1 \\ 0 & 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We find that a vector equation for the solution space is

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = t\begin{bmatrix}1/3\\1/2\\1/6\\1\end{bmatrix}, \quad t \in \mathbb{R}$$

To get the smallest positive integer values, we take $t = 6$. This gives $x_1 = 2$, $x_2 = 3$, $x_3 = 1$, and $x_4 = 6$. Thus, a balanced chemical equation is

$$2H_3PO_4 + 3Mg(OH)_2 \rightarrow Mg_3(PO_4)_2 + 6H_2O$$

EXAMPLE 2.4.7

Balance the chemical equation

$$(NH_4)_3PO_4 + Pb(NO_3)_4 \to Pb_3(PO_4)_4 + NH_4NO_3$$

Solution: We want to find constants x_1, x_2, x_3, x_4 such that

$$x_1(NH_4)_3PO_4 + x_2Pb(NO_3)_4 \to x_3Pb_3(PO_4)_4 + x_4NH_4NO_3$$

is balanced. Define vectors in $\mathbb{R}^5$ by

$$\begin{bmatrix} \text{\# of nitrogen atoms} \\ \text{\# of hydrogen atoms} \\ \text{\# of phosphorus atoms} \\ \text{\# of oxygen atoms} \\ \text{\# of lead atoms} \end{bmatrix}$$

We get the vector equation

$$x_1\begin{bmatrix} 3 \\ 12 \\ 1 \\ 4 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 0 \\ 0 \\ 12 \\ 1 \end{bmatrix} = x_3\begin{bmatrix} 0 \\ 0 \\ 4 \\ 16 \\ 3 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ 4 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Rearranging gives the homogeneous system

$$\begin{aligned} 3x_1 + 4x_2 - 2x_4 &= 0 \\ 12x_1 - 4x_4 &= 0 \\ x_1 - 4x_3 &= 0 \\ 4x_1 + 12x_2 - 16x_3 - 3x_4 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 3 & 4 & 0 & -2 \\ 12 & 0 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ 4 & 12 & -16 & -3 \\ 0 & 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & -1/12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We find that a vector equation for the solution space is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} 1/3 \\ 1/4 \\ 1/12 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

To get the smallest positive integer values, we take $t = 12$. This gives $x_1 = 4$, $x_2 = 3$, $x_3 = 1$, and $x_4 = 12$. Thus, a balanced chemical equation is

$$4(NH_4)_3PO_4 + 3Pb(NO_3)_4 \to Pb_3(PO_4)_4 + 12NH_4NO_3$$

Planar Trusses

It is common to use trusses, such as the one shown in Figure 2.4.6, in construction. For example, many bridges employ some variation of this design. When designing such structures, it is necessary to determine the **axial forces** in each **member** of the structure (that is, the force along the long axis of the member). To keep this simple, only two-dimensional trusses with hinged joints will be considered; it will be assumed that any displacements of the joints under loading are small enough to be negligible.

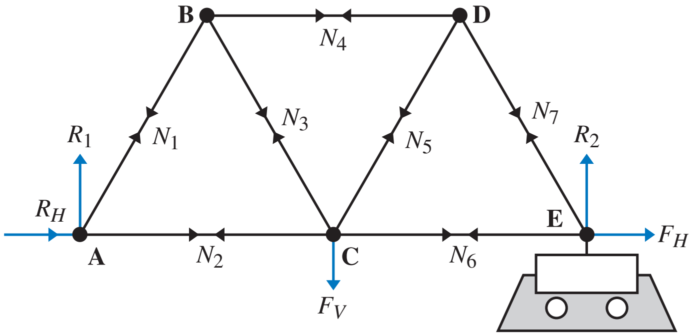


Figure 2.4.6 A planar truss. All triangles are equilateral with sides of length s.

The external loads (such as vehicles on a bridge, or wind or waves) are assumed to be given. The reaction forces at the supports (shown as R_1, R_2, and R_H in the figure) are also external forces; these forces must have values such that the total external force on the structure is zero. To get enough information to design a truss for a particular application, we must determine the forces in the members under various loadings. To illustrate the kinds of equations that arise, we shall consider only the very simple case of a vertical force F_V acting at C and a horizontal force F_H acting at E. Notice that in this figure, the right end of the truss is allowed to undergo small horizontal displacements; it turns out that if a reaction force were applied here as well, the equations would not uniquely determine all the unknown forces (the structure would be "statically indeterminate"), and other considerations would have to be introduced.

The geometry of the truss is assumed given: here it will be assumed that the triangles are equilateral with all sides equal to s metres.

First consider the equations that indicate that the total force on the structure is zero and that the total moment about some convenient point due to those forces is zero. Note that the axial force along the members does not appear in this first set of equations.

- Total horizontal force: $R_H + F_H = 0$
- Total vertical force: $R_1 + R_2 - F_V = 0$
- Moment about A: $-F_V(s) + R_2(2s) = 0$, so $R_2 = \frac{1}{2}F_V = R_1$

Next, we consider the system of equations obtained from the fact that the sum of the forces at each joint must be zero. The moments are automatically zero because the forces along the members act through the joints.

At a joint, each member at that joint exerts a force in the direction of the axis of the member. It will be assumed that each member is in tension, so it is "pulling" away from the joint; if it were compressed, it would be "pushing" at the joint. As indicated in the figure, the force exerted on this joint A by the upper-left-hand member has magnitude N_1; with the conventions that forces to the right are positive and forces up are positive,

the force vector exerted by this member on the joint A is $\begin{bmatrix} N_1/2 \\ \sqrt{3}N_1/2 \end{bmatrix}$. On the joint B, the same member will exert a force $\begin{bmatrix} -N_1/2 \\ -\sqrt{3}N_1/2 \end{bmatrix}$. If N_1 is positive, the force is a tension force; if N_1 is negative, there is compression.

For each of the joints A, B, C, D, and E, there are two equations—the first for the sum of horizontal forces and the second for the sum of the vertical forces:

$$
\begin{array}{llllllllll}
A1 & N_1/2 & +N_2 & & & & & & +R_H = & 0 \\
A2 & \sqrt{3}N_1/2 & & & & & & & +R_1 = & 0 \\
B1 & -N_1/2 & & +N_3/2 & +N_4 & & & & = & 0 \\
B2 & -\sqrt{3}N_1/2 & & -\sqrt{3}N_3/2 & & & & & = & 0 \\
C1 & & -N_2 & -N_3/2 & & +N_5/2 & +N_6 & & = & 0 \\
C2 & & & \sqrt{3}N_3/2 & & +\sqrt{3}N_5/2 & & & = & F_V \\
D1 & & & & -N_4 & -N_5/2 & & +N_7/2 & = & 0 \\
D2 & & & & & -\sqrt{3}N_5/2 & & -\sqrt{3}N_7/2 & = & 0 \\
E1 & & & & & & -N_6 & -N_7/2 & = & -F_H \\
E2 & & & & & & & \sqrt{3}N_7/2 & +R_2 = & 0
\end{array}
$$

Notice that if the reaction forces are treated as unknowns, this is a system of 10 equations in 10 unknowns. The geometry of the truss and its supports determines the coefficient matrix of this system, and it could be shown that the system is necessarily consistent with a unique solution. Notice also that if the horizontal force equations (A1, B1, C1, D1, and E1) are added together, the sum is the total horizontal force equation, and similarly the sum of the vertical force equations is the total vertical force equation. A suitable combination of the equations would also produce the moment equation, so if those three equations are solved as above, then the 10 joint equations will still be a consistent system for the remaining 7 axial force variables.

For this particular truss, the system of equations is quite easy to solve, since some of the variables are already leading variables. For example, if $F_H = 0$, from A2 and E2 it follows that $N_1 = N_7 = -\frac{1}{\sqrt{3}}F_V$ and then B2, C2, and D2 give $N_3 = N_5 = \frac{1}{\sqrt{3}}F_V$; then A1 and E1 imply that $N_2 = N_6 = \frac{1}{2\sqrt{3}}F_V$, and B1 implies that $N_4 = -\frac{1}{\sqrt{3}}F_V$. Note that the members AC, BC, CD, and CE are under tension, and AB, BD, and DE experience compression, which makes intuitive sense.

This is a particularly simple truss. In the real world, trusses often involve many more members and use more complicated geometry; trusses may also be three-dimensional. Therefore, the systems of equations that arise may be considerably larger and more complicated. It is also sometimes essential to introduce considerations other than the equations of equilibrium of forces in statics. To study these questions, you need to know the basic facts of linear algebra.

It is worth noting that in the system of equations above, each of the quantities N_1, $N_2, \dots, N_7$ appears with a non-zero coefficient in only some of the equations. Since each member touches only two joints, this sort of special structure will often occur in the equations that arise in the analysis of trusses. A deeper knowledge of linear algebra is important in understanding how such special features of linear equations may be exploited to produce efficient solution methods.

Linear Programming

Linear programming is a procedure for deciding the best way to allocate resources. "Best" may mean fastest, most profitable, least expensive, or best by whatever criterion is appropriate. For linear programming to be applicable, the problem must have some special features. These will be illustrated by an example.

In a primitive economy, a man decides to earn a living by making hinges and gate latches. He is able to obtain a supply of 25 kg per week of suitable metal at a price of 2 dollars per kg. His design requires 500 g to make a hinge and 250 g to make a gate latch. With his primitive tools, he finds that he can make a hinge in 1 hour, and it takes 3/4 hour to make a gate latch. He is willing to work 60 hours a week. The going price is 3 dollars for a hinge and 2 dollars for a gate latch. How many hinges and how many gate latches should he produce each week in order to maximize his net income?

To analyze the problem, let x be the number of hinges produced per week and let y be the number of gate latches. Then the amount of metal used is $(0.5x + 0.25y)$ kg. Clearly, this must be less than or equal to 25 kg:

$$0.5x + 0.25y \leq 25$$

Multiplying by 4 to clear the decimals gives

$$2x + y \leq 100$$

Such an inequality is called a **constraint** on x and y; it is a linear constraint because the corresponding equation is linear.

Our producer also has a time constraint: the time taken making hinges plus the time taken making gate latches cannot exceed 60 hours. Therefore,

$$1x + 0.75y \leq 60$$

which can be rewritten as

$$4x + 3y \leq 240$$

Obviously, also $x \geq 0$ and $y \geq 0$.

The producer's net revenue for selling x hinges and y gate latches is

$$R(x, y) = 3x + 2y - 2(25)$$

This is called the **objective function** for the problem. The mathematical problem can now be stated as follows:

Find the point (x, y) that maximizes the objective function

$$R(x, y) = 3x + 2y - 50$$

subject to the linear constraints

$$\begin{aligned} 2x + y &\leq 100 \\ 4x + 3y &\leq 240 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

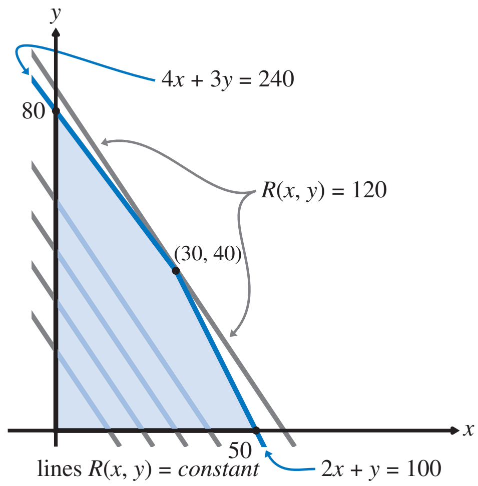


Figure 2.4.7 The feasible region for the linear programming example. The grey lines are level sets of the objective function R.

It is useful to introduce one piece of special vocabulary: the **feasible region** for the problem is the set of (x, y) satisfying all of the constraints. Our goal is to find which point in the feasible region gives us the maximum value of $R(x, y)$.

Consider sets of the form $R(x, y) = k$, where k is a constant. These sets form a family of parallel lines, called the **objective lines** as in Figure 2.4.7.

As we move further from the origin into the first quadrant, $R(x, y)$ increases. The biggest possible value for $R(x, y)$ will occur at a point where the set $R(x, y) = k$ (for some constant k to be determined) just touches the feasible region. For larger values of $R(x, y)$, the set $R(x, y) = k$ does not meet the feasible region at all, so there are no feasible points that give such bigger values of R. The touching must occur at a **vertex**—that is, at an intersection point of two of the boundary lines. (In general, the line $R(x, y) = k$ for the largest possible constant could touch the feasible region along a line segment that makes up part of the boundary. But such a line segment has two vertices as endpoints, so it is correct to say that the touching occurs at a vertex.)

For this particular problem, we see that the vertices of the feasible region are $(0, 0)$, $(50, 0)$, $(0, 80)$, and at the intersection of the two lines $2x + y = 100$ and $4x + 3y = 240$. We can find the intersection by solving

$$\begin{aligned} 2x + y &= 100 \\ 4x + 3y &= 240 \end{aligned}$$

The solution is $x = 30$, $y = 40$.

Now compare the values of $R(x, y)$ at all of these vertices:

$$\begin{aligned} R(0, 0) &= -50 \\ R(50, 0) &= 100 \\ R(0, 80) &= 110 \\ R(30, 40) &= 120 \end{aligned}$$

The vertex $(30, 40)$ gives the best net revenue, so the producer should make 30 hinges and 40 gate latches each week.

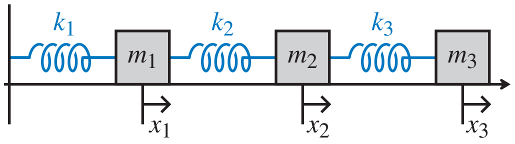
General Remarks Problems involving allocation of resources can be found in numerous areas such as management engineering, economics, politics, and biology. Problems such as scheduling ship transits through a canal can be analyzed this way. Oil companies must make choices about the grades of crude oil to use in their refineries, and about the amounts of various refined products to produce. Such problems often involve tens or even hundreds of variables—and similar numbers of constraints. The boundaries of the feasible region are hyperplanes in some $\mathbb{R}^n$, where n is large. Although the basic principles of the solution method remain the same as in this example (look for the best vertex), the problem is much more complicated because there are so many vertices. In fact, it is a challenge to find vertices; simply solving all possible combinations of systems of boundary equations is not good enough. Note in the simple two-dimensional example that the point $(60, 0)$ is the intersection point of two of the lines ($y = 0$ and $4x + 3y = 240$) that make up the boundary, but it is not a vertex of the feasible region because it fails to satisfy the constraint $2x + y \leq 100$. For higher-dimension problems, drawing pictures is not good enough, and an organized approach is called for.

The standard method for solving linear programming problems is the **simplex method**, which finds an initial vertex and then prescribes a method (very similar to row reduction) for moving to another vertex, improving the value of the objective function with each step.

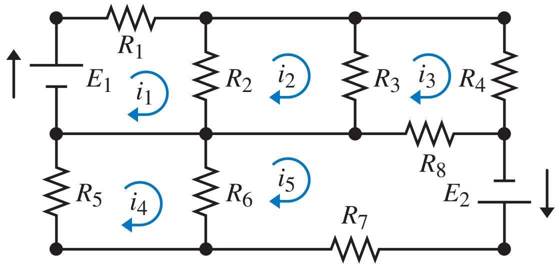
PROBLEMS 2.4

Practice Problems

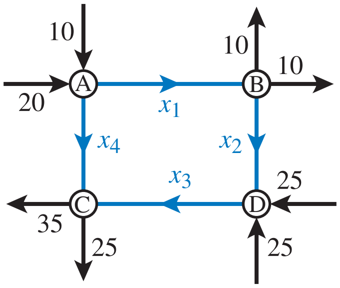
A1 For the spring-mass system below, assume that the spring constants are $k_1 = 2$ N/m, $k_2 = 1$ N/m, and $k_3 = 4$ N/m. Find the equilibrium displacements x_1, x_2, x_3 if forces $f_1 = 10$ N, $f_2 = 6$ N, and $f_3 = 8$ N are applied to masses m_1, m_2, and m_3 respectively.



A2 Write the system of linear equations required to solve for the loop currents in the diagram.



A3 Determine the amount of water flowing through each pipe.



A4 Find constants $A, B, C, D,$ and E such that

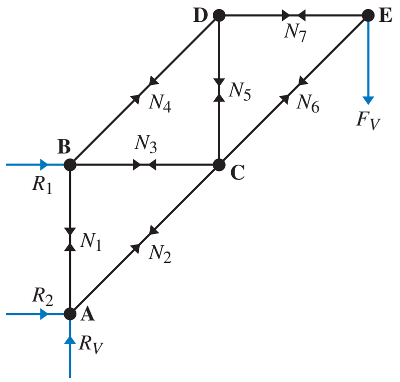
$$\frac{4x^4 + x^3 + x^2 + x + 1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$$

A5 Balance the chemical equation

$$Al(OH)_3 + H_2CO_3 \rightarrow Al_2(CO_3)_3 + H_2O$$

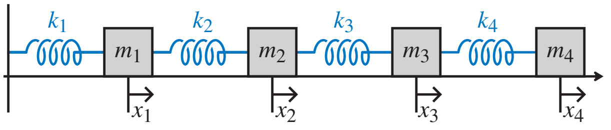
A6 Find the maximum value of the objective function $x+y$ subject to the constraints $0 \leq x \leq 100$, $0 \leq y \leq 80$, and $4x + 5y \leq 600$. Sketch the feasible region.

A7 Determine the system of equations for the reaction forces and axial forces in members for the truss shown in the diagram to the right. Assume that all triangles are right-angled and isosceles, with side lengths s, s, and $\sqrt{2}s$.

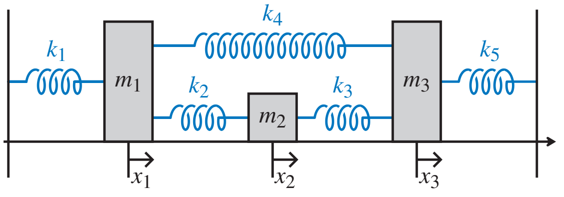


Homework Problems

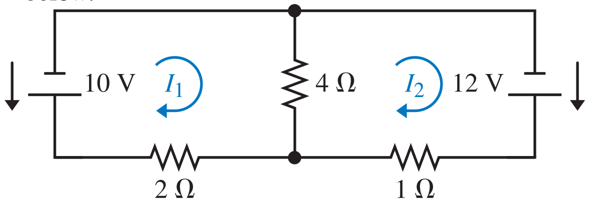
B1 For the spring-mass system below, assume that the spring constants are $k_1 = 3$ N/m, $k_2 = 2$ N/m, $k_3 = 1$ N/m, and $k_4 = 4$ N/m. Find the equilibrium displacements x_1, x_2, x_3, x_4 if constant forces $f_1 = 2$ N, $f_2 = 3$ N, $f_3 = 3$ N, and $f_4 = 4$ N are applied to masses m_1, m_2, m_3, and m_4 respectively.



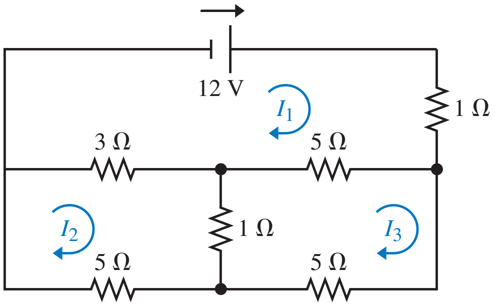
B2 For the spring-mass system below, assume that the spring constants are $k_1 = 2$ N/m, $k_2 = 3$ N/m, $k_3 = 1$ N/m, $k_4 = 2$ N/m, and $k_5 = 1$ N/m. Find the equilibrium displacements x_1, x_2, x_3 if a constant force of 1 N is applied to all three masses.



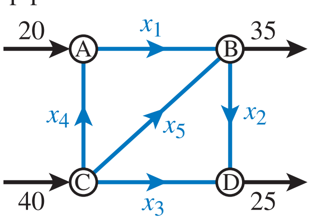
B3 Determine the currents I_1 and I_2 for the electric circuit below.



B4 Determine the currents I_1, I_2, and I_3 for the electric circuit below.



B5 Consider the pipe network below.



(a) Determine the amount of water flowing through each pipe.

(b) If x_4 and x_5 are both set to a flow rate of 20 L/s, what is the flow rate of x_1, x_2, and x_3?

B6 Find constants A, B, C, and D such that

$$\frac{2x^2 + 3x - 3}{(x^2 + 3)(x^2 + 3x + 3)} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{x^2 + 3x + 3}$$

B7 Find constants A, B, C, D, and E such that

$$\frac{x^3 + 2x^2 - 1}{(x - 1)(x^2 + 1)(x^2 + 3)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 3}$$

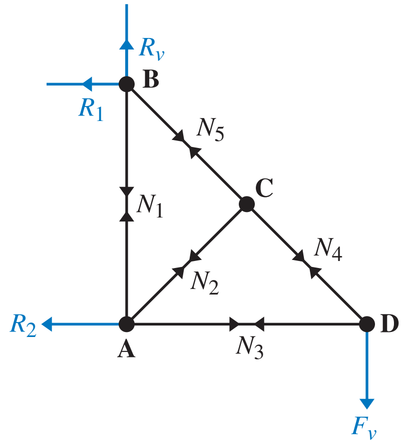
B8 Balance the chemical equation

$$C_3H_8 + O_2 \to CO_2 + H_2O$$

B9 Balance the chemical equation

$$Ca_3(PO)_4 + SiO_2 + C \to CaSiO_3 + CO + P$$

B10 Determine the system of equations for the reaction forces and axial forces in members for the truss shown in the diagram to the right. Assume that both triangles are right-angled and isosceles, with side lengths s, s, and $\sqrt{2}s$.



CHAPTER REVIEW

Suggestions for Student Review

Try to answer all of these questions before checking at the suggested locations. In particular, try to invent your own examples. These review suggestions are intended to help you carry out your review. They may not cover every idea you need to master. Working in small groups may improve your efficiency.

1 Describe the geometric interpretation of a system $\left[A \mid \vec{b}\right]$ of m linear equations in n variables and its solutions. What was the alternate view of a system and how does it relate to the content in Section 2.3? (Sections 2.1, 2.3)

2 When you row reduce an augmented matrix $\left[A \mid \vec{b}\right]$ to solve a system of linear equations, why can you stop when the matrix is in row echelon form? How do you use this form to decide if the system is consistent and if it has a unique solution? (Section 2.1)

3 How is reduced row echelon form different from row echelon form? (Section 2.2)

4 (a) Write the augmented matrix of a consistent non-homogeneous system of three linear equations in four variables, such that the coefficient matrix is in row echelon form (but not reduced row echelon form) and of rank 3.

(b) Determine the general solution of your system.

(c) Perform the following sequence of elementary row operations on your augmented matrix:

(i) Interchange the first and second rows.

(ii) Add the (new) second row to the first row.

(iii) Add twice the second row to the third row.

(iv) Add the third row to the second.

(d) Regard the result of (c) as the augmented matrix of a system and solve that system directly. (Don't just use the reverse operation in (c).) Check that your general solution agrees with (b).

5 State the definition of the rank of a matrix and the System-Rank Theorem. Explain how parts (2) and (3) of the System-Rank Theorem relate to both of our interpretations of a system of linear equations. (Sections 2.1, 2.2, 2.3)

6 For homogeneous systems, how can you use the row echelon form to determine whether there are non-trivial solutions and, if there are, how many parameters there are in the general solution? Is there any case where we know (by inspection) that a homogeneous system has non-trivial solutions? (Section 2.2)

7 Write a short explanation of how you use information about consistency of systems and uniqueness of solutions in testing for linear independence and in determining whether a vector $\vec{x}$ belongs to a given subspace of $\mathbb{R}^n$. (Section 2.3)

8 State the definition of a basis and the definition of the dimension of a subspace. What form must the reduced row echelon form of the coefficient matrix of the vector equation $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{b}$ have if the set is a basis for $\mathbb{R}^n$? (Section 2.3)

9 Consider a subspace $S = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^3$ where $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

(a) Explain the procedure for finding a homogeneous system that represents the set.

(b) Explain the two ways of determining if a vector $\vec{x} \in \mathbb{R}^3$ is in S.

(c) Explain in general the advantages and disadvantages of having a system of linear equations representing a set of points rather than a vector equation.

10 Search online for some applications of systems of linear equations to your desired field of study. How do these relate with the applications covered so far in this text? (Section 2.4)

Chapter Quiz

In Problems E1 and E2:

(a) Write the augmented matrix.

(b) Row reduce the augmented matrix and determine the rank of the augmented matrix.

(c) Find the general solution of the system or explain why the system is inconsistent.

E1
$$\begin{aligned} x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 - 2x_2 + 4x_3 - x_4 &= 10 \\ x_1 - x_2 + x_3 \qquad &= 2 \\ x_1 \qquad + x_3 \qquad &= 9 \end{aligned}$$

E2
$$\begin{aligned} 2x_1 + 4x_2 + x_3 - 6x_4 &= 7 \\ 4x_1 + 8x_2 - 3x_3 + 8x_4 &= -1 \\ -3x_1 - 6x_2 + 2x_3 - 5x_4 &= 0 \\ x_1 + 2x_2 + x_3 - 5x_4 &= 5 \end{aligned}$$

E3 Let $A = \begin{bmatrix} 0 & 3 & 3 & 0 & -1 \\ 1 & 1 & 3 & 3 & 1 \\ 2 & 4 & 9 & 6 & 1 \\ -2 & -4 & -6 & -3 & -1 \end{bmatrix}$.

(a) Find the reduced row echelon form of A and state the rank of A.

(b) Write the general solution of the homogeneous system $\left[A \mid \vec{0}\right]$.

(c) What is a basis and the dimension of the solution space of $\left[A \mid \vec{0}\right]$?

E4 Find a homogeneous system that defines Span $\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}\right\}$.

E5 The matrix $A = \left[\begin{array}{cccc|c} 1 & 2 & 3 & a & 2 \\ 0 & 2 & 1 & 0 & -3 \\ 0 & 0 & b+2 & 0 & b \\ 0 & 0 & 0 & c^2-1 & c+1 \end{array}\right]$ is the augmented matrix of a system of linear equations.

(a) Determine all values of (a, b, c) such that the system is consistent and all values of (a, b, c) such that the system is inconsistent.

(b) Determine all values of (a, b, c) such that the system has a unique solution.

E6 (a) Determine all vectors $\vec{x}$ in $\mathbb{R}^5$ that are orthogonal to

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 8 \\ 5 \\ 9 \end{bmatrix}$$

(b) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three vectors in $\mathbb{R}^5$. Explain why there must be non-zero vectors orthogonal to all of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

E7 Determine whether $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$ is linearly independent.

E8 Determine whether the set $\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right\}$ spans $\mathbb{R}^3$.

E9 Prove that $\mathcal{B} = \left\{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

For Problems E10–E15, indicate whether the statement is true or false. Justify your answer with a brief explanation or counterexample.

E10 $x_1^2 + 2x_1x_2 + x_2 = 0$ is a linear equation.

E11 A consistent system must have a unique solution.

E12 If there are more equations than variables in a non-homogeneous system of linear equations, then the system must be inconsistent.

E13 If $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is a solution of a system of linear equations, then $\begin{bmatrix}2\\2\\2\end{bmatrix}$ is also a solution of the system.

E14 Some homogeneous systems of linear equations have unique solutions.

E15 If there are more variables than equations in a system of linear equations, then the system cannot have a unique solution.

Further Problems

These problems are intended to be challenging.

F1 The purpose of this problem is to explore the relationship between the general solution of the system $\left[A \mid \vec{b}\,\right]$ and the general solution of the **corresponding homogeneous** system $\left[A \mid \vec{0}\,\right]$. This relation will be studied with different tools in Section 3.4. We begin by considering some examples where the coefficient matrix is in RREF.

(a) Let $R = \begin{bmatrix}1 & 0 & r_{13}\\0 & 1 & r_{23}\end{bmatrix}$. Show that the general solution of the homogeneous system $\left[R \mid \vec{0}\,\right]$ is

$$\vec{x}_H = t\vec{v}, \quad t \in \mathbb{R}$$

where $\vec{v}$ is expressed in terms of r_{13} and r_{23}. Show that the general solution of the non-homogeneous system $[R \mid \vec{c}\,]$ is

$$\vec{x}_N = \vec{p} + \vec{x}_H$$

where $\vec{p}$ is expressed in terms of $\vec{c}$.

(b) Let $R = \begin{bmatrix}1 & r_{12} & 0 & 0 & r_{15}\\0 & 0 & 1 & 0 & r_{25}\\0 & 0 & 0 & 1 & r_{35}\end{bmatrix}$. Show that the general solution of the homogeneous system $\left[R \mid \vec{0}\,\right]$ is

$$\vec{x}_H = t_1\vec{v}_1 + t_2\vec{v}_2, \quad t_i \in \mathbb{R}$$

where each of $\vec{v}_1$ and $\vec{v}_2$ can be expressed in terms of the entries r_{ij}. Express each $\vec{v}_i$ explicitly. Then show that the general solution of $[R \mid \vec{c}\,]$ can be written as

$$\vec{x}_N = \vec{p} + \vec{x}_H$$

where $\vec{p}$ is expressed in terms of the components $\vec{c}$, and $\vec{x}_H$ is the solution of the corresponding homogeneous system.

The pattern should now be apparent; if it is not, try again with another special case of R. In the next part of this exercise, create an effective labelling system so that you can clearly indicate what you want to say.

(c) Let R be a matrix in RREF, with m rows, n columns, and rank k. Show that the general solution of the homogeneous system $\left[R \mid \vec{0}\,\right]$ is

$$\vec{x}_H = t_1\vec{v}_1 + \cdots + t_{n-k}\vec{v}_{n-k}, \quad t_i \in \mathbb{R}$$

where each $\vec{v}_i$ is expressed in terms of the entries in R. Suppose that the system $[R \mid \vec{c}\,]$ is consistent and show that the general solution is

$$\vec{x}_N = \vec{p} + \vec{x}_H$$

where $\vec{p}$ is expressed in terms of the components of $\vec{c}$, and $\vec{x}_H$ is the solution of the corresponding homogeneous system.

(d) Use the result of (c) to discuss the relationship between the general solution of the consistent system $\left[A \mid \vec{b}\,\right]$ and the corresponding homogeneous system $\left[A \mid \vec{0}\,\right]$.

F2 *This problem involves comparing the efficiency of row reduction procedures.*

When we use a computer to solve large systems of linear equations, we want to keep the number of arithmetic operations as small as possible. This reduces the time taken for calculations, which is important in many industrial and commercial applications. It also tends to improve accuracy: every arithmetic operation is an opportunity to *lose* accuracy through truncation or round-off, subtraction of two nearly equal numbers, and so on.

We want to count the number of multiplications and/or divisions in solving a system by elimination. We focus on these operations because they are more time-consuming than addition or subtraction, and the number of additions is approximately the same as the number of multiplications. We make certain assumptions: the system $\left[A \mid \vec{b}\,\right]$ has n equations and n variables, and it is consistent with a unique solution. (Equivalently, A has n rows, n columns, and rank n.) We assume for simplicity that no row interchanges are required. (If row interchanges are required, they can be handled by renaming "addresses" in the computer.)

(a) How many multiplications and divisions are required to reduce $\left[A \mid \vec{b}\,\right]$ to a form $\left[C \mid \vec{d}\,\right]$ such that C is in row echelon form?

Hints

(1) To carry out the obvious first elementary row operation, compute $\frac{a_{21}}{a_{11}}$—one division. Since we know what will happen in the first column, we do not multiply a_{11} by $\frac{a_{21}}{a_{11}}$, but we must multiply every other element of the first row of $\left[A \mid \vec{b}\,\right]$ by this factor and subtract the product from the corresponding element of the second row—n multiplications.

(2) Obtain zeros in the remaining entries in the first column, then move to the $(n-1)$ by n blocks consisting of the reduced version of $\left[A \mid \vec{b}\,\right]$ with the first row and first column deleted.

(3) Note that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

(4) The biggest term in your answer should be $n^3/3$. Note that n^3 is much greater than n^2 when n is large.

(b) Determine how many multiplications and divisions are required to solve the system with the augmented matrix $\left[C \mid \vec{d}\,\right]$ of part (a) by back-substitution.

(c) Show that the number of multiplications and divisions required to row reduce $[R \mid \vec{c}\,]$ to reduced row echelon form is the same as the number used in solving the system by back-substitution. Conclude that the Gauss-Jordan procedure is as efficient as Gaussian elimination with back-substitution. For large n, the number of multiplications and divisions is roughly $\frac{n^3}{3}$.

(d) Suppose that we do a "clumsy" Gauss-Jordan procedure. We do not first obtain row echelon form; instead we obtain zeros in all entries above and below a pivot before moving on to the next column. Show that the number of multiplications and divisions required in this procedure is roughly $\frac{n^3}{2}$, so that this procedure requires approximately 50% more operations than the more efficient procedures.

CHAPTER 3

Matrices, Linear Mappings, and Inverses

CHAPTER OUTLINE

In many applications of linear algebra, we use vectors in $\mathbb{R}^n$ to represent quantities, such as forces, and then use the tools of Chapters 1 and 2 to solve various problems. However, there are many times when it is useful to translate a problem into other linear algebra objects. In this chapter, we look at two of these fundamental objects: matrices and linear mappings. We now explore the properties of these objects and show how they are tied together with the material from Chapters 1 and 2.

3.1 Operations on Matrices

We used matrices essentially as bookkeeping devices in Chapter 2. Matrices also possess interesting algebraic properties, so they have wider and more powerful applications than is suggested by their use in solving systems of equations. We now look at some of these algebraic properties.

Definition
Matrix
$M_{m\times n}(\mathbb{R})$

An $m \times n$ **matrix** A is a rectangular array with m rows and n columns. We denote the entry in the i-th row and j-th column by a_{ij}. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Two $m \times n$ matrices A and B are **equal** if $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. The set of all $m \times n$ matrices with real entries is denoted by $M_{m\times n}(\mathbb{R})$.

Remarks

1. When working with multiple matrices we sometimes denote the ij-th entry of a matrix A by $(A)_{ij} = a_{ij}$.
2. In the notation for the set of all $m \times n$ matrices, the $(\mathbb{R})$ indicates that the entries of the matrices are real. In Chapter 9 we will look at the set $M_{m\times n}(\mathbb{C})$, the set of all $m \times n$ matrices with complex entries.

Several special types of matrices arise frequently in linear algebra.

Definition
Square Matrix

A matrix in $M_{n\times n}(\mathbb{R})$ (the number of rows of the matrix is equal to the number of columns) is called a **square matrix**.

Definition
Upper Triangular
Lower Triangular

A matrix $U \in M_{n\times n}(\mathbb{R})$ is said to be **upper triangular** if $u_{ij} = 0$ whenever $i > j$. A matrix $L \in M_{n\times n}(\mathbb{R})$ is said to be **lower triangular** if $l_{ij} = 0$ whenever $i < j$.

EXAMPLE 3.1.1

The matrices $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ are upper triangular.

The matrices $\begin{bmatrix} -3 & 0 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ -2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ are lower triangular.

The matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is both upper and lower triangular.

Definition
Diagonal Matrix

A matrix $D \in M_{n\times n}(\mathbb{R})$ such that $d_{ij} = 0$ for all $i \neq j$ is called a **diagonal matrix** and is denoted by

$$D = \operatorname{diag}(d_{11}, d_{22}, \cdots, d_{nn})$$

EXAMPLE 3.1.2

We denote the diagonal matrix $D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -2 \end{bmatrix}$ by $D = \operatorname{diag}(\sqrt{3}, -2)$.

The notation $\operatorname{diag}(0, 3, 1)$ denotes the diagonal matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Observe that the columns of a diagonal matrix are just scalar multiples of the standard basis vectors for $\mathbb{R}^n$. That is,

$$\operatorname{diag}(d_{11}, d_{22}, \cdots, d_{nn}) = \begin{bmatrix} d_{11}\vec{e}_1 & \cdots & d_{nn}\vec{e}_n \end{bmatrix}$$

Matrices as Vectors

We have seen that matrices are useful in solving systems of linear equations. However, we shall see that matrices show up in different kinds of problems, and *it is important to be able to think of matrices as "things" that are worth studying and playing with—and these things may have no connection with a system of equations*. In particular, we now show that we can treat matrices in exactly the same way as we did vectors in $\mathbb{R}^n$ in Chapter 1.

Definition
Addition and Scalar Multiplication of Matrices

Let $A, B \in M_{m\times n}(\mathbb{R})$. We define **addition** of matrices by

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$$

We define **scalar multiplication** of A by a scalar $t \in \mathbb{R}$ by

$$(tA)_{ij} = t(A)_{ij}$$

Remark

As with vectors in $\mathbb{R}^n$, $A - B$ is to be interpreted as $A + (-1)B$.

EXAMPLE 3.1.3

Perform the following operations.

(a) $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$

Solution: $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 2+5 & 3+1 \\ 4+(-2) & 1+7 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 2 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$

Solution: $\begin{bmatrix} 3 & 0 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3-1 & 0-(-1) \\ 1-(-2) & -5-0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$

(c) $5\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Solution: $5\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5(2) & 5(3) \\ 5(4) & 5(1) \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 20 & 5 \end{bmatrix}$

(d) $2\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$

Solution: $2\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 12 & 0 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2+12 & 6+0 \\ 0+3 & -2+6 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 3 & 4 \end{bmatrix}$

Note that matrix addition is defined only if the matrices are the same size.

Properties of Matrix Addition and Scalar Multiplication

We now look at the properties of addition and scalar multiplication of matrices. It is very important to notice that these are the exact same ten properties discussed in Theorem 1.4.1 for addition and scalar multiplication of vectors in $\mathbb{R}^n$.

Theorem 3.1.1

For all $A, B, C \in M_{m\times n}(\mathbb{R})$ and $s, t \in \mathbb{R}$ we have

(1) $A + B \in M_{m\times n}(\mathbb{R})$ (closed under addition)

(2) $A + B = B + A$ (addition is commutative)

(3) $(A + B) + C = A + (B + C)$ (addition is associative)

(4) There exists a matrix $O_{m,n} \in M_{m\times n}(\mathbb{R})$, such that $A + O_{m,n} = A$ for all $A \in M_{m\times n}(\mathbb{R})$ (zero matrix)

(5) For each $A \in M_{m\times n}(\mathbb{R})$, there exists $(-A) \in M_{m\times n}(\mathbb{R})$ such that $A+(-A) = O_{m,n}$ (additive inverses)

(6) $sA \in M_{m\times n}(\mathbb{R})$ (closed under scalar multiplication)

(7) $s(tA) = (st)A$ (scalar multiplication is associative)

(8) $(s + t)A = sA + tA$ (a distributive law)

(9) $s(A + B) = sA + sB$ (another distributive law)

(10) $1A = A$ (scalar multiplicative identity)

These properties follow easily from the definitions of addition and multiplication by scalars. The proofs are left to the reader.

The matrix $O_{m,n}$, called the **zero matrix**, is the $m \times n$ matrix with all entries as zero. The additive inverse $(-A)$ of a matrix A is defined by $(-A)_{ij} = -(A)_{ij}$.

As before, we call a sum of linear combinations of matrices a **linear combination** of matrices.

Spanning and Linear Independence for Matrices

To stress the fact that matrices can be treated in the same way as vectors in $\mathbb{R}^n$, we now briefly look at the concepts of spanning and linear independence for sets of matrices.

Definition
Span

Let $\mathcal{B} = \{A_1, \dots, A_k\}$ be a set of $m \times n$ matrices. The **span** of $\mathcal{B}$ is defined as

$$\operatorname{Span} \mathcal{B} = \{t_1A_1 + \cdots + t_kA_k \mid t_1, \dots, t_k \in \mathbb{R}\}$$

Definition
Linearly Dependent
Linearly Independent

Let $\mathcal{B} = \{A_1, \dots, A_k\}$ be a set of $m \times n$ matrices. The set $\mathcal{B}$ is said to be **linearly dependent** if there exist real coefficients $c_1, \dots, c_k$ not all zero such that

$$c_1A_1 + \cdots + c_kA_k = O_{m,n}$$

The set $\mathcal{B}$ is said to be **linearly independent** if the only solution to

$$c_1A_1 + \cdots + c_kA_k = O_{m,n}$$

is the trivial solution $c_1 = \cdots = c_k = 0$.

EXAMPLE 3.1.4

Determine whether $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is in the span of $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$.

Solution: We want to find if there exists $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + t_4 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 & t_1 + t_3 + t_4 \\ t_3 & t_2 + t_4 \end{bmatrix}$$

Since two matrices are equal if and only if their corresponding entries are equal, this gives the system of linear equations

$$\begin{aligned} t_1 + t_2 &= 1 \\ t_1 + t_3 + t_4 &= 2 \\ t_3 &= 3 \\ t_2 + t_4 &= 4 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

We see that the system is consistent. Therefore, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is in the span of $\mathcal{B}$. In particular, we have $t_1 = -2$, $t_2 = 3$, $t_3 = 3$, and $t_4 = 1$.

EXAMPLE 3.1.5

Determine whether the set $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \right\}$ is linearly independent.

Solution: We consider the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} + t_2 \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} t_1 + 3t_2 & 2t_1 + 2t_2 \\ 2t_1 + t_2 + 2t_3 & -t_1 + t_2 + 2t_3 \end{bmatrix}$$

This gives the homogeneous system of equations

$$\begin{aligned} t_1 + 3t_2 &= 0 \\ 2t_1 + 2t_2 &= 0 \\ 2t_1 + t_2 + 2t_3 &= 0 \\ -t_1 + t_2 + 2t_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix of this system gives

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The only solution is the trivial solution $t_1 = t_2 = t_3 = 0$, so $\mathcal{C}$ is linearly independent.

EXERCISE 3.1.1

Determine whether $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \right\}$ is linearly independent. Is $X = \begin{bmatrix} 1 & 5 \\ -5 & 1 \end{bmatrix}$ in the span of $\mathcal{B}$?

EXERCISE 3.1.2

Consider $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Prove that $\mathcal{B}$ is linearly independent and show that Span $\mathcal{B} = M_{2\times 2}(\mathbb{R})$. Compare $\mathcal{B}$ with the standard basis for $\mathbb{R}^4$.

Transpose

We will soon see that we sometimes wish to treat the rows of an $m\times n$ matrix as vectors in $\mathbb{R}^n$. To preserve our convention of writing vectors in $\mathbb{R}^n$ as column vectors, we invent some notation for turning columns into rows and vice versa.

Definition
Transpose (vector)

Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. The **transpose** of $\vec{x}$, denoted $\vec{x}^T$, is the **row vector**

$$\vec{x}^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

EXAMPLE 3.1.6

If $\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, then what is the matrix $A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix}$?

Solution: Since $\vec{a}_1^T = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$ and $\vec{a}_2^T = \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$, we get $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{bmatrix}$.

EXERCISE 3.1.3

If $\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vec{a}_3^T \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 0 & 2 \\ 5 & 9 & -3 \end{bmatrix}$, then what are $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$?

We now extend this operation to matrices. We will see throughout the book that the transpose of a matrix is very useful in helping us solve a variety problems.

Definition
Transpose (matrix)

Let $A \in M_{m\times n}(\mathbb{R})$. The **transpose** of A is the $n \times m$ matrix, denoted A^T, whose ij-th entry is the ji-th entry of A. That is,

$$(A^T)_{ij} = (A)_{ji}$$

EXAMPLE 3.1.7

Determine the transpose of $A = \begin{bmatrix} -1 & 6 & -4 \\ 3 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 5 \end{bmatrix}$.

Solution: $A^T = \begin{bmatrix} -1 & 6 & -4 \\ 3 & 5 & 2 \end{bmatrix}^T = \begin{bmatrix} -1 & 3 \\ 6 & 5 \\ -4 & 2 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \end{bmatrix}$.

Theorem 3.1.2

If A and B are matrices, column vectors, or row vectors of the correct size so that the required operations are defined, and $s \in \mathbb{R}$, then

(1) $(A^T)^T = A$

(2) $(A + B)^T = A^T + B^T$

(3) $(sA)^T = sA^T$

Proof: We prove (2) and leave (1) and (3) as exercises. For (2) we have

$$((A + B)^T)_{ij} = (A + B)_{ji} = (A)_{ji} + (B)_{ji} = (A^T)_{ij} + (B^T)_{ij} = (A^T + B^T)_{ij}$$

■

EXERCISE 3.1.4

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 5 \end{bmatrix}$. Verify that $(A^T)^T = A$ and $(3A)^T = 3A^T$.

An Introduction to Matrix Multiplication

The whole purpose of algebra is to use symbols to make writing and manipulating mathematical equations easier. For example, rather than having to write

“The area of a circle is equal to the radius of the circle multiplied by itself and then multiplied by 3.141592653589...,” we instead just write

$$A = \pi r^2$$

We could now use our rules of operations on real numbers to manipulate this equation as desired.

Our goal is to define matrix-vector multiplication and matrix multiplication to facilitate working with linear equations.

Matrix-Vector Multiplication Using Rows

We motivate our first definition of matrix-vector multiplication with an example.

EXAMPLE 3.1.8

According to the USDA, 1 kg of 1% milk contains 34 grams of protein and 50 grams of sugar, and 1 kg of apples contains 3 grams of protein and 100 grams of sugar. If one drinks m kg of milk and eats a kg of apples, then the amount of protein p and the amount of sugar s they will have consumed is given by the system of linear equations

$$\begin{aligned} 34m + 3a &= p \\ 50m + 100a &= s \end{aligned}$$

Observe that we can write the amount of protein being consumed as the dot product

$$\begin{bmatrix} 34 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ a \end{bmatrix} = p$$

Similarly, the amount of sugar being consumed is

$$\begin{bmatrix} 50 \\ 100 \end{bmatrix} \cdot \begin{bmatrix} m \\ a \end{bmatrix} = s$$

If we let $\vec{a}_1 = \begin{bmatrix} 34 \\ 3 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} m \\ a \end{bmatrix}$, then, since vectors in $\mathbb{R}^n$ are equal if and only if they have equal entries, we can represent the system of linear equations in the form

$$\begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} p \\ s \end{bmatrix} \tag{3.1}$$

We now define **matrix-vector multiplication**, so that we can write equation (3.1) in the form of

$$A\vec{x} = \begin{bmatrix} p \\ s \end{bmatrix}$$

where $A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix}$ is the coefficient matrix of the original system of linear equations.

Definition
Matrix-Vector Multiplication

Let $A \in M_{m\times n}(\mathbb{R})$ whose rows are denoted $\vec{a}_i^T$ for $1 \leq i \leq m$. For any $\vec{x} \in \mathbb{R}^n$, we define $A\vec{x}$ by

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$$

It is important to note that if A is an $m\times n$ matrix, then $A\vec{x}$ is defined only if $\vec{x} \in \mathbb{R}^n$. Moreover, if $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} \in \mathbb{R}^m$.

EXAMPLE 3.1.9

Let $A = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 0 & 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Calculate $A\vec{x}$, $A\vec{y}$, and $A\vec{z}$.

Solution: Using the definition of matrix-vector multiplication gives

$$A\vec{x} = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 3(2) + 4(-1) + (-5)(6) \\ 1(2) + 0(-1) + (2)(6) \end{bmatrix} = \begin{bmatrix} -28 \\ 14 \end{bmatrix}$$

$$A\vec{y} = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3(1) + 4(0) + (-5)(0) \\ 1(1) + 0(0) + (2)(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A\vec{z} = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(0) + 4(0) + (-5)(1) \\ 1(0) + 0(0) + (2)(1) \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

EXERCISE 3.1.5

Calculate the following matrix-vector products.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -4 \\ 3 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

EXAMPLE 3.1.10

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \\ 9 & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix}$. Write the system of linear equations represented by $A\vec{x} = \vec{b}$.

Solution: By definition, we have that

$$A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & -4 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - 4x_2 \\ 9x_1 - x_2 \end{bmatrix}$$

Hence, $A\vec{x} = \vec{b}$ gives

$$\begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - 4x_2 \\ 9x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix}$$

Since vectors are equal if and only if their corresponding entries are equal, this gives the system of linear equations

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - 4x_2 &= 0 \\ 9x_1 - x_2 &= 8 \end{aligned}$$

EXERCISE 3.1.6

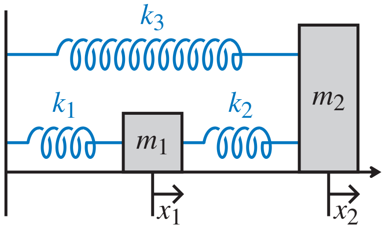
Write the following system of linear equations in the form $A\vec{x} = \vec{b}$.

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 \phantom{{}-3x_2} &= 15 \\ -2x_1 - 3x_2 &= -5 \end{aligned}$$

EXAMPLE 3.1.11

Consider the spring-mass system depicted below. Rather than finding the equilibrium displacement as we did in Section 2.4, we will now determine how to find the forces f_1 and f_2 acting on the masses m_1 and m_2 given displacements x_1 and x_2. By Hooke's Law, we find that the forces f_1 and f_2 satisfy

$$\begin{aligned} f_1 &= (k_1 + k_2)x_1 - k_2x_2 \\ f_2 &= -k_2x_1 + (k_2 + k_3)x_2 \end{aligned}$$



Using matrix-vector multiplication, we can rewrite this as

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

We call the matrix $K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$ the **stiffness matrix** of the system.

In most cases, we will now represent a system of linear equations $\left[A \mid \vec{b} \right]$ as $A\vec{x} = \vec{b}$. This will allow us to use properties of matrix multiplication (see Theorem 3.1.4 below) when dealing with systems of linear equations.

Matrix-Vector Multiplication Using Columns

In Example 3.1.8 we derived a definition of matrix-vector multiplication by focusing on each equation. However, in some cases it may make more sense to focus on the products themselves. In particular, observe that we instead could write the system

$$\begin{aligned} 34m + 3a &= p \\ 50m + 100a &= s \end{aligned}$$

in the form

$$\begin{bmatrix} 34m + 3a \\ 50m + 100a \end{bmatrix} = \begin{bmatrix} p \\ s \end{bmatrix}$$

Then, we could use operations on vectors to get

$$m\begin{bmatrix} 34 \\ 50 \end{bmatrix} + a\begin{bmatrix} 3 \\ 100 \end{bmatrix} = \begin{bmatrix} p \\ s \end{bmatrix}$$

We still want the left-hand side to be written in the form $A\vec{x}$. So, this gives us an alternative definition of matrix-vector multiplication.

Definition
Matrix-Vector Multiplication

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in M_{m\times n}(\mathbb{R})$. For any $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, we define $A\vec{x}$ by

$$A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

Remarks

1. As before, if A is an $m\times n$ matrix, then $A\vec{x}$ only makes sense if $\vec{x} \in \mathbb{R}^n$. Moreover, if $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} \in \mathbb{R}^m$.
2. By definition, for any $\vec{x} \in \mathbb{R}^n$, $A\vec{x}$ is a linear combination of the columns of A.

EXAMPLE 3.1.12

Let $A = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 0 & 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Calculate $A\vec{x}$ and $A\vec{y}$.

Solution: We have

$$A\vec{x} = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} -28 \\ 14 \end{bmatrix}$$

$$A\vec{y} = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

EXERCISE 3.1.7

Calculate the following matrix-vector products by expanding them as a linear combination of the columns of the matrix.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -4 \\ 3 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Observe that this exercise contained the same problems as Exercise 3.1.5. This was to demonstrate that both definitions of matrix-vector multiplication indeed give the same answer. At this point you may wonder why we have two different definitions for the same thing. The reason is that they both have different uses. We use the first method for computing matrix-vector products when we want to work with the rows of a matrix. When we are working with the columns of a matrix or linear combinations of vectors, then we use the second method.

The following is a simple but useful theorem.

Theorem 3.1.3

If $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in M_{m\times n}(\mathbb{R})$ and $\vec{e}_i$ is the i-th standard basis vector of $\mathbb{R}^n$, then

$$A\vec{e}_i = \vec{a}_i$$

The proof is left as Problem A34.

Matrix Multiplication

It is important to observe that matrix-vector multiplication behaves like a function. It inputs a vector in $\mathbb{R}^n$ and outputs a vector in $\mathbb{R}^m$. For instance, in Example 3.1.8 we formed the function which inputs the amount of milk and apples and outputs the amount of protein and sugar. We now want to define matrix multiplication to represent a composition of functions. We again use an example to demonstrate this.

EXAMPLE 3.1.13 Suppose that apple rice pudding contains 300 g of milk and 240 g of apples, while apple milk drink contains 250 g of milk and 100 g of apples. If we let $B = \begin{bmatrix} .3 & .25 \\ .24 & .1 \end{bmatrix}$, then we can represent the amount of milk m and apples a consumed by having r kg of apple rice pudding and d kg of apple milk drink by

$$\begin{bmatrix} m \\ a \end{bmatrix} = \begin{bmatrix} .3r + .25d \\ .24r + .1d \end{bmatrix} = \begin{bmatrix} .3 & .25 \\ .24 & .1 \end{bmatrix}\begin{bmatrix} r \\ d \end{bmatrix} = B\begin{bmatrix} r \\ d \end{bmatrix}$$

Using the matrix $A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix} = \begin{bmatrix} 34 & 3 \\ 50 & 100 \end{bmatrix}$ from Example 3.1.8, we get that the amount of protein and sugar consumed by having these is

$$\begin{aligned}
\begin{bmatrix} p \\ s \end{bmatrix} &= A\begin{bmatrix} m \\ a \end{bmatrix} \\
&= \begin{bmatrix} 34 & 3 \\ 50 & 100 \end{bmatrix}\begin{bmatrix} .3r + .25d \\ .24r + .1d \end{bmatrix} \\
&= \begin{bmatrix} 34(.3r + .25d) + 3(.24r + .1d) \\ 50(.3r + .25d) + 100(.24r + .1d) \end{bmatrix} \\
&= \begin{bmatrix} \big(34(.3) + 3(.24)\big)r + \big(34(.25) + 3(.1)\big)d \\ \big(50(.3) + 100(.24)\big)r + \big(50(.25) + 100(.1)\big)d \end{bmatrix} \\
&= \begin{bmatrix} 34(.3) + 3(.24) & 34(.25) + 3(.1) \\ 50(.3) + 100(.24) & 50(.25) + 100(.1) \end{bmatrix}\begin{bmatrix} r \\ d \end{bmatrix}
\end{aligned}$$

Since we also have that

$$\begin{bmatrix} p \\ s \end{bmatrix} = A\begin{bmatrix} m \\ a \end{bmatrix} = AB\begin{bmatrix} r \\ d \end{bmatrix}$$

This shows us how to define the matrix product AB.

Careful inspection of the result above shows that the product AB must be defined by the following rules:

- $(AB)_{11}$ is the dot product of the first row of A and the first column of B.
- $(AB)_{12}$ is the dot product of the first row of A and the second column of B.
- $(AB)_{21}$ is the dot product of the second row of A and the first column of B.
- $(AB)_{22}$ is the dot product of the second row of A and the second column of B.

EXAMPLE 3.1.14

Calculate $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$.

Solution: Taking dot products of the rows of the first matrix with columns of the second matrix gives

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 2(5)+3(-2) & 2(1)+3(7) \\ 4(5)+1(-2) & 4(1)+1(7) \end{bmatrix} = \begin{bmatrix} 4 & 23 \\ 18 & 11 \end{bmatrix}$$

To formalize the definition of **matrix multiplication**, it will be convenient to use $\vec{a}_i^T$ to represent the i-th row of A and $\vec{b}_j$ to represent the j-th column of B. Observe from our work above that we want the ij-th entry of AB to be the dot product of the i-th row of A and the j-th column of B. However, for this to be defined, $\vec{a}_i^T$ must have the same number of entries as $\vec{b}_j$. Hence, the number of entries in the rows of the matrix A (that is, the number of columns of A) must be equal to the number of entries in the columns of B (that is, the number of rows of B). We can now make a precise definition.

Definition
Matrix Multiplication

Let $A \in M_{m\times n}(\mathbb{R})$ with rows $\vec{a}_1^T, \dots, \vec{a}_m^T$ and let $B \in M_{n\times p}(\mathbb{R})$ with columns $\vec{b}_1, \dots, \vec{b}_p$. We define AB to be the $m \times p$ matrix whose ij-th entry is

$$(AB)_{ij} = \vec{a}_i \cdot \vec{b}_j$$

To emphasize the point, if A is an $m \times n$ matrix and B is a $q \times p$ matrix, then AB is defined only if $n = q$.

EXAMPLE 3.1.15

Calculate $\begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & 3 \\ 0 & 5 \end{bmatrix}$.

Solution: We have

$$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2(3)+3(1)+0(2)+1(0) & 2(1)+3(2)+0(3)+1(5) \\ 4(3)+1(1)+2(2)+1(0) & 4(1)+1(2)+2(3)+1(5) \\ 0(3)+0(1)+0(2)+1(0) & 0(1)+0(2)+0(3)+1(5) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 13 \\ 17 & 17 \\ 0 & 5 \end{bmatrix}$$

EXAMPLE 3.1.16

Calculate the following or explain why they are not defined.

(a) $\begin{bmatrix} 1 & 1 & 2 \\ -3 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 1 \\ 5 & 7 \end{bmatrix}$

Solution: For (a) we have

$$\begin{bmatrix} 1 & 1 & 2 \\ -3 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1(5)+1(4)+2(2) & 1(6)+1(7)+2(5) \\ (-3)(5)+1(4)+3(2) & (-3)(6)+1(7)+3(5) \\ 0(5)+0(4)+1(2) & 0(6)+0(7)+1(5) \end{bmatrix} = \begin{bmatrix} 13 & 23 \\ -5 & 4 \\ 2 & 5 \end{bmatrix}$$

For (b) we see that the product is not defined because the first matrix has two columns but the second matrix has three rows.

EXERCISE 3.1.8

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Calculate the following or explain why they are not defined.

(a) AB (b) BA (c) A^TA (d) BB^T

EXAMPLE 3.1.17

Let $A = \begin{bmatrix} 5 & 3 & -1 \\ 4 & 2 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Calculate $\vec{x}^T A$.

Solution: To compute this, we interpret the row vector $\vec{x}^T$ as a 1×2 matrix and use the definition of matrix multiplication. Since the number of columns of the first matrix equals the number of rows of the second matrix, the product is defined. We get

$$\begin{aligned} \vec{x}^T A &= \begin{bmatrix} -3 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 & -1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (-3)(5)+1(4) & (-3)(3)+1(2) & (-3)(-1)+1(1) \end{bmatrix} \\ &= \begin{bmatrix} -11 & -7 & 4 \end{bmatrix} \end{aligned}$$

EXAMPLE 3.1.18

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{y} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \in \mathbb{R}^3$. Compute $\vec{x}^T\vec{y}$.

Solution: Using the definition of matrix multiplication, we get

$$\vec{x}^T\vec{y} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1(6)+2(5)+3(4) \end{bmatrix} = \begin{bmatrix} 28 \end{bmatrix}$$

Observe that the vectors $\vec{x}$ and $\vec{y}$ in Example 3.1.18 satisfy $\vec{x} \cdot \vec{y} = 28$ which matches the result in the example. This should not be surprising since we have defined matrix multiplication in terms of the dot product. More generally, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have

$$\vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$$

where we interpret the 1×1 matrix on the right-hand side as a scalar. This formula will be used frequently later in the book.

Defining matrix multiplication with the dot product fits our first view of matrix-vector multiplication. We now look at how we could define matrix multiplication by using our alternate view of matrix-vector multiplication.

EXAMPLE 3.1.19

In Example 3.1.13 we found that $\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} 34(.3) + 3(.24) & 34(.25) + 3(.1) \\ 50(.3) + 100(.24) & 50(.25) + 100(.1) \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$. Observe that the first column of this matrix can be written as

$$\begin{bmatrix} 34(.3) + 3(.24) \\ 50(.3) + 100(.24) \end{bmatrix} = (.3)\begin{bmatrix} 34 \\ 50 \end{bmatrix} + (.24)\begin{bmatrix} 3 \\ 100 \end{bmatrix} = A\begin{bmatrix} .3 \\ .24 \end{bmatrix}$$

Similarly, the second column of the matrix is

$$\begin{bmatrix} 34(.25) + 3(.1) \\ 50(.25) + 100(.1) \end{bmatrix} = (.25)\begin{bmatrix} 34 \\ 50 \end{bmatrix} + (.1)\begin{bmatrix} 3 \\ 100 \end{bmatrix} = A\begin{bmatrix} .25 \\ .1 \end{bmatrix}$$

So, the i-th column of AB is the matrix-vector product of A and the i-th column of B.

Thus, we can alternatively define matrix multiplication in the following way.

Definition
Matrix Multiplication

For $A \in M_{m\times n}(\mathbb{R})$ and $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix} \in M_{n\times p}(\mathbb{R})$ we define AB to be the $m \times p$ matrix

$$AB = A\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix} \tag{3.2}$$

EXAMPLE 3.1.20

Calculate $\begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & 3 \\ 0 & 5 \end{bmatrix}$.

Solution: We have

$$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 17 \\ 5 \end{bmatrix}$$

Hence,

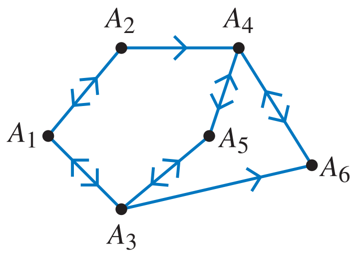
$$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 13 \\ 17 & 17 \\ 0 & 5 \end{bmatrix}$$

Both interpretations of matrix multiplication will be very useful, so it is important to know and understand both of them.

CONNECTION

We now see that linear combinations of vectors (and hence concepts such as spanning and linear independence), solving systems of linear equations, and matrix multiplication are all closely tied together. We will continue to see these connections later in this chapter and throughout the book.

EXAMPLE 3.1.21 The diagram indicates how six webpages are connected by hyperlinks. For example, the single directional arrow from A_3 to A_6 indicates there is a hyperlink on page A_3 that takes the user to page A_6. The double directional arrow between A_4 and A_5 indicates that there is a hyperlink on page A_4 to page A_5 and a hyperlink on page A_5 to A_4.



We create a matrix C to represent the network by putting a 1 in the i, j-th entry of C if webpage A_i has a hyperlink to page A_j. Thus, because of the connection from A_3 to A_6 we get $(C)_{3,6} = 1$. Since A_6 does not have a hyperlink to A_3, we have that $(C)_{6,3} = 0$. The bidirectional connection between A_4 and A_5 means that we have $(C)_{4,5} = 1$ and $(C)_{5,4} = 1$. Note, that in our example, no page links to itself (although that certainly happens), so we have $(C)_{ii} = 0$ for $1 \leq i \leq 6$. Completing the matrix we find that

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

We call this the **connectivity matrix** of the network as it indicates all the direct communications (travel) between webpages.

Matrix multiplication has been defined to represent a composition of functions. So, if the matrix C indicates travel from one webpage to another by following a single hyperlink, then $C^2 = CC$ indicates travel from one webpage to another by following two hyperlinks. We find that

$$C^2 = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The fact that $(C^2)_{3,4} = 2$ indicates that there are two ways of clicking a sequence of two hyperlinks from A_3 to A_4, while $(C^2)_{1,2} = 0$ indicates that there is no sequence of two hyperlinks (even though there is a direct link) from A_1 to A_2. A sequence of n hyperlinks links moving from one page to another is called a **walk of length** n.

The ij-th entry of $C + C^2 + C^3$ is the number of walks of length at most 3 from A_i to A_j. For example, we can calculate that $(C + C^2 + C^3)_{1,6} = 2$. This indicates that there are two ways of getting from page A_1 to page A_6 by clicking at most 3 hyperlinks. The value $(C + C^2 + C^3)_{6,2} = 0$ indicates that it would take clicking more than 3 hyperlinks to move from page A_6 to page A_2.

Properties of Matrix Multiplication

Theorem 3.1.4 If A, B, and C are matrices, column vectors, or row vectors of the correct size so that the required operations are defined, and $s \in \mathbb{R}$, then

(1) $A(B + C) = AB + AC$

(2) $(A + B)C = AC + BC$

(3) $s(AB) = (sA)B = A(sB)$

(4) $A(BC) = (AB)C$

(5) $(AB)^T = B^T A^T$

These properties follows easily from the definition of matrix multiplication. However, the proofs are not particularly illuminating and so are omitted.

Three Important Facts:

1. Matrix multiplication is not commutative: That is, in general, $AB \neq BA$. In fact, if BA is defined, it is not necessarily true that AB is even defined. For example, if B is 2×2 and A is 2×3, then BA is defined, but AB is not. However, even if both AB and BA are defined, they are usually not equal. $AB = BA$ is true only in very special circumstances.

EXAMPLE 3.1.22

Show that if $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$, then $AB \neq BA$.

Solution: $AB = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 23 \\ 22 & -3 \end{bmatrix}$,

but

$$BA = \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 14 \\ 24 & -13 \end{bmatrix}$$

When multiplying both sides of a matrix equation by a matrix we must either multiply it on the right of both sides of the equation or on the left of both sides of the equation. That is, if we have the matrix equation

$$AX = B$$

then multiplying by the matrix C can give either

$$CAX = CB \quad \text{or} \quad AXC = BC$$

2. The cancellation law is almost never valid for matrix multiplication: That is, if $AB = AC$, then we cannot guarantee that $B = C$.

EXAMPLE 3.1.23

Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = AC$$

so $AB = AC$ but $B \neq C$.

Remark

The fact that we do not have the cancellation law for matrix multiplication comes from the fact that **we do not have division for matrices**.

We must distinguish carefully between a general cancellation law and the following theorem, which we will use many times.

Theorem 3.1.5 Matrices Equal Theorem

If $A, B \in M_{m\times n}(\mathbb{R})$ such that $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$, then $A = B$.

You are asked to prove Theorem 3.1.5, with hints, in Problem C1.

Note that it is the assumption that equality holds for *every* $\vec{x} \in \mathbb{R}^n$ that distinguishes this from a cancellation law.

3. $AB = O_{m,n}$ does not imply that one of A or B is the zero matrix.

EXAMPLE 3.1.24

If $A = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
But neither A nor B is the zero matrix.

Identity Matrix

We have seen that the zero matrix $O_{m,n}$ is the additive identity for addition of $m \times n$ matrices. However, since we also have multiplication of matrices, it is important to determine whether we have a multiplicative identity. If we do, we need to determine what the multiplicative identity is. First, we observe that for there to exist a matrix A and a matrix I such that $AI = A = IA$, both A and I must be $n \times n$ matrices as otherwise either AI or IA is undefined. The multiplicative identity I is the $n \times n$ matrix that has this property for all $n \times n$ matrices A.

To find how to define I, we begin with a simple case. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want to find a matrix $I = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ such that $AI = A$. By matrix multiplication, we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Thus, we must have $a = ae + bg$, $b = af + bh$, $c = ce + dg$, and $d = cf + dh$. Although this system of equations is not linear, it is still easy to solve. We find that we must have $e = 1 = h$ and $f = g = 0$. Thus,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \operatorname{diag}(1, 1)$$

It is easy to verify that I also satisfies $IA = A$. Hence, I is the multiplicative identity for 2×2 matrices. We now extend this definition to the $n \times n$ case.

Definition
Identity Matrix

The $n \times n$ matrix $I = \text{diag}(1, 1, \dots, 1)$ is called the **identity matrix**.

EXAMPLE 3.1.25

The 3×3 identity matrix is $I = \text{diag}(1, 1, 1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The 4×4 identity matrix is $I = \text{diag}(1, 1, 1, 1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Remarks

1. In general, the size of I (the value of n) is clear from the given context. However, in some cases, we stress the size of the identity matrix by denoting it by I_n. For example, I_2 is the 2×2 identity matrix, and I_m is the $m \times m$ identity matrix.

2. The columns of the identity matrix should seem familiar. If $\{\vec{e}_1, \dots, \vec{e}_n\}$ is the standard basis for $\mathbb{R}^n$, then

$$I_n = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix}$$

Theorem 3.1.6

If $A \in M_{m\times n}(\mathbb{R})$, then $I_m A = A = A I_n$.

You are asked to prove this theorem in Problem C2. It implies that I_n is the multiplicative identity for the set of $n \times n$ matrices.

Remark

Often the best way of understanding a theorem or definition is to write down some simple examples. So, to help us understand Theorem 3.1.6, we randomly choose a matrix A, say $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}$. Since A is 2×3, to multiply this by I_m on the left we must have $m = 2$ so that the matrix multiplication is valid. Similarly, to multiply A on the right by I_n we must have $n = 3$. So, Theorem 3.1.6 says that we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EXERCISE 3.1.9

Let $A \in M_{m\times n}(\mathbb{R})$ such that the system of linear equations $A\vec{x} = \vec{e}_i$ is consistent for all $1 \leq i \leq m$.

(a) Prove that the system of equations $A\vec{x} = \vec{y}$ is consistent for all $\vec{y} \in \mathbb{R}^m$.

(b) What can you conclude about the rank of A from the result in part (a)?

(c) Prove that there exists a matrix B such that $AB = I_m$.

(d) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Use your method in part (c) to find a matrix B such that $AB = I_2$.

Block Multiplication

Observe that in our second interpretation of matrix multiplication, equation (3.2), we calculated the product AB in **blocks**. That is, we computed the smaller matrix products $A\vec{b}_1, A\vec{b}_2, \dots, A\vec{b}_p$ and put these in the appropriate positions to create AB. This is a very simple example of **block multiplication**. Observe that we could also regard the rows of A as blocks and write

$$AB = \begin{bmatrix} \vec{a}_1^T B \\ \vdots \\ \vec{a}_p^T B \end{bmatrix}$$

There are more general statements about the products of two matrices, each of which have been partitioned into blocks. In addition to clarifying the meaning of some calculations, block multiplication is used in organizing calculations with very large matrices.

Roughly speaking, as long as the sizes of the blocks are chosen so that the products of the blocks are defined and fit together as required, block multiplication is defined by an extension of the usual rules of matrix multiplication. We demonstrate this with a couple of examples.

EXAMPLE 3.1.26

Suppose that $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$ such that A and B are **partitioned** into blocks as indicated:

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$$

Say that A_1 is $r \times n$ so that A_2 is $(m - r) \times n$, while B_1 is $n \times q$ and B_2 is $n \times (p - q)$. Now, the product of a 2×1 matrix and a 1×2 matrix is given by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}$$

So, for the partitioned block matrices, we have

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{bmatrix}$$

Observe that all the products are defined and the size of the resulting matrix is $m \times p$.

EXAMPLE 3.1.27

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$.

Let $A_{11} = \begin{bmatrix} 1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 2 & -3 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A_{22} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ so that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$.

Let $B_{11} = \begin{bmatrix} 2 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 3 & 1 \end{bmatrix}$, $B_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $B_{22} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$, so that $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

Use block multiplication to calculate AB.

Solution: According to normal matrix-matrix multiplication rules we have

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Computing each entry we get

$$A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} 2 \end{bmatrix} + \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -13 \end{bmatrix} = \begin{bmatrix} 5 & -12 \end{bmatrix}$$

$$A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$A_{21}B_{12} + A_{22}B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 7 \end{bmatrix}$$

Hence,

$$AB = \begin{bmatrix} 2 & 5 & -12 \\ 0 & 3 & -3 \\ 2 & 3 & 7 \end{bmatrix}$$

Remark

To understand why block multiplication works, try multiplying out AB without using block multiplication. Carefully compare your calculations for each entry with the calculations in Example 3.1.27.

PROBLEMS 3.1

Practice Problems

For Problems A1–A9, compute the expression or explain why it is not defined. Let

$A = \begin{bmatrix} 2 & -2 & 3 \\ 4 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -4 & 1 \\ 2 & -5 & 3 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 3 \\ -1/2 & 1/3 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

A1 $A + B$ **A2** $3A - 2B + C^T$ **A3** $C\vec{x}$
A4 AB **A5** AC **A6** CB
A7 CD **A8** $D^TA\vec{x}$ **A9** $\vec{x}^T\vec{x}$

For Problems A10–A13, find a matrix A and vectors $\vec{x}$ and $\vec{b}$ such that $A\vec{x} = \vec{b}$ represents the given system.

A10
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 - x_2 + 5x_3 &= 5 \end{aligned}$$

A11
$$\begin{aligned} x_1 - 4x_2 + x_3 - 2x_4 &= 1 \\ x_1 - x_2 + 3x_3 &= 0 \end{aligned}$$

A12
$$\begin{aligned} \tfrac{1}{3}x_1 + 3x_2 - \tfrac{1}{4}x_3 &= 1 \\ x_1 + x_3 &= \tfrac{2}{3} \\ x_1 - x_2 \phantom{- \tfrac{1}{4}x_3} &= 3 \end{aligned}$$

A13
$$\begin{aligned} x_1 - x_2 &= 3 \\ 3x_1 + x_2 &= 4 \\ 5x_1 - 8x_2 &= 17 \end{aligned}$$

For Problems A14–A19, determine whether the statement is true or false. Justify your answer.

A14 If $A \in M_{3\times 2}(\mathbb{R})$ and $A\vec{x}$ is defined, then $\vec{x} \in \mathbb{R}^2$.

A15 If $A \in M_{2\times 4}(\mathbb{R})$ and $A\vec{x}$ is defined, then $A\vec{x} \in \mathbb{R}^4$.

A16 If $A \in M_{3\times 3}(\mathbb{R})$, then there is no matrix B such that $AB = BA$.

A17 If $A \in M_{m\times n}(\mathbb{R})$, then A^TA is a square matrix.

A18 The only 2×2 matrix A such that $A^2 = O_{2,2}$ is $O_{2,2}$. (NOTE: As usual, by A^2 we mean AA.)

A19 If A and B are 2×2 matrices such that $AB = O_{2,2}$, then either $A = O_{2,2}$ or $B = O_{2,2}$.

For Problems A20 and A21, check whether $A + B$ and AB are defined. If so, check that $(A + B)^T = A^T + B^T$ and/or $(AB)^T = B^TA^T$.

A20 $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & -3 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$

A21 $A = \begin{bmatrix} 2 & -4 & 5 \\ 4 & 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -4 \\ 5 & -2 \\ 1 & 3 \end{bmatrix}$

For Problems A22–A30, compute the product or explain why it is not defined. Let

$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & -4 \\ 3 & 5 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 \\ 1 & 3 \\ 4 & -3 \end{bmatrix}$,

$D = \begin{bmatrix} 4 & 3 & 2 & 1 \\ -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

A22 AB **A23** BA **A24** DC
A25 C^TD **A26** $\vec{x}^T\vec{y}$ **A27** $\vec{x}\vec{x}$
A28 $A(BC)$ **A29** $(AB)C$ **A30** $(AB)^T$

A31 Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & -1 & 4 \\ -1 & 0 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

(a) Determine $A\vec{x}$, $A\vec{y}$, and $A\vec{z}$ using both definitions of matrix-vector multiplication.

(b) Use the result of (a) to determine $A\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ 4 & -1 & 1 \end{bmatrix}$.

A32 Verify the following case of block multiplication by calculating both sides of the equation and comparing.

$$\left[\begin{array}{cc|cc} 2 & 3 & -4 & 5 \\ -4 & 1 & 2 & 1 \end{array}\right]\left[\begin{array}{cc} 6 & 3 \\ -2 & 4 \\ \hline 1 & 3 \\ -3 & 2 \end{array}\right]$$
$$= \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$$

A33 Let $\mathcal{B} = \left\{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\right\}$.

(a) Determine whether $A = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix}$ is in Span $\mathcal{B}$.

(b) Determine whether the set $\mathcal{B}$ is linearly independent.

A34 Prove if $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ is an $m \times n$ matrix and $\vec{e}_i$ is the i-th standard basis vector of $\mathbb{R}^n$, then $A\vec{e}_i = \vec{a}_i$.

Homework Problems

For Problems B1–B12, compute the expression or explain why it is not defined. Let $A = \begin{bmatrix} 8 & 1 \\ -2 & 1 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -4 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $E = \begin{bmatrix} 2 & 7 \\ 6 & -1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

B1 $A + C$ **B2** $2A + B - 2C$ **B3** $A + D$

B4 $C\vec{x}$ **B5** $C^T\vec{x}$ **B6** DE

B7 AE **B8** D^TB **B9** ED

B10 $E\vec{x}$ **B11** $AC^T\vec{x}$ **B12** $\vec{x}^T\vec{x}$

For Problems B13–B16, find a matrix A and vectors $\vec{x}$ and $\vec{b}$ such that $A\vec{x} = \vec{b}$ represents the given system.

B13 $\begin{aligned} x_1 - 3x_2 + x_3 - x_4 &= 1 \\ x_1 + x_2 + 3x_3 + 4x_4 &= 5 \end{aligned}$

B14 $\begin{aligned} 2x_1 + x_2 + 5x_3 &= 0 \\ 3x_1 - x_2 - 2x_3 &= 0 \end{aligned}$

B15 $\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 9 \\ 2x_1 - 3x_2 &= -3 \end{aligned}$

B16 $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 8x_1 - x_2 &= 1 \\ 7x_1 - 4x_2 &= 1 \end{aligned}$

For Problems B17–B22, determine whether the statement is true or false. Justify your answer.

B17 If $A, B \in M_{2\times 2}(\mathbb{R})$, then $AB = BA$.

B18 If $A \in M_{4\times 3}(\mathbb{R})$ and $A\vec{x}$ is defined, then $A\vec{x} \in \mathbb{R}^4$.

B19 If $A \in M_{2\times 3}(\mathbb{R})$, then there is no matrix B such that $AB = BA$.

B20 If $A, B \in M_{2\times 2}(\mathbb{R})$, then $(A + B)(A - B) = A^2 - B^2$.

B21 If $A \in M_{m\times n}(\mathbb{R})$ and $A^T = A$, then A is a square matrix.

B22 If $A, B \in M_{2\times 2}(\mathbb{R})$ and $A\vec{x} = B\vec{x}$ for some $\vec{x} \in \mathbb{R}^2$, then $A = B$.

For Problems B23 and B24, check whether $A + B$ and AB are defined. If so, check that $(A + B)^T = A^T + B^T$ and/or $(AB)^T = B^TA^T$.

B23 $A = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$

B24 $A = \begin{bmatrix} 6 & 9 \\ -2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & -5 \end{bmatrix}$

For Problems B25–B36, compute the product or explain why it is not defined. Let $A = \begin{bmatrix} 6 & 3 & 3 \\ -1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 3 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

B25 AB **B26** BA **B27** AC

B28 CA **B29** $A\vec{x}$ **B30** $B\vec{z}$

B31 $\vec{x}^T\vec{y}$ **B32** $\vec{y}^T\vec{z}$ **B33** $\vec{z}^TC\vec{z}$

B34 $A(BC)$ **B35** $(CA)^T$ **B36** A^TC^T

B37 Let $A = \begin{bmatrix} 2 & -8 & 5 \\ 0 & 1 & 5 \\ -7 & 3 & 9 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Determine $A\vec{x}$, $A\vec{y}$, and $A\vec{z}$ using both definitions of matrix-vector multiplication.

(b) Use the result of (a) to determine $A\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 0 \\ 3 & -2 & 1 \end{bmatrix}$.

B38 Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 1 & -4 & 6 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ -7 \\ -5 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Determine $A\vec{x}$, $A\vec{y}$, and $A\vec{z}$ using both definitions of matrix-vector multiplication.

(b) Use the result of (a) to determine $A\begin{bmatrix} 2 & 0 & -1 \\ -7 & 3 & 1 \\ -5 & 2 & 1 \end{bmatrix}$.

B39 Verify the following case of block multiplication by calculating both sides of the equation and comparing.

$$\left[\begin{array}{c|cc} 6 & 3 & -2 \\ 1 & 2 & -1 \end{array}\right]\left[\begin{array}{cc} -4 & 8 \\ \hline 2 & 1 \\ 0 & 6 \end{array}\right] = \begin{bmatrix} 6 \\ 1 \end{bmatrix}\begin{bmatrix} -4 & 8 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix}$$

B40 Let $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -4 & -1 \end{bmatrix}\right\}$.

(a) Determine whether $A = \begin{bmatrix} 2 & 3 \\ 10 & -1 \end{bmatrix}$ is in Span $\mathcal{B}$.

(b) Determine whether the set $\mathcal{B}$ is linearly independent.

B41 Let $C = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.

(a) Determine whether $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is in Span C.

(b) Determine whether the set C is linearly independent.

B42 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$.

(a) Determine whether $A = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$ is in Span $\mathcal{B}$.

(b) Determine whether the set $\mathcal{B}$ is linearly independent.

B43 Denote the number of individuals in year t of two adjacent cities Neville and Maggiton by n_t and m_t respectively. Suppose that the only changes in these populations occur via individuals moving from one city to the other. Suppose that each year, $\frac{1}{10}$ of the population of Neville moves to Maggiton, while $\frac{1}{5}$ of the population of Maggiton migrates to Neville. We can express the year-to-year change in the populations as:

$$n_{t+1} = \frac{9}{10}x_t + \frac{1}{5}y_t$$
$$m_{t+1} = \frac{1}{10}x_t + \frac{4}{5}y_t$$

(a) Express the year-to-year change in the population as a matrix-vector product.

$$\begin{bmatrix} n_{t+1} \\ m_{t+1} \end{bmatrix} = A \begin{bmatrix} n_t \\ m_t \end{bmatrix}$$

(b) Suppose that in year $t = 0$ the populations are given by $n_0 = 1000$, $m_0 = 2000$. Use your formula from part (a) to calculate the populations in years $t = 1, 2$, and 3. Confirm that in each year, the total population $(n_t + m_t)$ is 3000. Why should this be expected?

(c) Calculate $A^3 = AAA$ and use it to calculate $A^3 \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$. What would $A^{10} \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$ represent?

For Problems B44 and B45:

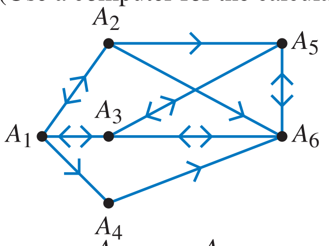
(a) Find the connectivity matrix.

(b) Determine how many paths of length 3 there are from A_2 to A_6.

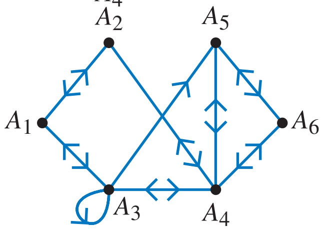
(c) Determine how many paths of length at most 3 there are from A_5 to A_1.

(Use a computer for the calculations.)

B44



B45



B46 The Fibonacci sequence f_n is defined by

$$f_1 = 1, \quad f_2 = 1, \quad f_{n+2} = f_n + f_{n+1}$$

We get the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. This is called a *two-step recursion*, because each term depends on the two preceding terms. The sequence can also be constructed as a pair of one-step recursions as follows:

$$x_{n+1} = y_n$$
$$y_{n+1} = x_n + y_n$$

(a) Verify that using this rule and taking $x_0 = 1$, $y_0 = 1$, the x_n sequence is the Fibonacci sequence, while the y_n sequence is a shifted version of the Fibonacci sequence.

(b) An advantage of the one-step recursion form of the sequence definition is that it can be written as a matrix product. Find the matrix **A** so that the recursion takes the form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

Conceptual Problems

C1 Prove Theorem 3.1.5, using the following hints. To prove $A = B$, prove that $A - B = O_{m,n}$; note that $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$ if and only if $(A - B)\vec{x} = \vec{0}$ for every $\vec{x} \in \mathbb{R}^n$. Now, suppose that $C\vec{x} = \vec{0}$ for every $\vec{x} \in \mathbb{R}^n$. Consider the case where $\vec{x} = \vec{e}_i$ and conclude that each column of C must be the zero vector.

C2 Prove Theorem 3.1.6.

C3 Let $A \in M_{m\times n}(\mathbb{R})$ such that the system of linear equations $A\vec{x} = \vec{e}_i$ is consistent for all $1 \leq i \leq m$.

(a) Prove that the system of equations $A\vec{x} = \vec{y}$ is consistent for all $\vec{y} \in \mathbb{R}^m$.

(b) What can you conclude about the rank of A from the result in part (a)?

(c) Prove there exists a matrix B such that $AB = I_m$.

(d) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Use your method in part (c) to find a matrix B such that $AB = I_2$.

C4 Find a formula to calculate the ij-th entry of AA^T and of A^TA. Explain why it follows that if AA^T or A^TA is the zero matrix, then A is the zero matrix.

C5 A square matrix A is called **symmetric** if $A^T = A$. Prove that for any $B \in M_{m\times n}(\mathbb{R})$ both BB^T and B^TB are symmetric.

C6 (a) Construct $A \in M_{2\times 2}(\mathbb{R})$ that is not the zero matrix yet satisfies $A^2 = O_{2,2}$.

(b) Find $A, B \in M_{2\times 2}(\mathbb{R})$ with $A \neq B$ and neither $A = O_{2,2}$ nor $B = O_{2,2}$, such that

$$A^2 - AB - BA + B^2 = O_{2,2}$$

C7 Find as many 2×2 matrices A as you can that satisfy $A^2 = I$.

C8 We define the **trace** of $A \in M_{n\times n}(\mathbb{R})$ by

$$\operatorname{tr} A = a_{11} + a_{22} + \cdots + a_{nn}$$

Let $A, B \in M_{n\times n}(\mathbb{R})$. Prove that

$$\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$$

C9 Let $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in M_{n\times n}(\mathbb{R})$.

(a) Show that $D^2 = \operatorname{diag}(\lambda_1^2, \dots, \lambda_n^2)$.

(b) Use induction to prove that $D^k = \operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k)$.

C10 Let $A \in M_{m\times n}(\mathbb{R})$ such that $n > m$. Prove that if $\operatorname{rank} A = m$, then there are infinitely many matrices $B \in M_{n\times m}(\mathbb{R})$ such that $AB = I_m$.

C11 Let $A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix}$ and $B = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_k \end{bmatrix}$ be $n\times k$ matrices. Prove that if

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \operatorname{Span}\{\vec{u}_1, \dots, \vec{u}_k\}$$

then there exists a matrix C such that $A = BC$.

C12 Let $A \in M_{m\times n}(\mathbb{R})$ with one row of zeros and $B \in M_{n\times p}(\mathbb{R})$.

(a) Use block multiplication to prove that AB also has at least one row of zeros.

(b) Give an example where AB has more than one row of zeros.

3.2 Matrix Mappings and Linear Mappings

Functions are a fundamental concept in mathematics. Recall that a **function** f *is a rule that assigns to every element* x *of an initial set called the* **domain** *of the function a unique value* y *in another set called the* **codomain** *of* f*. We say that* f **maps** x *to* y *or that* y *is the* **image** *of* x *under* f*, and we write* f(x) = y*. If* f *is a function with domain* U *and codomain* V*, then we say that* f *maps* U *to* V *and denote this by* f:U → V*. In your earlier mathematics, you met functions* f : ℝ → ℝ *such as* f(x) = x² *and looked at their various properties. In this section, we will start by looking at more general functions* f : $\mathbb{R}^n \to \mathbb{R}^m$*, commonly called mappings or transformations. We will also look at a class of functions called linear mappings that are very important in linear algebra and its applications.*

Matrix Mappings

We saw in the preceding section that our rule for matrix-vector multiplication behaves like a function whose domain is $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$. In particular, for any $m \times n$ matrix A and vector $\vec{x} \in \mathbb{R}^n$, the product $A\vec{x}$ is a vector in $\mathbb{R}^m$.

Definition
Matrix Mapping

For any $A \in M_{m\times n}(\mathbb{R})$, we define a function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ called the **matrix mapping** corresponding to A by

$$f_A(\vec{x}) = A\vec{x}, \text{ for all } \vec{x} \in \mathbb{R}^n$$

Remark

Although a matrix mapping sends vectors to vectors, it is much more common to view functions as mapping points to points. Thus, when dealing with mappings in this text, we will often write

$$f(x_1, \dots, x_n) = (y_1, \dots, y_m), \quad \text{or} \quad f(x_1, \dots, x_n) = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

EXAMPLE 3.2.1

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$. Find $f_A(1, 2)$ and $f_A(-1, 4)$.

Solution: We have

$$f_A(1,2) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 2 \end{bmatrix}$$

$$f_A(-1,4) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 17 \\ 4 \end{bmatrix}$$

EXERCISE 3.2.1

Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 2 & 6 \\ 3 & 2 & 1 & 7 \end{bmatrix}$. Find $f_A(-1,1,1,0)$ and $f_A(-3,1,0,1)$.

EXERCISE 3.2.2

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. Find $f_A(1,0)$, $f_A(0,1)$, and $f_A(2,3)$.
What is the relationship between the value of $f_A(2,3)$ and the values of $f_A(1,0)$ and $f_A(0,1)$?

Based on our results in Exercise 3.2.2, is such a relationship always true? A good way to explore this is to look at a more general example.

EXAMPLE 3.2.2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and find the values of $f_A(1,0)$, $f_A(0,1)$, and $f_A(x_1,x_2)$.

Solution: We have

$$f_A(1,0) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$$
$$f_A(0,1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

Then we get

$$f_A(x_1,x_2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

We can now clearly see the relationship between the image of the standard basis vectors in $\mathbb{R}^2$ and the image of any vector $\vec{x}$. We conjecture that this works for any $m \times n$ matrix A.

Theorem 3.2.1

Let $\vec{e}_1, \dots, \vec{e}_n$ be the standard basis vectors of $\mathbb{R}^n$. If $A \in M_{m\times n}(\mathbb{R})$ and $f_A : \mathbb{R}^n \to \mathbb{R}^m$ is the corresponding matrix mapping, then for any vector $\vec{x} \in \mathbb{R}^n$ we have

$$f_A(\vec{x}) = x_1 f_A(\vec{e}_1) + x_2 f_A(\vec{e}_2) + \cdots + x_n f_A(\vec{e}_n)$$

Proof: Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$. Then,

$$\begin{aligned} f_A(\vec{x}) = A\vec{x} &= x_1\vec{a}_1 + \cdots + x_n\vec{a}_n && \text{by definition of matrix-vector multiplication} \\ &= x_1 A\vec{e}_1 + \cdots + x_n A\vec{e}_n && \text{by Theorem 3.1.3} \\ &= x_1 f_A(\vec{e}_1) + \cdots x_n f_A(\vec{e}_n) \end{aligned}$$

■

Since the images of the standard basis vectors are just the columns of A, we see that the image of any vector $\vec{x} \in \mathbb{R}^n$ is a linear combination of the columns of A. This should not be surprising as this is one of our interpretations of matrix-vector multiplication. However, it does make us think about how a matrix mapping will affect a linear combination of vectors in $\mathbb{R}^n$.

Theorem 3.2.2

If $A \in M_{m\times n}(\mathbb{R})$ with corresponding matrix mapping $f_A : \mathbb{R}^n \to \mathbb{R}^m$, then for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ and any $t \in \mathbb{R}$ we have

$$f_A(\vec{x} + \vec{y}) = f_A(\vec{x}) + f_A(\vec{y}) \tag{3.3}$$

$$f_A(t\vec{x}) = tf_A(\vec{x}) \tag{3.4}$$

Proof: Using properties of matrix multiplication, we get

$$f_A(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = f_A(\vec{x}) + f_A(\vec{y})$$

and

$$f_A(t\vec{x}) = A(t\vec{x}) = tA\vec{x} = tf_A(\vec{x})$$

■

A function that satisfies equation (3.3) is said to **preserve addition**. Similarly, a function satisfying equation (3.4) is said to **preserve scalar multiplication**. Notice that a function that satisfies both properties will in fact **preserve linear combinations**—that is,

$$f_A(t_1\vec{x}_1 + \cdots + t_n\vec{x}_n) = t_1 f_A(\vec{x}_1) + \cdots + t_n f_A(\vec{x}_n)$$

We call such functions **linear mappings**.

Linear Mappings

Definition
Linear Mapping
Linear Operator

A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear mapping** (or **linear transformation**) if for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ it satisfies

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

A linear mapping whose domain and codomain are the same is sometimes called a **linear operator**.

Remarks

1. *Linear transformation* and *linear mapping* mean exactly the same thing. Some people prefer one or the other, but we shall use both.
2. Since a linear operator L has the same domain and codomain, we often speak of a **linear operator** L **on** $\mathbb{R}^n$ to indicate that L is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.
3. For the time being, we have defined only linear mappings whose domain is $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$. In Chapter 4, we will look at other sets that can be the domain and/or codomain of linear mappings.
4. Two linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ are **equal** if $L(\vec{x}) = M(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$. In this case, we can write $L = M$.

EXAMPLE 3.2.3

Show that the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (2x_1 + x_2, -3x_1 + 5x_2)$ is linear.

Solution: For any $\vec{y}, \vec{z} \in \mathbb{R}^2$, we have

$$\begin{aligned}
f(s\vec{y} + t\vec{z}) &= f(sy_1 + tz_1, sy_2 + tz_2) \\
&= \begin{bmatrix} 2(sy_1 + tz_1) + (sy_2 + tz_2) \\ -3(sy_1 + tz_1) + 5(sy_2 + tz_2) \end{bmatrix} \\
&= s\begin{bmatrix} 2y_1 + y_2 \\ -3y_1 + 5y_2 \end{bmatrix} + t\begin{bmatrix} 2z_1 + z_2 \\ -3z_1 + 5z_2 \end{bmatrix} \\
&= sf(\vec{y}) + tf(\vec{z})
\end{aligned}$$

Thus, f is closed under linear combinations and therefore is a linear operator.

EXAMPLE 3.2.4

Determine whether the mapping $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(\vec{x}) = \|\vec{x}\|$ is linear.

Solution: Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ and consider

$$f(\vec{x} + \vec{y}) = \|\vec{x} + \vec{y}\| \quad \text{and} \quad f(\vec{x}) + f(\vec{y}) = \|\vec{x}\| + \|\vec{y}\|$$

Are these equal? By the triangle inequality, we get

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

and we expect equality only when one of $\vec{x}, \vec{y}$ is a multiple of the other. Therefore, we believe that these are not always equal, and consequently f is not closed under addition. To demonstrate this, we give a counterexample: if $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then

$$f(\vec{x} + \vec{y}) = f(1, 1, 0) = \left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{2}$$

but

$$f(\vec{x}) + f(\vec{y}) = \left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\| = 1 + 1 = 2$$

Thus, $f(\vec{x} + \vec{y}) \neq f(\vec{x}) + f(\vec{y})$ for all pairs of vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^3$, hence f is not linear.

EXERCISE 3.2.3

Determine whether the following mappings are linear.

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (x_1^2, x_1 + x_2)$

(b) $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x_1, x_2) = (x_2, x_1 - x_2)$

Is Every Linear Mapping a Matrix Mapping?

We saw that every matrix determines a corresponding linear mapping. It is natural to ask whether every linear mapping can be represented as a matrix mapping.

EXAMPLE 3.2.5

Can the linear mapping f defined by $f(x_1, x_2) = (2x_1 + x_2, -3x_1 + 5x_2)$, be represented as a matrix-mapping?

Solution: If $f(\vec{x}) = A\vec{x}$, then our work with matrix mappings suggests that the columns of A are the images of the standard basis vectors. We have

$$f(1,0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \qquad f(0,1) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Thus, we define

$$A = \begin{bmatrix} f(1,0) & f(0,1) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}$$

This gives

$$f_A(x_1, x_2) = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ -3x_1 + 5x_2 \end{bmatrix} = f(x_1, x_2)$$

Hence, f can be represented as a matrix mapping.

This example not only gives us a good reason to believe it is always true but indicates how we can find the matrix for a given linear mapping L.

Theorem 3.2.3

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. If we define

$$[L] = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) & \cdots & L(\vec{e}_n) \end{bmatrix}$$

then we have

$$L(\vec{x}) = [L]\vec{x}$$

Proof: Let $\vec{x} \in \mathbb{R}^n$. Writing $\vec{x}$ as a linear combination of the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$ gives

$$\begin{aligned} L(\vec{x}) &= L(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \\ &= x_1L(\vec{e}_1) + \cdots + x_nL(\vec{e}_n) && \text{since } L \text{ is linear} \\ &= \begin{bmatrix} L(\vec{e}_1) & \cdots & L(\vec{e}_n) \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} && \text{by definition of matrix-vector multiplication} \\ &= [L]\vec{x} \end{aligned}$$

■

Remarks

1. The matrix $[L]$ in Theorem 3.2.3 is called the **standard matrix** of L.

2. Combining Theorem 3.2.3 with Theorem 3.2.2 shows that a mapping is linear if and only if it is a matrix mapping.

EXAMPLE 3.2.6

Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find the standard matrix of the mapping $\text{proj}_{\vec{v}} : \mathbb{R}^2 \to \mathbb{R}^2$ and use it to find $\text{proj}_{\vec{v}}(\vec{u})$.

Solution: Since $\text{proj}_{\vec{v}}$ is linear (see Section 1.5), we can apply Theorem 3.2.3 to find $[\text{proj}_{\vec{v}}]$. The first column of the matrix is the image of the first standard basis vector under $\text{proj}_{\vec{v}}$:

$$\text{proj}_{\vec{v}}(\vec{e}_1) = \frac{\vec{e}_1 \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{1(3) + 0(4)}{3^2 + 4^2}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9/25 \\ 12/25 \end{bmatrix}$$

Similarly, the second column is the image of the second basis vector:

$$\text{proj}_{\vec{v}}(\vec{e}_2) = \frac{\vec{e}_2 \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{0(3) + 1(4)}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 12/25 \\ 16/25 \end{bmatrix}$$

Hence, the standard matrix of the linear mapping is

$$[\text{proj}_{\vec{v}}] = \begin{bmatrix} \text{proj}_{\vec{v}}(\vec{e}_1) & \text{proj}_{\vec{v}}(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$$

Therefore, we have

$$\text{proj}_{\vec{v}}(\vec{u}) = [\text{proj}_{\vec{v}}]\vec{u} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 33/25 \\ 44/25 \end{bmatrix}$$

EXAMPLE 3.2.7

Let $G : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $G(x_1, x_2, x_3) = (x_1, x_2)$. Prove that G is linear and find the standard matrix of G.

Solution: We first prove that G is linear. For any $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$, we have

$$\begin{aligned} G(s\vec{x} + t\vec{y}) &= G(sx_1 + ty_1, sx_2 + ty_2, sx_3 + ty_3) \\ &= \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \end{bmatrix} \\ &= s\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + t\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= sG(\vec{x}) + tG(\vec{y}) \end{aligned}$$

Hence, G is linear. Thus, we can apply Theorem 3.2.3 to find its standard matrix. The images of the standard basis vectors are

$$G(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad G(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad G(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, by definition, $[G] = \begin{bmatrix} G(\vec{e}_1) & G(\vec{e}_2) & G(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Did we really need to prove that G was linear first? Could we have not just constructed $[G]$ using the image of the standard basis vectors and then said that G is linear because it is a matrix mapping? *No*! We must always check the hypotheses of a theorem before using it. Theorem 3.2.3 says that *if* f is linear, *then* $[f]$ can be constructed from the images of the standard basis vectors. The converse is not true!

For example, consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix}$. The images of the standard basis vectors are $f(\vec{e}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $f(\vec{e}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so we can construct the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But this matrix does not represent the mapping! In particular, observe that $f(1,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence, even though we can create a matrix using the images of the standard basis vectors, it does not imply that the matrix will represent that mapping, unless we already know the mapping is linear.

EXERCISE 3.2.4

Let $H : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $H(x_1, x_2, x_3, x_4) = (x_3 + x_4, x_1)$. Prove that H is linear and find the standard matrix of H.

Linear Combinations and Compositions of Linear Mappings

We now look at the usual operations on functions and how these affect linear mappings.

Definition
Addition and Scalar Multiplication of Linear Mappings

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings. We define $(L + M) : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(L + M)(\vec{x}) = L(\vec{x}) + M(\vec{x}), \qquad \text{for all } \vec{x} \in \mathbb{R}^n$$

For any $c \in \mathbb{R}$, we define $(cL) : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(cL)(\vec{x}) = cL(\vec{x}), \qquad \text{for all } \vec{x} \in \mathbb{R}^n$$

EXAMPLE 3.2.8

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ and $M : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L(x_1, x_2) = (x_1, x_1 + x_2, -x_2)$ and $M(x_1, x_2) = (x_1, x_1, x_2)$. Calculate $L + M$ and $3L$.

Solution: $L + M$ is the mapping defined by

$$(L + M)(\vec{x}) = L(\vec{x}) + M(\vec{x}) = (x_1, x_1 + x_2, -x_2) + (x_1, x_1, x_2) = (2x_1, 2x_1 + x_2, 0)$$

and $3L$ is the mapping defined by

$$(3L)(\vec{x}) = 3L(\vec{x}) = 3(x_1, x_1 + x_2, -x_2) = (3x_1, 3x_1 + 3x_2, -3x_2)$$

By analyzing Example 3.2.8 we can learn more than just how to add linear mappings and how to multiply them by a scalar. We first observe that $L + M$ and $3L$ are both linear mappings. This makes us realize that we can rewrite the calculations in the example in terms of standard matrices. For example,

$$(L+M)(\vec{x}) = L(\vec{x}) + M(\vec{x}) = [L]\vec{x} + [M]\vec{x} = ([L]+[M])\vec{x}$$
$$= \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_1 + x_2 \\ 0 \end{bmatrix}$$

This matches our result above. Generalizing what we did here gives the following theorem.

Theorem 3.2.4

If $L, M : \mathbb{R}^n \to \mathbb{R}^m$ are linear mappings and $c \in \mathbb{R}$, then $L + M : \mathbb{R}^n \to \mathbb{R}^m$ and $cL : \mathbb{R}^n \to \mathbb{R}^m$ are linear mappings. Moreover, we have

$$[L+M] = [L] + [M] \quad \text{and} \quad [cL] = c[L]$$

Proof: We will prove the result for cL. The result for $L+M$ is left as Problem C3. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned}
(cL)(s\vec{x} + t\vec{y}) &= cL(s\vec{x} + t\vec{y}) && \text{by definition of } cL \\
&= c\left(sL(\vec{x}) + tL(\vec{y})\right) && \text{since } L \text{ is linear} \\
&= csL(\vec{x}) + ctL(\vec{y}) && \text{by properties V9,V8 of Theorem 1.4.1} \\
&= s(cL)(\vec{x}) + t(cL)(\vec{y}) && \text{by definition of } cL
\end{aligned}$$

Hence, (cL) is linear. Moreover, for any $\vec{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}
[cL]\vec{x} &= (cL)(\vec{x}) && \text{by definition of the standard matrix of } cL \\
&= c(L(\vec{x})) && \text{by definition of } cL \\
&= c[L]\vec{x} && \text{by definition of the standard matrix of } L
\end{aligned}$$

Thus, by the Matrices Equal Theorem, $[cL] = c[L]$. ■

Definition
Composition of Linear Mappings

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear mappings. The **composition** $M \circ L : \mathbb{R}^n \to \mathbb{R}^p$ is defined by

$$(M \circ L)(\vec{x}) = M(L(\vec{x}))$$

for all $\vec{x} \in \mathbb{R}^n$.

Note that the definition makes sense only if the domain of the second map M contains the codomain of the first map L, as we are evaluating M at $L(\vec{x})$. Moreover, observe that the order of the mappings in the definition is important.

EXAMPLE 3.2.9

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ and $M : \mathbb{R}^3 \to \mathbb{R}^1$ be the linear mappings defined by $L(x_1, x_2) = (x_1 + x_2, 0, x_1 - 2x_2)$ and $M(x_1, x_2, x_3) = (x_1 + x_2 + x_3)$. Find $M \circ L$.

Solution: We have

$$(M \circ L)(x_1, x_2) = M(x_1 + x_2, 0, x_1 - 2x_2) = ((x_1 + x_2) + 0 + (x_1 - 2x_2)) = (2x_1 - x_2)$$

EXAMPLE 3.2.10

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ and $M : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mappings defined by $L(x_1, x_2) = (2x_1 + x_2, x_1 + x_2)$ and $M(x_1, x_2) = (x_1 - x_2, -x_1 + 2x_2)$. Then, $M \circ L$ is mapping defined by

$$\begin{aligned}(M \circ L)(x_1, x_2) &= M(2x_1 + x_2, x_1 + x_2)\\ &= \Big((2x_1 + x_2) - (x_1 + x_2), -(2x_1 + x_2) + 2(x_1 + x_2)\Big)\\ &= (x_1, x_2)\end{aligned}$$

As with addition and scalar multiplication, we observe that a composition of linear mappings is linear. Moreover, we recall that we defined matrix multiplication to represent a composition of functions. Hence, the next theorem is not surprising.

Theorem 3.2.5

If $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ are linear mappings, then $M \circ L : \mathbb{R}^n \to \mathbb{R}^p$ is a linear mapping and

$$[M \circ L] = [M][L]$$

We leave the proof as Problem C4.

EXAMPLE 3.2.11

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ and $M : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mappings defined by $L(x_1, x_2) = (2x_1 + x_2, x_1 + x_2)$ and $M(x_1, x_2) = (x_1 - x_2, -x_1 + 2x_2)$. Find $[M \circ L]$.

Solution: We have

$$[M \circ L] = [M][L] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In Example 3.2.10, $M \circ L$ is the mapping such that $(M \circ L)(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. We see from Example 3.2.11 that the standard matrix of this mapping is the identity matrix. Thus, we make the following definition.

Definition
Identity Mapping

The **identity mapping** is the linear mapping $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\mathrm{Id}(\vec{x}) = \vec{x}$$

CONNECTION

The mappings L and M in Example 3.2.10 also satisfy $L \circ M = \mathrm{Id}$ and hence are said to be **inverses** of each other, as are the matrices $[L]$ and $[M]$. We will look at inverse mappings and matrices in Section 3.5.

PROBLEMS 3.2

Practice Problems

A1 Let $A = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix}$ and let $f_A(\vec{x}) = A\vec{x}$ be the corresponding matrix mapping.

(a) Determine the domain and codomain of f_A.

(b) Determine $f_A(2, -5)$ and $f_A(-3, 4)$.

(c) Find the images of the standard basis vectors for the domain under f_A.

(d) Determine $f_A(\vec{x})$.

(e) Check your answers in (c) and (d) by calculating $[f_A(\vec{x})]$ using Theorem 3.2.3.

A2 Let $A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix}$ and let $f_A(\vec{x}) = A\vec{x}$.

(a) Determine the domain and codomain of f_A.

(b) Determine $f_A(2, -2, 3, 1)$ and $f_A(-3, 1, 4, 2)$.

(c) Find the images of the standard basis vectors for the domain under f_A.

(d) Determine $f_A(\vec{x})$.

(e) Check your answers in (c) and (d) by calculating $[f_A(\vec{x})]$ using Theorem 3.2.3.

For Problems A3–A16, state the domain and codomain of the mapping. Either prove that the mapping is linear or give a counterexample to show why it cannot be linear.

A3 $f(x_1, x_2) = (\sin x_1, e^{x_2^2})$

A4 $f(x_1, x_2, x_3) = (0, x_1x_2x_3)$

A5 $g(x_1, x_2, x_3) = (1, 1, 1)$

A6 $g(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2)$

A7 $h(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, x_1x_2)$

A8 $k(x_1, x_2, x_3) = (x_1 + x_2, 0, x_2 - x_3)$

A9 $\ell(x_1, x_2, x_3) = (x_2, |x_1|)$

A10 $m(x_1) = (x_1, 1, 0)$

A11 $L(x_1, x_2, x_3) = (2x_1, x_1 - x_2 + 3x_3)$

A12 $L(x_1, x_2) = (x_1^2 - x_2^2, x_1)$

A13 $M(x_1, x_2, x_3) = (x_1 + 2, x_2 + 2)$

A14 $M(x_1, x_2, x_3) = (x_1 + 3x_3, x_2 - 2x_3, x_1 + x_2)$

A15 $N(x_1, x_2) = (-x_1, 0, x_1)$

A16 $N(x_1, x_2, x_3) = (x_1x_2, x_1x_3, x_2x_3)$

For Problems A17–A19, determine $[\text{proj}_{\vec{v}}]$.

A17 $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **A18** $\vec{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ **A19** $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

For Problems A20–A22, determine $[\text{perp}_{\vec{v}}]$.

A20 $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **A21** $\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ **A22** $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

For Problems A23–A31, determine the domain, the codomain, and the standard matrix of the linear mapping.

A23 $L(x_1, x_2) = (-3x_1 + 5x_2, -x_1 - 2x_2)$

A24 $L(x_1, x_2) = (x_1, x_2, x_1 + x_2)$

A25 $L(x_1) = (x_1, 0, 3x_1)$

A26 $M(x_1, x_2, x_3) = (x_1 - x_2 + \sqrt{2}x_3)$

A27 $M(x_1, x_2, x_3) = (2x_1 - x_3, 2x_1 - x_3)$

A28 $N(x_1, x_2, x_3) = (0, 0, 0, 0)$

A29 $L(x_1, x_2, x_3) = (2x_1 - 3x_2 + x_3, x_2 - 5x_3)$

A30 $K(x_1, x_2, x_3, x_4) = (5x_1 + 3x_3 - x_4, x_2 - 7x_3 + 3x_4)$

A31 $M(x_1, x_2, x_3, x_4) = (x_1 - x_3 + x_4, x_1 + 2x_2 - 3x_4, x_2 + x_3)$

For Problems A32–A41, find the standard matrix of the linear mapping with the given properties.

A32 $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(\vec{e}_1) = (3, 5)$, $L(\vec{e}_2) = (1, -2)$.

A33 $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(\vec{e}_1) = (-1, 11)$, $L(\vec{e}_2) = (13, -21)$.

A34 $L : \mathbb{R}^2 \to \mathbb{R}^3$, $L(\vec{e}_1) = (1, 0, 1)$, $L(\vec{e}_2) = (1, 0, 1)$.

A35 $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\vec{e}_1) = (1, 0, 1)$, $L(\vec{e}_2) = (1, 0, 1)$, $L(\vec{e}_3) = (1, 0, 1)$.

A36 $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\vec{e}_1) = (5, 0, 0)$, $L(\vec{e}_2) = (0, 3, 0)$, $L(\vec{e}_3) = (0, 0, 2)$.

A37 $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L(\vec{e}_1) = (-1, \sqrt{2})$, $L(\vec{e}_2) = (1/2, 0)$, $L(\vec{e}_3) = (-1, -1)$.

A38 $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\vec{e}_1) = (2, 1, 1)$, $L(\vec{e}_2) = (1, -2, 1)$, $L(\vec{e}_3) = (0, 1, -2)$.

A39 $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(1, 0) = (3, 5)$, $L(1, 1) = (5, -2)$.

A40 $L : \mathbb{R}^2 \to \mathbb{R}^3$, $L(1, 1) = (3, 2, 0)$, $L(1, -1) = (1, 0, -2)$.

A41 $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L(1, 0, 1) = (2, 0)$, $L(1, 1, 1) = (4, 5)$, $L(1, 1, 0) = (5, 6)$.

A42 Suppose that S and T are linear mappings with standard matrices

$$[S] = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}, \quad [T] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Determine the domain and codomain of S and T.
(b) Determine $[S + T]$ and $[2S - 3T]$.

A43 Suppose that S and T are linear mappings with standard matrices

$$[S] = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix}, \quad [T] = \begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 2 & -1 \\ 3 & -4 \end{bmatrix}$$

(a) Determine the domain and codomain of S and T.
(b) Determine $[S \circ T]$ and $[T \circ S]$.

For Problems A44–A49, suppose that L, M, and N are linear mappings with standard matrices $[L] = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$, $[M] = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & -1 \end{bmatrix}$, $[N] = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ -3 & 0 \\ 1 & 4 \end{bmatrix}$. Determine whether the given composition is defined. If so, calculate the standard matrix of the composition.

A44 $L \circ M$ **A45** $M \circ L$ **A46** $L \circ N$
A47 $N \circ L$ **A48** $M \circ N$ **A49** $N \circ M$

A50 (a) Invent a linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^3$ such that $L(1,0) = (3,-1,4)$ and $L(0,1) = (1,-5,9)$.
(b) Invent a *non*-linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^3$ such that $L(1,0) = (3,-1,4)$ and $L(0,1) = (1,-5,9)$.

Homework Problems

B1 Let $A = \begin{bmatrix} 6 & 2 & 3 \\ -1 & 6 & 4 \end{bmatrix}$ and let $f_A(\vec{x}) = A\vec{x}$ be the corresponding matrix mapping.
(a) Determine the domain and codomain of f_A.
(b) Determine $f_A(2,1,-1)$ and $f_A(3,-8,9)$.
(c) Find the images of the standard basis vectors for the domain under f_A.
(d) Determine $f_A(\vec{x})$.
(e) Check your answers in (c) and (d) by calculating $[f_A(\vec{x})]$ using Theorem 3.2.3.

B2 Let $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 4 & -1 & 0 & 1 \end{bmatrix}$ and let $f_A(\vec{x}) = A\vec{x}$.
(a) Determine the domain and codomain of f_A.
(b) Determine $f_A(1,1,1,1)$ and $f_A(3,1,-2,3)$.
(c) Find the images of the standard basis vectors for the domain under f_A.
(d) Determine $f_A(\vec{x})$.
(e) Check your answers in (c) and (d) by calculating $[f_A(\vec{x})]$ using Theorem 3.2.3.

For Problems B3–B9, state the domain and codomain of the mapping. Either prove that the mapping is linear or give a counterexample to show why it cannot be linear.

B3 $f(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2)$

B4 $g(x_1, x_2) = (2x_1 - x_2, 3x_1 + x_2)$

B5 $h(x_1, x_2) = (0, 0)$

B6 $k(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_2)$

B7 $\ell(x_1, x_2, x_3, x_4) = (1, 1)$

B8 $m(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 + x_4)$

B9 $L(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3 + 5)$

For Problems B10–B12, determine $[\text{proj}_{\vec{v}}]$.

B10 $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **B11** $\vec{v} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ **B12** $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For Problems B13–B15, determine $[\text{perp}_{\vec{v}}]$.

B13 $\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ **B14** $\vec{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ **B15** $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For Problems B16–B21, determine the domain, the codomain, and the standard matrix of the linear mapping.

B16 $L(x_1, x_2) = (7x_1 - 2x_2, \pi x_1)$

B17 $L(x_1, x_2, x_3) = (x_1, x_2)$

B18 $L(x_1, x_2, x_3) = (x_1 - x_3, x_2 + x_3, x_1 + x_2 + x_3)$

B19 $K(x_1, x_2) = (2x_1 + 4x_2, x_1 + 2x_2, x_1 - x_2, x_1 + 3x_2)$

B20 $M(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, x_1 - x_3, x_1 + 2x_3)$

B21 $N(x_1, x_2, x_3) = (x_3, x_1, x_2)$

For Problems B22–B25, find the standard matrix of the linear mapping with the given properties.

B22 $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(1,0) = (8,3)$, $L(0,1) = (7,-1)$.

B23 $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L(1,0,0) = (1,5)$, $L(0,1,0) = (1,2)$, $L(0,0,1) = (4,2)$.

B24 $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(1,1) = (7,2)$, $L(1,0) = (1,-5)$.

B25 $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(1,2) = (-1,6)$, $L(2,1) = (3,4)$.

B26 Suppose that S and T are linear mappings with standard matrices

$$[S] = \begin{bmatrix} 3 & 5 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad [T] = \begin{bmatrix} 1 & -2 \\ -3 & 9 \\ 5 & 5 \end{bmatrix}$$

(a) Determine the domain and codomain of S and T.
(b) Determine $[S + T]$ and $[-S + T]$.

B27 Suppose that S and T are linear mappings with standard matrices

$$[S] = \begin{bmatrix} 3 & 7 \\ 2 & 4 \\ -1 & 1 \end{bmatrix}, [T] = \begin{bmatrix} 4 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

(a) Determine the domain and codomain of S and T.
(b) Determine $[S \circ T]$ and $[T \circ S]$.

For Problems B28–B33, suppose that L, M, and N are linear mappings with standard matrices $[L] = \begin{bmatrix} 5 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$, $[M] = \begin{bmatrix} 2 & 2 \\ 1 & 2 \\ -1 & 6 \end{bmatrix}$, $[N] = \begin{bmatrix} 2 & 1 \\ 7 & -4 \end{bmatrix}$. Determine whether the given composition is defined. If so, calculate the standard matrix of the composition.

B28 $L \circ M$ **B29** $M \circ L$ **B30** $L \circ N$
B31 $N \circ L$ **B32** $M \circ N$ **B33** $N \circ M$

Conceptual Problems

C1 Let $L : \mathbb{R}^n \to \mathbb{R}^m$. Show that for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$, L satisfies

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}) \quad \text{and} \quad L(s\vec{x}) = sL(\vec{x})$$

if and only if

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

C2 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping.
(a) Prove that $L(\vec{0}) = \vec{0}$.
(b) Explain what (a) says about a mapping $M : \mathbb{R}^n \to \mathbb{R}^m$ such that $M(\vec{0}) \neq \vec{0}$.

C3 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings. Prove that $(L + M)$ is linear and that

$$[L + M] = [L] + [M]$$

C4 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear mappings. Prove that $(M \circ L)$ is linear and that

$$[M \circ L] = [M][L]$$

C5 Let $\vec{v} \in \mathbb{R}^3$ be a fixed vector and define a mapping $\text{CROSS}_{\vec{v}}$ by

$$\text{CROSS}_{\vec{v}}(\vec{x}) = \vec{v} \times \vec{x}$$

Verify that $\text{CROSS}_{\vec{v}}$ is a linear mapping and determine its domain, codomain, and standard matrix.

C6 Let $\vec{v} \in \mathbb{R}^n$ be a fixed vector and define a mapping $\text{DOT}_{\vec{v}}$ by

$$\text{DOT}_{\vec{v}}(\vec{x}) = \vec{v} \cdot \vec{x}$$

Verify that $\text{DOT}_{\vec{v}}$ is a linear mapping. What is its domain and codomain? Verify that the matrix of this linear mapping can be written as $\vec{v}^T$.

C7 If $\vec{u}$ is a unit vector, show that $[\text{proj}_{\vec{u}}] = \vec{u}\vec{u}^T$.

C8 (a) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Prove that if $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ is a linearly independent set in $\mathbb{R}^m$, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.
(b) Give an example of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ where $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent in $\mathbb{R}^n$, but $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ is linearly dependent.

C9 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and let $\mathbb{S}$ be a subspace of $\mathbb{R}^m$. Prove that the set

$$T = \{\vec{v} \in \mathbb{R}^n \mid L(\vec{v}) \in \mathbb{S}\}$$

is a subspace of $\mathbb{R}^n$. T is called the **pre-image** of $\mathbb{S}$.

3.3 Geometrical Transformations

Geometrical transformations have long been of great interest to mathematicians. They have many important applications. Physicists and engineers often rely on simple geometrical transformations to gain understanding of the properties of materials or structures they wish to examine. For example, structural engineers use stretches, shears, and rotations to understand the deformation of materials. Material scientists use rotations and reflections to analyze crystals and other fine structures. Many of these simple geometrical transformations in $\mathbb{R}^2$ *and* $\mathbb{R}^3$ *are linear. The following is a brief partial catalogue of some of these transformations and their matrix representations. (*$\text{proj}_{\vec{v}}$ *and* $\text{perp}_{\vec{v}}$ *belong to the list of geometrical transformations, too, but they were discussed in Chapter 1 and so are not included here).*

Rotations in the Plane

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote the function that rotates a vector $\vec{x} \in \mathbb{R}^2$ about the origin through an angle θ as depicted in Figure 3.3.1. Using trigonometric identities, we can show that

$$R_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$$

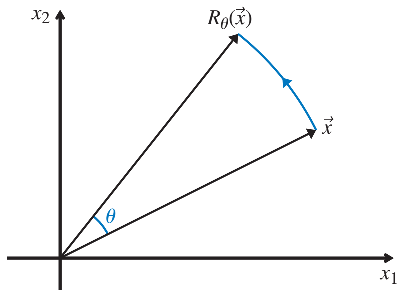


Figure 3.3.1 Counterclockwise rotation through angle θ in the plane.

From this, it is easy to prove that R_θ is linear. Hence, we can use the definition of the standard matrix to calculate $[R_\theta]$. We have

$$R_\theta(1,0) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$R_\theta(0,1) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Hence,

$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Definition
Rotation Matrix

A matrix $[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is called a **rotation matrix**.

EXAMPLE 3.3.1

What is the matrix of rotation of $\mathbb{R}^2$ through angle $2\pi/3$?

Solution: We have

$$[R_{2\pi/3}] = \begin{bmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

EXERCISE 3.3.1

Determine $[R_{\pi/4}]$ and use it to calculate $R_{\pi/4}(1, 1)$. Illustrate with a sketch.

CONNECTION

Other than the application of using rotation matrices to rotate computer graphics, one might think that rotation matrices are not particularly important. However, since the columns of a rotation matrix are both unit vectors and are orthogonal to each other, they form an **orthonormal basis** for $\mathbb{R}^2$ (see Problem C13 in Section 1.5). Such bases, and hence rotation matrices, are very useful in solving many problems. We will explore this in Chapter 8.

Rotation Through Angle θ About the x_3-axis in $\mathbb{R}^3$

Figure 3.3.2 demonstrates a counterclockwise rotation with respect to the right-handed standard basis. This rotation leaves x_3 unchanged, so that if the transformation is denoted by R, then $R(0, 0, 1) = (0, 0, 1)$. Together with the previous case, this tells us that the standard matrix of this rotation is

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These ideas can be adapted to give rotations about the other coordinate axes. We shall see how to determine the standard matrix of a rotation about an arbitrary axis in Chapter 7.

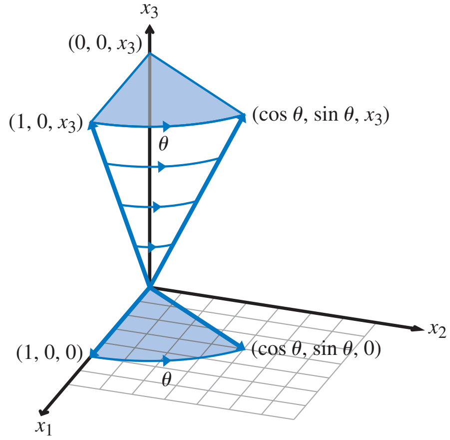


Figure 3.3.2 A right-handed counterclockwise rotation about the x_3-axis in $\mathbb{R}^3$.

Stretches

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be denote the function that multiplies all lengths in the x_1-direction by a scalar factor $t > 0$, while lengths in the x_2-direction are left unchanged (Figure 3.3.3). This linear operator, called a "stretch by factor t in the x_1-direction," has standard matrix

$$[S] = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

(If $t < 1$, you might prefer to call this a *shrink*.) Stretches can also be defined in the x_2-direction and in higher dimensions. Stretches are important in understanding the deformation of solids (see Section 8.4).

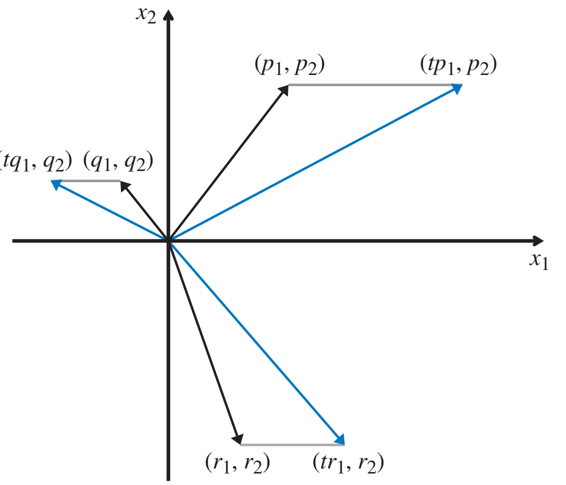


Figure 3.3.3 A stretch by a factor t in the x_1-direction.

EXERCISE 3.3.2

Let S be the stretch by factor 3 in the x_2-direction. Write the standard matrix $[S]$ of S and use it to calculate $S(\vec{x})$ where $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Illustrate with a sketch.

Contractions and Dilations

A linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix

$$[T] = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$$

transforms vectors in all directions by the same factor. Thus, for example, a circle of radius 1 centred at the origin is mapped to a circle of radius t at the origin. If $0 < t < 1$, such a transformation is called a **contraction**; if $t > 1$, it is a **dilation**.

EXERCISE 3.3.3

Let T be the dilation by factor 3. Write the standard matrix $[T]$ of T and use it to calculate $T(\vec{x})$ where $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Illustrate with a sketch.

Shears

Sometimes a force applied to a rectangle will cause it to deform into a parallelogram, as shown in Figure 3.3.4. The change can be described by the function $H : \mathbb{R}^2 \to \mathbb{R}^2$, such that $H(1,0) = (1,0)$ and $H(0,1) = (s,1)$. Although the deformation of a real solid may be more complicated, it is customary to assume that the transformation H is linear. Such a linear transformation is called a **horizontal shear** by amount s. Since the action of H on the standard basis vectors is known, we find that its standard matrix is

$$[H] = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

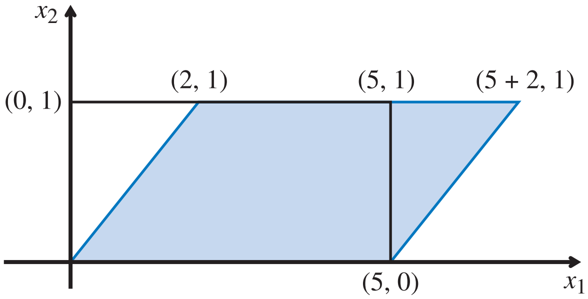


Figure 3.3.4 A horizontal shear by amount 2.

The standard matrix of a **vertical shear** $V : \mathbb{R}^2 \to \mathbb{R}^2$ by amount s will be

$$[V] = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

EXERCISE 3.3.4

Let V be a vertical shear by amount 2. Write the standard matrix $[V]$ of V. Determine $V(1,0)$ and $V(0,1)$, and illustrate with a sketch how the unit square is deformed into a parallelogram by V.

CONNECTION

Recall that the 2×2 identity matrix is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since $I_2\vec{x} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^2$, geometrically I_2 does nothing. If we perform the elementary row operation "multiply a row by a non-zero scalar t" on I_2, then we get the standard matrix of a stretch. Similarly, if we perform the elementary row operation "add s times row 2 to row 1" on I_2, then we get a horizontal shear. That is, elementary row operations have very simple geometrical interpretations. What is the geometry of "swapping row 1 and row 2"?

Reflections in Coordinate Axes in $\mathbb{R}^2$ or Coordinate Planes in $\mathbb{R}^3$

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection over the x_1-axis. See Figure 3.3.5. Then each vector corresponding to a point above the axis is mapped by F to the mirror image vector below. Hence,

$$F(x_1, x_2) = (x_1, -x_2)$$

It follows that this transformation is linear with standard matrix

$$[F] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

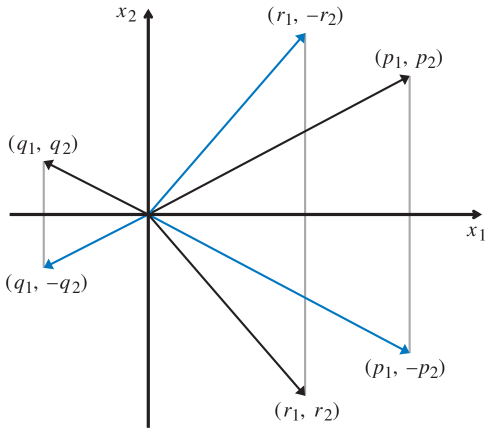


Figure 3.3.5 A reflection in $\mathbb{R}^2$ over the x_1-axis.

EXERCISE 3.3.5 Let F be a reflection in the x_2-axis. Write the standard matrix $[F]$ of F and use it to calculate $F(\vec{x})$ where $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Illustrate with a sketch.

Next, consider the reflection $F : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects in the x_1x_2-plane (that is, the plane $x_3 = 0$). Points above the plane are reflected to points below the plane. The standard matrix of this reflection is

$$[F] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

EXERCISE 3.3.6 Write the matrices for the reflections in the other two coordinate planes in $\mathbb{R}^3$.

General Reflections

We consider only reflections in (or "across") lines in $\mathbb{R}^2$ or planes in $\mathbb{R}^3$ that pass through the origin. Reflections in lines or planes not containing the origin involve translations (which are not linear) as well as linear mappings.

Consider the plane in $\mathbb{R}^3$ with equation $\vec{n} \cdot \vec{x} = 0$. Since a reflection is related to $\text{proj}_{\vec{n}}$, a **reflection in the plane with normal vector** $\vec{n}$ will be denoted $\text{refl}_{\vec{n}}$. If a vector $\vec{p}$ corresponds to a point P that does not lie in the plane, its image under $\text{refl}_{\vec{n}}$ is the vector that corresponds to the point on the opposite side of the plane, lying on a line through P perpendicular to the plane of reflection, at the same distance from the plane as P. Figure 3.3.6 shows reflection in a line. From the figure, we see that

$$\text{refl}_{\vec{n}}(\vec{p}) = \vec{p} - 2\,\text{proj}_{\vec{n}}(\vec{p})$$

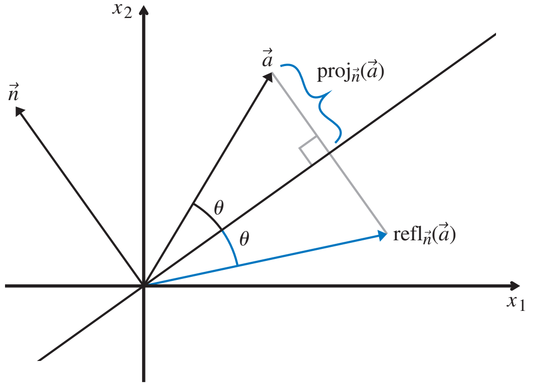


Figure 3.3.6 A reflection in $\mathbb{R}^2$ over the line with normal vector $\vec{n}$.

EXAMPLE 3.3.2

Prove that $\text{refl}_{\vec{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear mapping.

Solution: Since $\text{proj}_{\vec{n}}$ is a linear mapping we get

$$\begin{aligned}
\text{refl}_{\vec{n}}(b\vec{x} + c\vec{y}) &= (b\vec{x} + c\vec{y}) - 2\,\text{proj}_{\vec{n}}(b\vec{x} + c\vec{y}) \\
&= b\vec{x} + c\vec{y} - 2(b\,\text{proj}_{\vec{n}}(\vec{x}) + c\,\text{proj}_{\vec{n}}(\vec{y})) \\
&= b(\vec{x} - 2\,\text{proj}_{\vec{n}}(\vec{x})) + c(\vec{y} - 2\,\text{proj}_{\vec{n}}(\vec{y})) \\
&= b\,\text{refl}_{\vec{n}}(\vec{x}) + c\,\text{refl}_{\vec{n}}(\vec{y})
\end{aligned}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$ as required.

It is important to note that $\text{refl}_{\vec{n}}$ is a reflection over a line or plane passing through the origin with normal vector $\vec{n}$. The calculations for reflection in a line in $\mathbb{R}^2$ are similar to those for a plane, provided that the equation of the line is given in scalar form $\vec{n} \cdot \vec{x} = 0$. If the vector equation of the line is given as $\vec{x} = t\vec{d}$, then either we must find a normal vector $\vec{n}$ and proceed as above, or, in terms of the direction vector $\vec{d}$, the reflection will map $\vec{p}$ to $(\vec{p} - 2\,\text{perp}_{\vec{d}}(\vec{p}))$.

EXAMPLE 3.3.3

Consider a reflection $\text{refl}_{\vec{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ over the plane with normal vector $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Determine the standard matrix $[\text{refl}_{\vec{n}}]$.

Solution: We have

$$\text{refl}_{\vec{n}}(\vec{e}_1) = \vec{e}_1 - 2\frac{\vec{e}_1 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\left(\frac{1}{6}\right)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\text{refl}_{\vec{n}}(\vec{e}_2) = \vec{e}_2 - 2\frac{\vec{e}_2 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\left(\frac{-1}{6}\right)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\text{refl}_{\vec{n}}(\vec{e}_3) = \vec{e}_3 - 2\frac{\vec{e}_3 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\left(\frac{2}{6}\right)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

Hence,

$$[\text{refl}_{\vec{n}}] = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}$$

PROBLEMS 3.3

Practice Problems

For Problems A1–A4, determine the matrix of the rotation in the plane through the given angle.

A1 $\frac{\pi}{2}$ **A2** π **A3** $-\frac{\pi}{4}$ **A4** $\frac{2\pi}{5}$

For Problems A5–A9, in $\mathbb{R}^2$, let V be a vertical shear by amount 3 and S be a stretch by a factor 5 in the x_2-direction.

A5 Determine $[V]$ and $[S]$.

A6 Calculate the composition of S *followed by* V.

A7 Calculate the composition of S *following* V.

A8 Calculate the composition of S *followed by* a rotation through angle θ.

A9 Calculate the composition of S *following* a rotation through angle θ.

For Problems A10–A14, in $\mathbb{R}^2$, let H be a horizontal shear by amount 1 and V be a vertical shear by amount -2.

A10 Determine $[H]$ and $[V]$.

A11 Calculate the composition of H *followed by* V.

A12 Calculate the composition of H *following* V.

A13 Calculate the composition of H *followed by* a reflection over the x_1-axis.

A14 Calculate the composition of H *following* a reflection over the x_1-axis.

For Problems A15–A18, determine the matrix of the reflection over the given line in $\mathbb{R}^2$.

A15 $x_1 + 3x_2 = 0$ **A16** $2x_1 - x_2 = 0$

A17 $-4x_1 + x_2 = 0$ **A18** $3x_1 - 5x_2 = 0$

For Problems A19–A22, determine the matrix of the reflection over the given plane in $\mathbb{R}^3$.

A19 $x_1 + x_2 + x_3 = 0$ **A20** $2x_1 - 2x_2 - x_3 = 0$

A21 $x_1 - x_3 = 0$ **A22** $x_1 + 2x_2 - 3x_3 = 0$

A23 Let $D : \mathbb{R}^3 \to \mathbb{R}^3$ be the dilation with factor $t = 5$ and let $\text{inj} : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear mapping defined by $\text{inj}(x_1, x_2, x_3) = (x_1, x_2, 0, x_3)$. Determine the matrix of $\text{inj} \circ D$.

A24 (a) Let $P : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear mapping defined by $P(x_1, x_2, x_3) = (x_2, x_3)$ and let S be the shear in $\mathbb{R}^3$ such that $S(x_1, x_2, x_3) = (x_1, x_2, x_3 + 2x_1)$. Determine the matrix of $P \circ S$.

(b) Can you define a shear $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T \circ P = P \circ S$, where P and S are as in part (a)?

(c) Let $Q : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear mapping defined by $Q(x_1, x_2, x_3) = (x_1, x_2)$. Determine the matrix of $Q \circ S$, where S is the mapping in part (a).

Homework Problems

For Problems B1–B4, determine the matrix of the rotation in the plane through the given angle.

B1 $\frac{3\pi}{2}$ **B2** $\pi/3$ **B3** $-\frac{3\pi}{4}$ **B4** $\frac{5\pi}{6}$

For Problems B5–B9, in $\mathbb{R}^2$, let H be a horizontal shear by amount -2 and S be a stretch by a factor 3 in the x_1-direction.

B5 Determine $[H]$ and $[S]$.

B6 Calculate the composition of S *followed by* H.

B7 Calculate the composition of S *following* H.

B8 Calculate the composition of S *followed by* a rotation through angle $\frac{\pi}{3}$.

B9 Calculate the composition of S *following* a rotation through angle $\frac{\pi}{3}$.

For Problems B10–B14, in $\mathbb{R}^2$, let V be a vertical shear by amount -1 and T be a contraction by a factor $t = 1/2$.

B10 Determine $[V]$ and $[T]$.

B11 Calculate the composition of T *followed by* V.

B12 Calculate the composition of T *following* V.

B13 Find a linear mapping J such that $V \circ J = \text{Id}$.

B14 Find a linear mapping K such that $K \circ T = \text{Id}$.

For Problems B15–B18, determine the matrix of the reflection over the given line in $\mathbb{R}^2$.

B15 $2x_1 + 3x_2 = 0$ **B16** $5x_1 - x_2 = 0$

B17 $4x_1 - 3x_2 = 0$ **B18** $x_1 - 5x_2 = 0$

For Problems B19–B22, determine the matrix of the reflection over the given plane in $\mathbb{R}^3$.

B19 $x_1 + 2x_2 - 4x_3 = 0$ **B20** $x_1 + 2x_2 - 3x_3 = 0$

B21 $x_1 + 2x_3 = 0$ **B22** $2x_1 - 2x_2 + 3x_3 = 0$

B23 (a) Let $C : \mathbb{R}^3 \to \mathbb{R}^3$ be a contraction with factor $1/3$ and let $\text{inj} : \mathbb{R}^3 \to \mathbb{R}^5$ be the linear mapping defined by $\text{inj}(x_1, x_2, x_3) = (0, x_1, 0, x_2, x_3)$. Determine the matrix of $\text{inj} \circ C$.

(b) Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the shear defined by $S(x_1, x_2, x_3) = (x_1, x_2 - 2x_3, x_3)$. Determine the matrices $C \circ S$ and $S \circ C$, where C is the contraction in part (a).

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the shear defined by $T(x_1, x_2, x_3) = (x_1 + 3x_2, x_2, x_3)$. Determine the matrix of $S \circ T$ and $T \circ S$, where S is the mapping in part (b).

Conceptual Problems

C1 Let R denote the reflection in Problem A15 and let S denote the reflection in Problem A16. Show that the composition of R and S can be identified as a rotation. Determine the angle of the rotation. Draw a picture illustrating how the composition of these reflections is a rotation.

C2 In $\mathbb{R}^3$, calculate the matrix of the composition of a reflection in the x_2x_3-plane followed by a reflection in the x_1x_2-plane and identify it as a rotation about some coordinate axis. What is the angle of the rotation?

C3 Consider rotations R_θ and R_α in $\mathbb{R}^2$.

(a) From geometrical considerations, we know that $R_\alpha \circ R_\theta = R_{\alpha+\theta}$. Verify the corresponding matrix equation.

(b) Prove that $[R_{-\theta}]^T = [R_\theta]$.

(c) Prove that the columns of R_θ form an orthonormal basis for $\mathbb{R}^2$.

C4 Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$. A matrix A is said to be symmetric if $A^T = A$.

(a) Use the result of Section 3.2 Problem C7 to prove that $[\text{proj}_{\vec{u}}]$ is symmetric.

(b) Prove that $[\text{refl}_{\vec{n}}] = I - 2[\text{proj}_{\vec{n}}]$.

(c) Prove that $[\text{refl}_{\vec{n}}]$ is symmetric.

(d) From geometrical considerations, we know that $\text{refl}_{\vec{n}} \circ \text{refl}_{\vec{n}} = \text{Id}$. Verify the corresponding matrix equation. (Hint: use part (b) and the fact that $\text{proj}_{\vec{n}}$ satisfies the projection property (L2) from Section 1.5.)

C5 (a) Construct a 2×2 matrix $A \neq I$ such that $A^3 = I$. (Hint: think geometrically.)

(b) Construct a 2×2 matrix $A \neq I$ such that $A^5 = I$.

3.4 Special Subspaces

When working with a function we are often interested in the subset of the codomain which contains all possible images under the function. This is called the **range** *of the function. In this section we will look not only at the range of a linear mapping but also at a special subset of the domain. We will then use the connection between linear mappings and their standard matrices to derive four important subspaces of a matrix.*

The other main purpose of this section is to review and connect many of the concepts we have covered so far in the text. In addition to using what we learned about matrices and linear mappings in Section 3.1 and 3.2, we will be using:
- *subspaces and bases (Section 1.4)*
- *dot products and orthogonality (Section 1.5)*
- *rank of a matrix (Section 2.2)*
- *dimension of a subspace (Section 2.3)*

Special Subspaces of Linear Mappings

Range

Definition
Range

The **range** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined to be the set

$$\text{Range}(L) = \{L(\vec{x}) \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\}$$

EXAMPLE 3.4.1

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear mapping defined by $L(x_1, x_2) = (2x_1 - x_2, 0, x_1 + x_2)$. Find Range($L$).

Solution: By definition of the range, if $L(\vec{x})$ is any vector in the range, then

$$L(\vec{x}) = \begin{bmatrix} 2x_1 - x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This is valid for any $x_1, x_2 \in \mathbb{R}$. Thus, $\text{Range}(L) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

EXAMPLE 3.4.2

Find the range of a rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ through an angle θ.

Solution: Geometrically, it is clear that the range of the rotation is all of $\mathbb{R}^2$. Indeed, if we pick any vector $\vec{y} \in \mathbb{R}^2$, then we can get $\vec{y} = R_\theta(\vec{x})$ by taking $\vec{x}$ to be the vector which we get by rotating $\vec{y}$ by an angle of $-\theta$.

EXAMPLE 3.4.3

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Find the range of the linear mappings $\text{proj}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$.

Solution: Geometrically, the range of the projection should be the line spanned by $\vec{v}$. Indeed, by definition, every image $\text{proj}_{\vec{v}}(\vec{x})$ is a vector on the line. Moreover, for any vector $c\vec{v}$ on the line we have

$$\text{proj}_{\vec{v}}(c\vec{v}) = c\vec{v}$$

Hence, $\text{Range}(\text{proj}_{\vec{v}}) = \text{Span}\{\vec{v}\}$.

EXERCISE 3.4.1

Find a spanning set for the range of the linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x_1, x_2, x_3) = (x_1 - x_2, -2x_1 + 2x_2 + x_3)$.

Not surprisingly, we see in the examples that the range of a linear mapping is a subspace of the codomain. To prove that in general, we need the following useful result.

Theorem 3.4.1 If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $L(\vec{0}) = \vec{0}$.

Proof: Since $0\vec{x} = \vec{0}$ for any $\vec{x} \in \mathbb{R}^n$, we have

$$\vec{0} = 0L(\vec{x}) = L(0\vec{x}) = L(\vec{0})$$

■

Theorem 3.4.2 If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\text{Range}(L)$ is a subspace of $\mathbb{R}^m$.

Nullspace

Definition
Nullspace

The **nullspace** of a linear mapping L is the set of all vectors whose image under L is the zero vector $\vec{0}$. We write

$$\text{Null}(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{0}\}$$

Remark

The word **kernel**—and the notation $\ker(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{0}\}$—is often used in place of *nullspace*.

EXAMPLE 3.4.4

Let $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Find the nullspace of $\text{proj}_{\vec{v}} : \mathbb{R}^2 \to \mathbb{R}^2$.

Solution: Let $\vec{x} \in \text{Null}(\text{proj}_{\vec{v}})$. By definition of the nullspace, we have that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \text{proj}_{\vec{v}}(\vec{x}) = \frac{\vec{x} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{3x_1 - x_2}{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

This implies that $3x_1 - x_2 = 0$. Thus, every $\vec{x} \in \text{Null}(\text{proj}_{\vec{v}})$ satisfies

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Thus, $\text{Null}(\text{proj}_{\vec{v}}) = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

EXAMPLE 3.4.5

Find the nullspace of the linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L(x_1, x_2) = (2x_1 - x_2, 0, x_1 + x_2)$$

Solution: Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Null}(L)$. Then, we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix}$$

This gives us the homogeneous system

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This implies that the only solution is $\vec{x} = \vec{0}$. Thus, $\text{Null}(L) = \{\vec{0}\}$.

EXERCISE 3.4.2

Find a spanning set for the nullspace of the linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$L(x_1, x_2, x_3) = (x_1 - x_2, -2x_1 + 2x_2 + x_3)$$

Theorem 3.4.3

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\text{Null}(L)$ is a subspace of $\mathbb{R}^n$.

You are asked to prove this in Problem C4.

The Four Fundamental Subspaces of a Matrix

We now look at the relationship of the standard matrix of a linear mapping to its range and nullspace. In doing so, we will derive four important subspaces of a matrix.

Column Space

The connection between the range of a linear mapping and its standard matrix follows easily from our second interpretation of matrix-vector multiplication.

Theorem 3.4.4

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping with standard matrix $[L] = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$, then

$$\text{Range}(L) = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$

Proof: For any $\vec{x} \in \mathbb{R}^n$ we have

$$L(\vec{x}) = [L]\vec{x} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

Thus, a vector $\vec{y}$ is in Range(L) if and only if it is in $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$. Hence, $\text{Range}(L) = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$ as required. ■

Therefore, the range of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is the subspace of $\mathbb{R}^m$ spanned by the columns of its standard matrix.

Definition
Column Space

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in M_{m\times n}(\mathbb{R})$. The **column space** of A is the subspace of $\mathbb{R}^m$ defined by

$$\text{Col}(A) = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \{A\vec{x} \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\}$$

EXAMPLE 3.4.6

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$. Determine whether $\vec{c} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}$ are in the Col(A).

Solution: By definition, $\vec{c} \in \text{Col}(A)$ if there exists $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x} = \vec{c}$. Similarly, $\vec{d} \in \text{Col}(A)$ if there exists $\vec{y} \in \mathbb{R}^2$ such that $A\vec{y} = \vec{d}$. Since the coefficient matrix is the same for the two systems, we can answer both questions simultaneously by row reducing the doubly augmented matrix $\left[\begin{array}{c|c|c} A & \vec{c} & \vec{d} \end{array}\right]$:

$$\left[\begin{array}{cc|c|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 3 & -1 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

If we ignore the second augmented column, then this corresponds to solving the system $\left[\begin{array}{c|c} A & \vec{c} \end{array}\right]$. Hence, we see that $A\vec{x} = \vec{c}$ is consistent, so $\vec{c} \in \text{Col}(A)$.
Similarly, if we ignore the first augmented column, then this corresponds to solving the system $\left[\begin{array}{c|c} A & \vec{d} \end{array}\right]$. From this we see that $A\vec{x} = \vec{d}$ is inconsistent and hence $\vec{d} \notin \text{Col}(A)$.

EXAMPLE 3.4.7

Find a basis for the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \end{bmatrix}$ and state its dimension.

Solution: By definition, we have that

$$\operatorname{Col}(A) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$$

We now just need to determine if the spanning set is linearly independent. So, we consider

$$c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As we saw in Section 2.3, this gives us the homogeneous system with coefficient matrix A. Row reducing A to RREF gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & 7/3 \end{bmatrix}$$

The general solution is $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = t\begin{bmatrix} 5/3 \\ -7/3 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. Taking $t = 1$, gives $c_1 = 5/3$, $c_2 = -7/3$, and $c_3 = 1$. Hence,

$$\frac{5}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{7}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Hence, by Theorem 1.4.3 we get that

$$\operatorname{Col}(A) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

Since $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ is also linearly independent, we have that $\mathcal{B}$ is a basis for $\operatorname{Col}(A)$.

By definition, the dimension of a subspace is the number of vectors in a basis. Therefore, since $\mathcal{B}$ contains 2 vectors, we get that

$$\dim \operatorname{Col}(A) = 2$$

Example 3.4.7 shows us that all we need to do to find a basis for the column space of a matrix A is to determine which columns of A can be written as linear combinations of the others and remove them. Let R be the reduced row echelon form of A. As in the example, the columns of A that correspond to columns in R without leading ones will be linear combinations of the columns of A corresponding to the columns in R with leading ones. We get the following theorem.

Theorem 3.4.5 If R is the reduced row echelon form of a matrix A, then the columns of A that correspond to the columns of R with leading ones form a basis of the column space of A. Moreover,

$$\dim \operatorname{Col}(A) = \operatorname{rank}(A)$$

The idea of the proof of Theorem 3.4.5 is the same as the method outlined in the solution of Example 3.4.7.

EXAMPLE 3.4.8

Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 4 & 2 & 3 \\ 3 & 6 & 4 & 3 \end{bmatrix}$. Find a basis for Col($A$) and state its dimension.

Solution: Row reducing A gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 4 & 2 & 3 \\ 3 & 6 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The first, third, and fourth columns of R contain leading ones. Therefore, by Theorem 3.4.5, the first, third, and fourth columns of matrix A form a basis for Col(A). So, a basis for Col(A) is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}$$

We have

$$\dim \operatorname{Col}(A) = 3 = \operatorname{rank}(A)$$

Notice in Example 3.4.8 that last entry of every vector in the reduced row echelon form R of A is 0. Hence, $\operatorname{Col}(A) \neq \operatorname{Col}(R)$. That is, the first, third, and fourth columns of R *do not* form a basis for Col(A).

EXERCISE 3.4.3

Let $A = \begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 1 & -1 & 0 & 2 & -3 \\ -1 & 2 & 1 & -3 & -2 \end{bmatrix}$. Find a basis for Col($A$).

Nullspace

We now connect the standard matrix of a linear mapping L with Null(L).

Theorem 3.4.6 If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping with standard matrix $[L]$, then $\vec{x} \in \text{Null}(L)$ if and only if $[L]\vec{x} = \vec{0}$.

Proof: If $\vec{x} \in \text{Null}(L)$, then $\vec{0} = L(\vec{x}) = [L]\vec{x}$.

On the other hand, if $[L]\vec{x} = \vec{0}$, then $\vec{0} = [L]\vec{x} = L(\vec{x})$, so $\vec{x} \in \text{Null}(L)$. ■

This motivates the following definition.

Definition
Nullspace

Let $A \in M_{m\times n}(\mathbb{R})$. The **nullspace** (**kernel**) of A is defined by

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

Remark

Since the nullspace of A is just the solution space of a homogeneous system, we get by Theorem 2.2.3 that the nullspace of A is a subspace of $\mathbb{R}^n$.

EXAMPLE 3.4.9

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Determine whether $\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is in Null(A).

Solution: We have $A\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Since $A\vec{x} \neq \vec{0}$, $\vec{x} \notin \text{Null}(A)$.

EXAMPLE 3.4.10

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & -2 & 2 \end{bmatrix}$. Find a basis for the nullspace of A.

Solution: To find a basis for Null(A), we just need to find a basis for the solution space of $A\vec{x} = \vec{0}$ using the methods of Chapter 2.

Row reducing A gives

$$\begin{bmatrix} 1 & 3 & 1 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$$

Thus, the solution space of the system $A\vec{x} = \vec{0}$ is

$$\vec{x} = t\begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Hence, $\mathcal{B} = \left\{\begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}\right\}$ spans Null(A). Since $\mathcal{B}$ contains only one non-zero vector, it is also linearly independent. Therefore, $\mathcal{B}$ is a basis for Null(A).

EXAMPLE 3.4.11

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 0 & -4 \\ -2 & 6 & 4 \end{bmatrix}$. Find a basis for the nullspace of A.

Solution: Row reducing A gives

$$\begin{bmatrix} 1 & 2 & -1 \\ 5 & 0 & -4 \\ -2 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the only solution of the system $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. Therefore, $\text{Null}(A) = \{\vec{0}\}$. By definition, a basis for this subspace is the empty set.

As indicated in the examples, to find a basis for the nullspace of a matrix we just use the methods of Chapter 2 to solve the homogeneous system $A\vec{x} = \vec{0}$. You may have already observed that the vector equation of the solution space we get from this method always contains a set of linearly independent vectors. We demonstrate this more carefully with an example.

EXAMPLE 3.4.12

Let $A = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{bmatrix}$. Find a basis for Null($A$) and relate the dimension of Null(A) to rank(A).

Solution: Observe that A is already in reduced row echelon form. We find that the general solution to $A\vec{x} = \vec{0}$ is

$$\vec{x} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -4 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \quad t_1, t_2, t_3 \in \mathbb{R}$$

To see that the set containing these three vectors is linearly independent, observe that the vector corresponding to the free variable x_2 has a 1 as its second entry while the other vectors have a 0. Similarly, the vector corresponding to x_4 has a 1 as its fourth entry while the other vectors have 0, and so forth for the vector corresponding to x_5.

Since the vectors we got from the free variables formed a linearly independent set, we have that the dimension of the nullspace is the number of free variables. Hence,

$$\dim \text{Null}(A) = 3 = (\text{ \# of columns}) - \text{rank}(A)$$

Following the method in the example, we get the following theorem.

Theorem 3.4.7

If $A \in M_{m\times n}(\mathbb{R})$ with $\text{rank}(A) = r$, then

$$\dim \text{Null}(A) = n - r$$

Row Space and Left Nullspace

In a variety of applications, for example analyzing incidence matrices for electric circuits or the stoichiometry matrix for chemical reactions, we find that the column space and nullspace of A^T are are also important.

Definition
Row Space
Left Nullspace

Let $A \in M_{m\times n}(\mathbb{R})$. The **row space** of A is the subspace of $\mathbb{R}^n$ defined by

$$\operatorname{Row}(A) = \{A^T\vec{x} \in \mathbb{R}^n \mid \vec{x} \in \mathbb{R}^m\} = \operatorname{Col}(A^T)$$

The **left nullspace** of A is the subspace of $\mathbb{R}^m$ defined by

$$\operatorname{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m \mid A^T\vec{x} = \vec{0}\}$$

To find a basis for the left nullspace, we can just use our procedure for finding a basis for the nullspace on A^T. That is, we solve $A^T\vec{x} = \vec{0}$. To find a basis for the row space, we use the following theorem.

Theorem 3.4.8

If R is the reduced row echelon form of a matrix A, then the non-zero rows of R form a basis of the row space of A. Moreover,

$$\dim \operatorname{Row}(A) = \operatorname{rank}(A)$$

EXAMPLE 3.4.13

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Find a basis for $\operatorname{Row}(A)$ and a basis for $\operatorname{Null}(A^T)$.

Solution: Row reducing A gives

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Hence, by Theorem 3.4.8, a basis for $\operatorname{Row}(A)$ is formed by taking the non-zero rows of R. So, a basis for $\operatorname{Row}(A)$ is $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

To find a basis for $\operatorname{Null}(A^T)$ we row reduce A^T and solve $A^T\vec{x} = \vec{0}$. We get

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for $\operatorname{Null}(A^T)$ is $\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$.

These four special subspaces of matrix are extremely important so we make the following definition.

Definition
Four Fundamental Subspaces

Let $A \in M_{m\times n}(\mathbb{R})$. We call $\mathrm{Col}(A)$, $\mathrm{Null}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Null}(A^T)$ the **Four Fundamental Subspaces** of A.

Rank-Nullity Theorem

Combining Theorem 3.4.5 and Theorem 3.4.7 gives the following theorem.

Theorem 3.4.9 Rank-Nullity Theorem

If $A \in M_{m\times n}(\mathbb{R})$, then

$$\mathrm{rank}(A) + \dim(\mathrm{Null}(A)) = n$$

EXAMPLE 3.4.14

Find a basis for each of the four fundamental subspaces of $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 6 & 1 & -1 \\ -2 & -4 & 2 & 6 \end{bmatrix}$ and verify the Rank-Nullity Theorem for both A and A^T.

Solution: Row reducing A and A^T gives

$$A \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^T \sim \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus:

A basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$, and a basis for $\mathrm{Null}(A^T)$ is $\left\{ \begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix} \right\}$.

A basis for $\mathrm{Row}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$, and a basis for $\mathrm{Null}(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

For A, we have that $\mathrm{rank}(A) = 2$ and $\dim \mathrm{Null}(A) = 2$, so indeed

$$\mathrm{rank}(A) + \dim \mathrm{Null}(A) = 2 + 2 = 4 = \text{ \# of columns of } A$$

as predicted by the Rank-Nullity Theorem.

Similarly, for A^T, we have that $\mathrm{rank}(A^T) = 2$ and $\dim \mathrm{Null}(A^T) = 1$ so

$$\mathrm{rank}(A^T) + \dim \mathrm{Null}(A^T) = 2 + 1 = 3 = \text{ \# of columns of } A^T$$

We can observe something amazing about the bases for the fundamental subspaces of A in Example 3.4.14. First, each basis vector of $\mathrm{Col}(A)$ is orthogonal to each basis vector of $\mathrm{Null}(A^T)$. Similarly, each basis vector of $\mathrm{Row}(A)$ is orthogonal to each basis vector of $\mathrm{Null}(A)$. This implies that if we combine the basis vectors for $\mathrm{Col}(A)$ and $\mathrm{Null}(A^T)$ we will get a basis for $\mathbb{R}^3$, and if we combine the basis vectors for $\mathrm{Row}(A)$ and $\mathrm{Null}(A)$ we will get a basis for $\mathbb{R}^4$.

EXERCISE 3.4.4

Find a basis for each of the four fundamental subspaces of $A = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 2 & 3 & -8 & 4 \\ 0 & 1 & -2 & 3 \end{bmatrix}$ and verify the Rank-Nullity Theorem for A and A^T.

Fundamental Theorem of Linear Algebra

We now prove that our observations about the fundamental subspaces of the matrix A in Example 3.4.14 hold for all $m \times n$ matrices.

Theorem 3.4.10 Fundamental Theorem of Linear Algebra (FTLA)

If $A \in M_{m\times n}(\mathbb{R})$ with $\operatorname{rank}(A) = k$, then

(1) $\operatorname{Null}(A) = \{\vec{n} \in \mathbb{R}^n \mid \vec{n} \cdot \vec{r} = 0 \text{ for all } \vec{r} \in \operatorname{Row}(A)\}$.

(2) $\operatorname{Null}(A^T) = \{\vec{\ell} \in \mathbb{R}^m \mid \vec{\ell} \cdot \vec{c} = 0 \text{ for all } \vec{c} \in \operatorname{Col}(A)\}$.

(3) If $\{\vec{w}_1, \dots, \vec{w}_k\}$ is a basis for $\operatorname{Row}(A)$ and $\{\vec{w}_{k+1}, \dots, \vec{w}_n\}$ is a basis for $\operatorname{Null}(A)$, then $\{\vec{w}_1, \dots, \vec{w}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}$ is a basis for $\mathbb{R}^n$.

(4) If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for $\operatorname{Col}(A)$ and $\{\vec{v}_{k+1}, \dots, \vec{v}_m\}$ is a basis for $\operatorname{Null}(A^T)$, then $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_m\}$ is a basis for $\mathbb{R}^m$.

Proof: We prove (3) and (4) and leave (1) and (2) as Problem C10.

(3): Consider

$$c_1\vec{w}_1 + \cdots + c_k\vec{w}_k + c_{k+1}\vec{w}_{k+1} + \cdots + c_n\vec{w}_n = \vec{0} \tag{3.5}$$

Rearranging we can get

$$c_1\vec{w}_1 + \cdots + c_k\vec{w}_k = -c_{k+1}\vec{w}_{k+1} - \cdots - c_n\vec{w}_n$$

Observe that $c_1\vec{w}_1 + \cdots + c_k\vec{w}_k \in \operatorname{Row}(A)$ and $-c_{k+1}\vec{w}_{k+1} - \cdots - c_n\vec{w}_n \in \operatorname{Null}(A)$. Hence, $\vec{x} = c_1\vec{w}_1 + \cdots + c_k\vec{w}_k$ is a vector in both $\operatorname{Row}(A)$ and $\operatorname{Null}(A)$. Hence, by (1) we have that $\vec{x} \cdot \vec{x} = 0$. By Theorem 1.5.1 (2), this implies that $\vec{x} = \vec{0}$. Thus, we have

$$c_1\vec{w}_1 + \cdots + c_k\vec{w}_k = \vec{0} = -c_{k+1}\vec{w}_{k+1} - \cdots - c_n\vec{w}_n$$

So, $c_1 = \cdots = c_k = 0$ since $\{\vec{w}_1, \dots, \vec{w}_k\}$ is linearly independent, and $c_{k+1} = \cdots = c_n = 0$ since $\{\vec{w}_{k+1}, \dots, \vec{w}_n\}$ is linearly independent. Consequently, the only solution to equation (3.5) is $c_1 = \cdots = c_n = 0$ and so $\{\vec{w}_1, \dots, \vec{w}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}$ is linearly independent.

Since equation (3.5) has a unique solution, the coefficient matrix of this system, $\begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_n \end{bmatrix}$ has rank n by the System-Rank Theorem (2). Hence, by the System-Rank Theorem (3), the system

$$c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = \vec{b}$$

is consistent for all $\vec{b} \in \mathbb{R}^n$. Therefore, $\{\vec{w}_1, \dots, \vec{w}_n\}$ also spans $\mathbb{R}^n$ and hence is a basis for $\mathbb{R}^n$.

(4): Follows immediately from (3) by substituting A^T in for A. ■

Part (1) says the nullspace of A is exactly the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in the row space of A. We say that Null(A) and Row(A) are **orthogonal complements** of each other.

Part (2) says that the left nullspace of A and the column space of A are also orthogonal complements.

Part (3) can be interpreted as saying that every vector $\vec{x}_0 \in \mathbb{R}^n$ can be written as a sum of a vector $\vec{r}_0 \in \text{Row}(A)$ and a vector $\vec{n}_0 \in \text{Null}(A)$. Combining this with Part (1), we can think of $\mathbb{R}^n$ as being split up by a pair of orthogonal axes, where one axis is the nullspace and the other axis is the row space. We depict this in Figure 3.4.1.

Part (4) indicates that every vector $\vec{y}_0 \in \mathbb{R}^m$ can be written as a sum of a vector $\vec{c}_0 \in \text{Col}(A)$ and a vector $\vec{\ell}_0 \in \text{Null}(A^T)$. Hence, combining this with Part (2), we can think of $\mathbb{R}^m$ as having axes of the left nullspace and the column space as depicted in Figure 3.4.1.

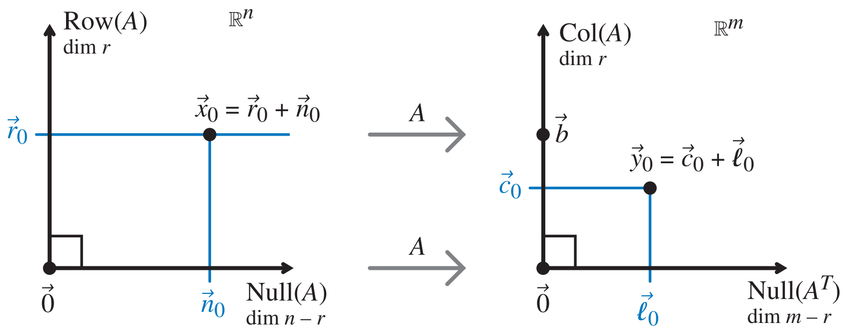


Figure 3.4.1 Graphical representation of the Fundamental Theorem of Linear Algebra.

The figure gives us a visualization of much of what we have done with solving systems of linear equations and more. For example, it shows that each horizontal line

$$\vec{x} = \vec{r}_0 + \vec{n}, \qquad \text{for all } \vec{n} \in \text{Null}(A)$$

in $\mathbb{R}^n$ is mapped by A to a unique vector $\vec{b} \in \text{Col}(A)$. That is, the matrix mapping $A\vec{x}$ is a one-to-one and onto mapping from Row(A) to Col(A).

This shows us that:

1. The general solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{r}_0 + \vec{n}$ for all $\vec{n} \in \text{Null}(A)$.
2. $A\vec{x} = \vec{b}$ has a unique solution if and only if $\text{Null}(A) = \{\vec{0}\}$.
3. $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^m$ if and only if $\text{Col}(A) = \mathbb{R}^m$ (i.e. $\text{rank}(A) = m$).
4. If $\vec{x}$ and $\vec{p}$ are both solutions of $A\vec{x} = \vec{b}$, then $\vec{x} - \vec{p} \in \text{Null}(A)$.

A Summary of Facts About Rank

For an $m \times n$ matrix A:

$$\begin{aligned}
\text{rank}(A) &= \text{the number of leading ones in the reduced row echelon form of } A \\
&= \text{the number of non-zero rows in any row echelon form of } A \\
&= \dim \text{Row}(A) \\
&= \dim \text{Col}(A) \\
&= n - \dim \text{Null}(A) \\
&= m - \dim \text{Null}(A^T) \\
&= \text{rank}(A^T)
\end{aligned}$$

PROBLEMS 3.4

Practice Problems

For Problems A1–A3, let L be the linear mapping with the given standard matrix.

(a) Is $\vec{y}_1 \in \text{Range}(L)$? If so, find $\vec{x}$ such that $L(\vec{x}) = \vec{y}_1$.

(b) Is $\vec{y}_2 \in \text{Range}(L)$? If so, find $\vec{x}$ such that $L(\vec{x}) = \vec{y}_2$.

(c) Is $\vec{v} \in \text{Null}(L)$?

A1 $[L] = \begin{bmatrix} 3 & 5 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \vec{y}_1 = \begin{bmatrix} 12 \\ 3 \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$

A2 $[L] = \begin{bmatrix} 7 & -2 \\ 7 & -8 \\ 1 & 1 \end{bmatrix}, \vec{y}_1 = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} 8 \\ -10 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

A3 $[L] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 5 & 1 \end{bmatrix}, \vec{y}_1 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For Problems A4–A14, find a basis for the range and a basis for the nullspace of the linear mapping.

A4 $L(x_1, x_2) = (3x_1 + 5x_2, x_1 - x_2, x_1 - x_2)$

A5 $L(x_1, x_2) = (2x_1 - x_2, 4x_1 - 2x_2)$

A6 $L(x_1, x_2) = (x_1 - 7x_2, x_1 + x_2)$

A7 $L(x_1, x_2) = (x_1, 2x_1, 3x_1)$

A8 $L(x_1, x_2, x_3) = (x_1 + x_2, 0)$

A9 $L(x_1, x_2, x_3) = (0, 0, 0)$

A10 $L(x_1, x_2, x_3) = (2x_1, -x_2 + 2x_3)$

A11 $L(x_1, x_2, x_3) = (x_1 + 7x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3)$

A12 $L(x_1, x_2, x_3) = (-x_2 + 2x_3, x_2 - 4x_3, -2x_2 + 4x_3)$

A13 $L(x_1, x_2, x_3, x_4) = (x_4, x_3, 0, x_2, x_1 + x_2 - x_3)$

A14 $L(x_1, x_2, x_3, x_4) = (x_1 - x_2 + 2x_3 + x_4, x_2 - 3x_3 + 3x_4)$

For Problems A15–A19, find a basis for each of the four fundamental subspaces of the standard matrix $[L]$ of L.

A15 The linear mapping in A4.

A16 The linear mapping in A8.

A17 The linear mapping in A9.

A18 The linear mapping in A10.

A19 The linear mapping in A13.

For Problems A20–A22, use a geometrical argument to give a basis for the nullspace and a basis for the range of the linear mapping.

A20 $\text{proj}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

A21 $\text{perp}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

A22 $\text{refl}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For Problems A23–A26, determine the standard matrix of a linear mapping L with the given nullspace and range.

A23 $\text{Null}(L) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \text{Range}(L) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \right\}$

A24 $\text{Null}(L) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \text{Range}(L) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

A25 $\text{Null}(L) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}, \text{Range}(L) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

A26 $\text{Null}(L) = \text{Span}\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}, \text{Range}(L) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

A27 Give a mathematical proof of the statement "If $\vec{r}_0$ is a solution of $A\vec{x} = \vec{b}$, then $\vec{r}_0 + \vec{n}$ is a solution of $A\vec{x} = \vec{b}$ for all $\vec{n} \in \text{Null}(A)$."

For Problems A28 and A29, suppose that the matrix is the coefficient matrix of a homogeneous system of equations. State the following:

(a) The number of variables in the system.

(b) The rank of the matrix.

(c) The dimension of the solution space.

A28 $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

A29 $\begin{bmatrix} 1 & -2 & 0 & 0 & 5 \\ 0 & 1 & 3 & 4 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

For Problems A30–A42, find a basis for each of the four fundamental subspaces, and verify the Rank-Nullity Theorem for A and A^T.

A30 $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

A31 $\begin{bmatrix} -7 & 3 \\ -3 & 1 \end{bmatrix}$

A32 $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$

A33 $\begin{bmatrix} 0 & -2 & 4 & 2 \\ 0 & 3 & -6 & -3 \end{bmatrix}$

A34 $\begin{bmatrix} 1 & 1 & -3 & 1 \\ 2 & 3 & -8 & 4 \\ 0 & 1 & -2 & 3 \end{bmatrix}$

A35 $\begin{bmatrix} 1 & 2 & 8 \\ 1 & 1 & 5 \\ 1 & 0 & -2 \end{bmatrix}$

A36 $\begin{bmatrix} 2 & 1 & 3 \\ 2 & -2 & 6 \\ 4 & 3 & 5 \end{bmatrix}$

A37 $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$

A38 $\begin{bmatrix} 1 & 2 & 9 \\ 0 & 1 & 7 \\ -2 & -2 & -4 \\ 1 & 1 & 2 \end{bmatrix}$

A39 $\begin{bmatrix} 3 & -1 & 6 \\ 1 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix}$

A40 $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

A41 $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

A42 $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & 1 & 7 & 1 \\ 2 & 4 & 0 & 6 & 1 \\ 3 & 6 & 1 & 13 & 2 \end{bmatrix}$

For Problems A43–A48, let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 2 & 3 & 1 & 2 & 11 \\ 1 & 1 & 1 & 3 & 7 \\ 1 & 2 & 0 & -1 & 4 \end{bmatrix}$ which has RREF $R = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Let

$$\vec{u} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 3 \\ -4 \\ 10 \\ 2 \\ 7 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

A43 Is $\vec{u} \in \operatorname{Null}(A^T)$?

A44 Is $\vec{v} \in \operatorname{Col}(A)$?

A45 Is $\vec{w} \in \operatorname{Null}(A)$?

A46 Is $\vec{z} \in \operatorname{Row}(A)$?

A47 Find a basis for Null(A).

A48 Find all solutions of $A\vec{x} = \vec{b}$.

Homework Problems

For Problems B1 and B2, let L be the linear mapping with the given standard matrix.

(a) Is $\vec{y}_1 \in \operatorname{Range}(L)$? If so, find $\vec{x}$ such that $L(\vec{x}) = \vec{y}_1$.

(b) Is $\vec{y}_2 \in \operatorname{Range}(L)$? If so, find $\vec{x}$ such that $L(\vec{x}) = \vec{y}_2$.

(c) Is $\vec{v} \in \operatorname{Null}(L)$?

B1 $[L] = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & -1 & 1 \end{bmatrix}, \vec{y}_1 = \begin{bmatrix} 1 \\ 6 \\ -9 \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$

B2 $[L] = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 4 & 1 \\ 1 & 3 & 0 \end{bmatrix}, \vec{y}_1 = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

For Problems B3–B11, find a basis for the range and a basis for the nullspace of the linear mapping.

B3 $L(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$

B4 $L(x_1, x_2, x_3, x_4) = (x_1, x_2 + x_4)$

B5 $L(x_1, x_2) = (0, x_1 + x_2)$

B6 $L(x_1, x_2, x_3, x_4) = (-x_1 + x_4, 2x_2 + 3x_3, x_1 - x_4)$

B7 $L(x_1) = (x_1, 2x_1, -x_1)$

B8 $L(x_1, x_2, x_3) = (2x_1 + x_3, x_1 - x_3, x_1 + x_3)$

B9 $L(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_1 + 6x_2 + x_3, x_1 + 2x_2 + x_3)$

B10 $L(x_1, x_2, x_3) = (x_1 - x_3, x_2 - x_3, x_1 - x_2)$

B11 $L(x_1, x_2, x_3, x_4) = (x_1 + x_3, 2x_1 + 2x_3 + x_4)$

For Problems B12–B17, find a basis for each of the four fundamental subspaces of the standard matrix $[L]$ of L.

B12 The linear mapping in B3.

B13 The linear mapping in B4.

B14 The linear mapping in B5.

B15 The linear mapping in B6.

B16 The linear mapping in B7.

B17 The linear mapping in B8.

For Problems B18–B20, use a geometrical argument to give a basis for the nullspace and a basis for the range of the linear mapping.

B18 $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

B19 $\operatorname{proj}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

B20 $\operatorname{refl}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For Problems B21–B24, determine the standard matrix of a linear mapping L with the given nullspace and range.

B21 $\operatorname{Null}(L) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}, \operatorname{Range}(L) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

B22 $\operatorname{Null}(L) = \operatorname{Span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}, \operatorname{Range}(L) = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$

B23 $\operatorname{Null}(L) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \operatorname{Range}(L) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

B24 $\operatorname{Null}(L) = \mathbb{R}^4, \operatorname{Range}(L) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

For Problems B25–B28, suppose that the matrix is the coefficient matrix of a homogeneous system of equations. State the following:

(a) The number of variables in the system.

(b) The rank of the matrix.

(c) The dimension of the solution space.

B25 $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -1 \end{bmatrix}$ **B26** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

B27 $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **B28** $\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

For Problems B29–B37, find a basis for each of the four fundamental subspaces, and verify the Rank-Nullity Theorem for A and A^T.

B29 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **B30** $\begin{bmatrix} 0 & 3 \\ 0 & -11 \end{bmatrix}$ **B31** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

B32 $\begin{bmatrix} 2 & 2 & 1 \\ -2 & -2 & -1 \\ 1 & 4 & 2 \end{bmatrix}$ **B33** $\begin{bmatrix} 1 & 1 & 2 & 1 \\ -2 & -2 & -5 & 1 \\ -1 & -1 & -3 & 2 \end{bmatrix}$

B34 $\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \\ 1 & 2 & -4 \end{bmatrix}$ **B35** $\begin{bmatrix} 3 & 2 & 0 & -1 \\ 3 & 2 & 0 & -1 \\ 3 & 2 & 0 & -1 \end{bmatrix}$

B36 $\begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & -4 \\ 0 & 1 & 5 \end{bmatrix}$ **B37** $\begin{bmatrix} 2 & 7 & -1 & 9 \\ 1 & 14 & 3 & 6 \\ 1 & 8 & 1 & -7 \end{bmatrix}$

For Problems B38–B43, let $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 2 & 1 \\ 3 & 6 & 2 & 1 & 5 & 2 \end{bmatrix}$

which has RREF $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Let

$\vec{u} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

B38 Is $\vec{u} \in \text{Null}(A^T)$?

B39 Is $\vec{v} \in \text{Col}(A)$?

B40 Is $\vec{w} \in \text{Null}(A)$?

B41 Is $\vec{z} \in \text{Row}(A)$?

B42 Find a basis for Null(A).

B43 Find all solution of $A\vec{x} = \vec{b}$.

Conceptual Problems

C1 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Prove that

$$\dim(\text{Range}(L)) + \dim(\text{Null}(L)) = n$$

C2 Suppose that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set in $\mathbb{R}^n$ and that $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping with $\text{Null}(L) = \{\vec{0}\}$. Prove that $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ is linearly independent.

C3 Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Prove that $\text{Null}(L) = \{\vec{0}\}$ if and only if $\text{Range}(L) = \mathbb{R}^n$.

C4 Prove if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then Null(L) is a subspace of $\mathbb{R}^n$.

C5 Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ are linear mappings.

(a) Show that the range of $M \circ L$ is a subspace of the range of M.

(b) Give an example such that the range of $M \circ L$ is not equal to the range of M.

(c) Show that the nullspace of L is a subspace of the nullspace of $M \circ L$.

C6 If A is a 5×7 matrix and rank(A) $= 4$, then what is the nullity of A, and what is the dimension of the column space of A?

C7 If A is a 5×4 matrix, then what is the largest possible dimension of the nullspace of A? What is the largest possible rank of A?

C8 If A is a 4×5 matrix and nullity(A) $= 3$, then what is the dimension of the row space of A?

C9 Let $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 = O_{n,n}$. Prove that the column space of A is a subspace of the nullspace of A.

C10 Prove parts (1) and (2) of the Fundamental Theorem of Linear Algebra.

C11 Let $A \in M_{m\times n}(\mathbb{R})$. If $A\vec{x} = \vec{b}$ is inconsistent, then there exists $\vec{y} \in \mathbb{R}^m$ such that $A^T\vec{y} = \vec{0}$ with $\vec{y}^T\vec{b} \neq 0$.

C12 Assume $\vec{r}_0$ is a solution of $A\vec{x} = \vec{b}$. Prove that if $\vec{y}$ is also a solution of $A\vec{x} = \vec{b}$, then $\vec{y} = \vec{r}_0 + \vec{n}$ for some $\vec{n} \in \text{Null}(A)$.

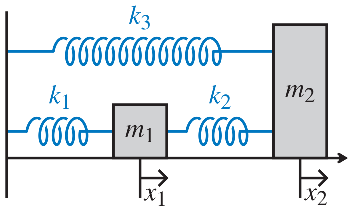
3.5 Inverse Matrices and Inverse Mappings

In this section we will look at inverses of matrices and linear mappings. We will make many connections with the material we have covered so far and provide useful tools for the material contained in the rest of the book.

EXAMPLE 3.5.1

In Example 3.1.11 on page 156 we saw that stiffness matrix $K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$ of the spring-mass system below allowed us to calculate the constant forces f_1, f_2 required to achieve desired equilibrium displacements x_1 and x_2 via the formula

$$K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{3.6}$$



Say that we were instead interested in finding the equilibrium displacements x_1, x_2 for given forces f_1 and f_2. That is, we would like to find a matrix K' such that

$$K' \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.7}$$

Substituting equation (3.7) into equation (3.6) gives

$$K\left(K' \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

which we can rewrite as

$$(KK') \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = I \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where I is the 2×2 identity matrix (see Section 3.1). Since this is valid for all $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathbb{R}^2$, the Matrices Equal Theorem gives

$$KK' = I$$

Similarly, if we instead substitute equation (3.6) into equation (3.7) we find that

$$K'K = I$$

Thus, the matrix K' is the multiplicative inverse of K. Since it computes the equilibrium displacements given the forces on the masses, it is called the **flexibility matrix** of the system.

Definition
Inverse

Let $A \in M_{n\times n}(\mathbb{R})$. If there exists $B \in M_{n\times n}(\mathbb{R})$ such that $AB = I = BA$, then A is said to be **invertible**, and B is called the **inverse** of A (and A is the inverse of B). The inverse of A is denoted A^{-1}.

EXAMPLE 3.5.2

The matrix $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ because

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Theorem 3.5.1 If A is invertible, then A^{-1} is unique.

Proof: If B and C are both inverses of A, then $AB = I = BA$ and $AC = I = CA$. Hence,

$$B = BI = B(AC) = (BA)C = IC = C$$

Therefore, the inverse is unique. ■

Notice in the proof that we actually only need to assume that $I = AC$ and that $BA = I$. In such a case, we call C a **right inverse** of A, and we call B a **left inverse** of A. Observe that the distinction is important since matrix multiplication is not commutative. The proof shows that for a square matrix, any left inverse must be equal to any right inverse. However, non-square matrices may have only a right inverse or a left inverse, but not both (see Problem C8). We will now show that for square matrices, a right inverse is automatically a left inverse.

Theorem 3.5.2 If $A, B \in M_{n\times n}(\mathbb{R})$ such that $AB = I$, then $BA = I$ and hence $B = A^{-1}$. Moreover, $\operatorname{rank}(B) = n = \operatorname{rank}(A)$.

Proof: Consider the homogeneous system $B\vec{x} = \vec{0}$. Since $AB = I$, we find that

$$\vec{x} = I\vec{x} = (AB)\vec{x} = A(B\vec{x}) = A(\vec{0}) = \vec{0}$$

Thus, $B\vec{x} = \vec{0}$ has a unique solution which, by the System-Rank Theorem (2), implies that rank $B = n$.
Let $\vec{y} \in \mathbb{R}^n$. Since $\operatorname{rank}(B) = n$, by the System-Rank Theorem (3), we get that there exists a vector $\vec{x}$ such that $B\vec{x} = \vec{y}$. Hence,

$$BA\vec{y} = BA(B\vec{x}) = B(AB)\vec{x} = BI\vec{x} = B\vec{x} = \vec{y} = I\vec{y}$$

Therefore, $BA = I$ by the Matrices Equal Theorem. Therefore, $AB = I$ and $BA = I$, so that $B = A^{-1}$.
Since we have $BA = I$, we see that $\operatorname{rank}(A) = n$, by the same argument we used to prove $\operatorname{rank}(B) = n$. ■

Theorem 3.5.2 makes it very easy to prove some useful properties of the matrix inverse. In particular, to show that $A^{-1} = B$, we only need to show that $AB = I$.

Theorem 3.5.3 If A and B are invertible matrices and t is a non-zero real number, then
(1) $(tA)^{-1} = \frac{1}{t}A^{-1}$
(2) $(AB)^{-1} = B^{-1}A^{-1}$
(3) $(A^T)^{-1} = (A^{-1})^T$

Proof: We have

$$(tA)\left(\frac{1}{t}A^{-1}\right) = \left(\frac{t}{t}\right)AA^{-1} = 1I = I$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$
$$(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
■

Finding the Inverse of a Matrix

For any given square matrix A, we would like to determine whether it has an inverse and, if so, what the inverse is. Fortunately, one procedure answers both questions. We begin by trying to solve the matrix equation $AX = I$ for the unknown square matrix X. If a solution X can be found, then $X = A^{-1}$ by Theorem 3.5.2. If no such matrix X can be found, then A is not invertible.

To keep it simple, the procedure is examined in the case where A is 3×3, but it should be clear that it can be applied to any square matrix. Write the matrix equation $AX = I$ in the form

$$\begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & A\vec{x}_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix}$$

Hence, we have

$$A\vec{x}_1 = \vec{e}_1, \quad A\vec{x}_2 = \vec{e}_2, \quad A\vec{x}_3 = \vec{e}_3$$

So, it is necessary to solve three systems of equations, one for each column of X. Note that each system has a different standard basis vector as its right-hand side, but all have the same coefficient matrix. Since the procedure for solving systems of equations requires that we row reduce the coefficient matrix, we might as well write out a "triple-augmented matrix" and solve all three systems at once. Therefore, write

$$\left[\begin{array}{c|ccc} A & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}\right] = \left[\begin{array}{c|c} A & I \end{array}\right]$$

and row reduce to reduced row echelon form to solve.

Suppose that A **is row equivalent to** I, so that the reduction gives

$$\left[\begin{array}{c|c} A & I \end{array}\right] \sim \left[\begin{array}{c|ccc} I & \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array}\right]$$

This tells us that $\vec{b}_1$ is a solution of $A\vec{x} = \vec{e}_1$, $\vec{b}_2$ is a solution of $A\vec{x} = \vec{e}_2$, and $\vec{b}_3$ is a solution of $A\vec{x} = \vec{e}_3$. That is, we have

$$A\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} = I$$

Thus,

$$A^{-1} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$$

If the reduced row echelon form of A **is not** I, then $\text{rank}(A) < n$. Hence, A is not invertible, since Theorem 3.5.2 tells us that if A is invertible, then $\text{rank}(A) = n$.

First, we summarize the procedure and then give an example.

Algorithm 3.5.1
Finding A^{-1}

To find the inverse of a square matrix A,

(1) Row reduce the multi-augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$ so that the left block is in reduced row echelon form.

(2) If the reduced row echelon form is $\left[\begin{array}{c|c} I & B \end{array}\right]$, then $A^{-1} = B$.

(3) If the reduced row echelon form of A is not I, then A is not invertible.

EXAMPLE 3.5.3

Determine whether $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ is invertible, and if it is, determine its inverse.

Solution: Write the matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$ and row reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & -2 & 0 & 1 \end{array}\right] \sim$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

Hence, A is invertible and $A^{-1} = \begin{bmatrix} 2 & -5 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$.

You should check that the inverse has been correctly calculated by verifying that $AA^{-1} = I$.

EXAMPLE 3.5.4

Let $A \in M_{2\times 2}(\mathbb{R})$. Find a condition on the entries of A that guarantees that A is invertible and then, assuming that A is invertible, find A^{-1}.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. For A to be invertible, we require that rank $A = 2$. Thus, we cannot have $a = 0 = c$. Assume that $a \neq 0$. Row reducing $[A \mid I]$ we get

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - \frac{c}{a}R_1 \end{array} \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \begin{array}{c} \\ aR_2 \end{array} \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right]$$

Since we need rank $A = 2$, we now require that $ad - bc \neq 0$. Assuming this, we continue row reducing to get

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Thus, we get that A is invertible if and only if $ad - bc \neq 0$. Moreover, if A is invertible, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Repeating the argument in the case that $c \neq 0$ gives the same result.

CONNECTION

The quantity $ad - bc$ determines whether a 2×2 matrix is invertible or not. It is called the **determinant** of the matrix. We will examine this in more detail in Chapter 5.

EXAMPLE 3.5.5

Determine whether $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is invertible, and if it is, determine its inverse.

Solution: Write the matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$ and row reduce:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Hence, A is not invertible.

EXERCISE 3.5.1

Assume that the stiffness matrix for a spring-mass system is $K = \begin{bmatrix} 6 & -4 \\ -4 & 7 \end{bmatrix}$. Find K^{-1} and use it to find the equilibrium displacements x_1, x_2 when the constant forces are $f_1 = 10$ and $f_2 = 5$. Compare with Example 2.4.1.

Invertible Matrix Theorem

In Theorem 3.5.2 and in the description of the procedure for finding the inverse matrix, we used some facts about systems of equations with square matrices. It is worth stating them clearly as a theorem. This theorem is widely considered as one of the most important theorems in linear algebra since it shows us how a variety of concepts are tied together. Note that most of the conclusions follow from previous results.

Theorem 3.5.4

Invertible Matrix Theorem

If $A \in M_{n\times n}(\mathbb{R})$, then the following statements are equivalent (that is, one is true if and only if each of the others is true).

(1) A is invertible.
(2) $\operatorname{rank}(A) = n$.
(3) The reduced row echelon form of A is I.
(4) For all $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ is consistent and has a unique solution.
(5) $\operatorname{Null}(A) = \{\vec{0}\}$
(6) $\operatorname{Col}(A) = \mathbb{R}^n$
(7) $\operatorname{Row}(A) = \mathbb{R}^n$
(8) $\operatorname{Null}(A^T) = \{\vec{0}\}$
(9) A^T is invertible

You are asked to prove the Invertible Matrix Theorem in Problem C16.

Amongst many other things, the Invertible Matrix Theorem tells us that if a matrix A is invertible, then the system $A\vec{x} = \vec{b}$ is consistent with a unique solution. In particular, if we multiply both sides of the equation by A^{-1} on the left we get

$$\begin{aligned} A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

EXAMPLE 3.5.6

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Find the solution of $A\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Solution: By the result of Example 3.5.4, we have that $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

Thus, the solution of $A\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

CONNECTION

It might seem that we solved the system of linear equations in Example 3.5.6 without performing any elementary row operations. However, with some thought, one realizes that the elementary row operations are "contained" inside the inverse of the matrix (we obtained the inverse by row reducing). In the next section, we will see more of the connection between matrix multiplication and elementary row operations.

EXERCISE 3.5.2

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$. Determine A^{-1} and use it to solve $A\vec{x} = \vec{b}$.

It likely seems very inefficient to solve Exercise 3.5.2 by the method described. One would think that simply row reducing the augmented matrix of the system would make more sense. However, if we wanted to solve many systems of equations with the same coefficient matrix A, then we would only need to compute A^{-1} once and each system could then be solved by simple matrix-vector multiplication.

Inverse Linear Mappings

It is useful to introduce the **inverse of a linear mapping** here because many geometrical transformations provide nice examples of inverses. Note that just as the inverse matrix is defined only for square matrices, the inverse of a linear mapping is defined only for linear operators. Recall that the identity transformation $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$ is the linear mapping defined by $\mathrm{Id}(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Definition
Inverse of a Linear Mapping

If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping and there exists another linear mapping $M : \mathbb{R}^n \to \mathbb{R}^n$ such that $M \circ L = \mathrm{Id} = L \circ M$, then L is said to be **invertible**, and M is called the **inverse** of L, usually denoted L^{-1}.

Observe that if M is the inverse of L and $L(\vec{v}) = \vec{w}$, then

$$M(\vec{w}) = M(L(\vec{v})) = (M \circ L)(\vec{v}) = \mathrm{Id}(\vec{v}) = \vec{v}$$

Similarly, if $M(\vec{w}) = \vec{v}$, then

$$L(\vec{v}) = L(M(\vec{w})) = (L \circ M)(\vec{w}) = \mathrm{Id}(\vec{w}) = \vec{w}$$

So, we have $L(\vec{v}) = \vec{w}$ if and only if $M(\vec{w}) = \vec{v}$.

Theorem 3.5.5 If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping with standard matrix $[L] = A$ and $M : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping with standard matrix $[M] = B$, then M is the inverse of L if and only if B is the inverse of A.

Proof: By Theorem 3.2.5, $[M \circ L] = [M][L]$. Hence, $L \circ M = \mathrm{Id} = M \circ L$ if and only if $AB = I = BA$. ■

For many of the geometrical transformations of Section 3.3, an inverse transformation is easily found by geometrical arguments, and these provide many examples of inverse matrices.

EXAMPLE 3.5.7 For each of the following geometrical transformations, determine the inverse transformation. Verify that the product of the standard matrix of the transformation and its inverse is the identity matrix.

(a) The rotation R_θ of the plane

(b) In the plane, a stretch T by a factor of $t > 0$ in the x_1-direction

Solution: (a) The inverse transformation is a rotation by angle $-\theta$. That is, $(R_\theta)^{-1} = R_{-\theta}$. We have

$$[R_{-\theta}] = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

since $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. Hence,

$$\begin{aligned}[R_\theta][R_{-\theta}] &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

(b) The inverse transformation T^{-1} is a stretch by a factor of $\frac{1}{t}$ in the x_1-direction:

$$[T][T^{-1}] = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

CONNECTION

Observe that $[R_\theta]^{-1} = [R_\theta]^T$. This amazing fact is a result of the columns of R_θ being unit vectors that are orthogonal to each other. See Section 1.5 Problem C12. We shall look at other **orthonormal sets** in Chapter 7.

EXERCISE 3.5.3 For each of the following geometrical transformations, determine the inverse transformation. Verify that the product of the standard matrix of the transformation and its inverse is the identity matrix.

(a) A reflection over the line $x_2 = x_1$ in the plane

(b) A shear in the plane by a factor of t in the x_1-direction

Observe that if $\vec{y} \in \mathbb{R}^n$ is in the domain of the inverse M, then it must be in the range of the original L. Therefore, it follows that if L has an inverse, the range of L must be all of the codomain $\mathbb{R}^n$. Moreover, if $L(\vec{x}_1) = \vec{y} = L(\vec{x}_2)$, then by applying M to both sides, we have

$$M(L(\vec{x}_1)) = M(L(\vec{x}_2)) \Rightarrow x_1 = x_2$$

Hence, we have shown that for any $\vec{y} \in \mathbb{R}^n$, there exists a unique $\vec{x} \in \mathbb{R}^n$ such that $L(\vec{x}) = \vec{y}$. This property is the linear mapping version of statement (4) of Theorem 3.5.4 about square matrices.

EXAMPLE 3.5.8

Prove that the linear mapping $\text{proj}_{\vec{v}}$ is not invertible for any $\vec{v} \in \mathbb{R}^n$, $n \geq 2$.

Solution: By definition, $\text{proj}_{\vec{v}}(\vec{x}) = \dfrac{\vec{x} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$. So, any vector $\vec{y} \in \mathbb{R}^n$ that is not a scalar multiple of $\vec{v}$ cannot be in the range of $\text{proj}_{\vec{v}}$. Thus, $\text{Range}(\text{proj}_{\vec{v}}) \neq \mathbb{R}^n$, and hence $\text{proj}_{\vec{v}}$ is not invertible.

EXAMPLE 3.5.9

Prove that the linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(x_1, x_2, x_3) = (2x_1 + x_2, x_3, x_2 - 2x_3)$$

is invertible.

Solution: Assume that $\vec{x}$ is in the nullspace of L. Then $L(\vec{x}) = \vec{0}$, so by definition of L, we have

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

The only solution to this system is $x_1 = x_2 = x_3 = 0$. Thus, $\text{Null}(L) = \{\vec{0}\}$. By Theorem 3.4.6, this implies that $\text{Null}([L]) = \{\vec{0}\}$. Thus, by the Invertible Matrix Theorem $[L]$ is invertible and hence by Theorem 3.5.5, L is also invertible.

Finally, recall that the matrix condition $AB = I = BA$ implies that the matrix inverse can be defined only for square matrices. Here is an example that illustrates for linear mappings that the domain and codomain of L must be the same if it is to have an inverse.

EXAMPLE 3.5.10

Consider the linear mappings $P : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $P(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$ and $\text{inj} : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $\text{inj}(x_1, x_2, x_3) = (x_1, x_2, x_3, 0)$.

It is easy to see that $P \circ \text{inj} = \text{Id}$ but that $\text{inj} \circ P \neq \text{Id}$. Thus, P is not an inverse for inj. Notice that P satisfies the condition that its range is all of its codomain, but it fails the condition that its nullspace is trivial. On the other hand, inj satisfies the condition that its nullspace is trivial but fails the condition that its range is all of its codomain.

PROBLEMS 3.5

Practice Problems

For Problems A1–A19, either show that the matrix is not invertible or find its inverse. Check by multiplication.

A1 $\begin{bmatrix} 3 & 8 \\ -3 & -8 \end{bmatrix}$ **A2** $\begin{bmatrix} 4 & 6 \\ -5 & 1 \end{bmatrix}$ **A3** $\begin{bmatrix} 4 & 6 \\ -2 & 3 \end{bmatrix}$

A4 $\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$ **A5** $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ **A6** $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 3 & 1 & 7 \end{bmatrix}$

A7 $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ **A8** $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **A9** $\begin{bmatrix} 1 & -2 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & -3 \end{bmatrix}$

A10 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **A11** $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & -5 & -6 \end{bmatrix}$ **A12** $\begin{bmatrix} 1 & 3 & -3 \\ 2 & 1 & 4 \\ 2 & -1 & 8 \end{bmatrix}$

A13 $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **A14** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **A15** $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 4 \\ 2 & 3 & 2 \end{bmatrix}$

A16 $\begin{bmatrix} 3 & 2 & 4 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 2 & 4 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix}$ **A17** $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 7 & 1 \\ 0 & 6 & 3 & 1 \end{bmatrix}$

A18 $\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -7 & 3 \\ 0 & 1 & -2 & 2 \\ -1 & 2 & -9 & 2 \end{bmatrix}$ **A19** $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

A20 Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$. Find B^{-1} and use it to find the solution of

(a) $B\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $B\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(c) $B\vec{x} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$ (d) $B\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

A21 Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

(a) Find A^{-1} and B^{-1}.
(b) Calculate AB and $(AB)^{-1}$ and check that $(AB)^{-1} = B^{-1}A^{-1}$.
(c) Calculate $(3A)^{-1}$ and check that it equals $\frac{1}{3}A^{-1}$.
(d) Calculate $(A^T)^{-1}$ and check that $A^T(A^T)^{-1} = I$.
(e) Show that $(A+B)^{-1} \neq A^{-1} + B^{-1}$

A22 The mappings in this problem are from $\mathbb{R}^2 \to \mathbb{R}^2$.
(a) Determine the matrix of the rotation $R_{\pi/6}$ and the matrix of $R_{\pi/6}^{-1}$.
(b) Determine the matrix of a vertical shear S by amount 2 and the matrix of S^{-1}.
(c) Determine the matrix of the reflection R in the line $x_1 - x_2 = 0$ and the matrix of R^{-1}.
(d) Determine the matrix of $(R \circ S)^{-1}$ and the matrix of $(S \circ R)^{-1}$ (without determining the matrices of $R \circ S$ and $S \circ R$).

A23 Suppose that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping and that $M : \mathbb{R}^n \to \mathbb{R}^n$ is a function (not assumed to be linear) such that $\vec{x} = M(\vec{y})$ if and only if $\vec{y} = L(\vec{x})$. Show that M is also linear.

For Problems A24–A29, use a geometrical argument to determine the inverse of the matrix. What do you notice about the matrix and its inverse?

A24 $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ **A25** $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ **A26** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A27 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ **A28** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **A29** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

For Problems A30 and A31, find the inverse of the linear mapping.

A30 $L(x_1, x_2) = (2x_1 + 3x_2, x_1 + 5x_2)$

A31 $L(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, x_2 + x_3, 2x_1 + 3x_2 + 6x_3)$

For Problems A32 and A33, solve the matrix equation for the matrix X.

A32 $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & -1 \end{bmatrix}$ **A33** $X \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ -2 & 1 \end{bmatrix}$

For Problems A34–A36, let $A, B \in M_{n\times n}(\mathbb{R})$. Prove or disprove the statement.

A34 If A is invertible, then the columns of A^T form a linearly independent set.

A35 If A and B are invertible matrices, then $A + B$ is invertible.

A36 If $AX = XC$ and X is invertible, then $A = C$.

Homework Problems

For Problems B1–B16, either show that the matrix is not invertible or find its inverse. Check by multiplication.

B1 $\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$ **B2** $\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$ **B3** $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 2 & 3 & 8 \end{bmatrix}$

B4 $\begin{bmatrix} 2 & 0 \\ 5 & 0 \end{bmatrix}$ **B5** $\begin{bmatrix} 7 & 8 \\ 2 & 0 \end{bmatrix}$ **B6** $\begin{bmatrix} 4 & 0 & -4 \\ 1 & 1 & -1 \\ 6 & 2 & -5 \end{bmatrix}$

B7 $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$ **B8** $\begin{bmatrix} 3 & -3 & -3 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ **B9** $\begin{bmatrix} 1 & 2 & -3 \\ -2 & -3 & 4 \\ 2 & 1 & 0 \end{bmatrix}$

B10 $\begin{bmatrix} 1 & 2 \\ -5 & -8 \end{bmatrix}$ **B11** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$ **B12** $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 9 & 8 \\ -1 & -3 & -3 \end{bmatrix}$

B13 $\begin{bmatrix} 1 & 0 & 3 & -5 \\ 1 & 1 & 3 & -4 \\ 0 & 0 & 1 & -2 \\ 2 & 2 & 6 & -8 \end{bmatrix}$ **B14** $\begin{bmatrix} 1 & -1 & -1 & 4 \\ 0 & 1 & 1 & -2 \\ -2 & 2 & 3 & -11 \\ 0 & 1 & 3 & -7 \end{bmatrix}$

B15 $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \end{bmatrix}$ **B16** $\begin{bmatrix} 0 & 3 & -1 & 1 \\ 2 & 3 & -1 & 1 \\ 0 & 6 & -3 & 2 \\ 2 & -3 & 2 & 0 \end{bmatrix}$

B17 Let $B = \begin{bmatrix} 0 & 1 & 6 \\ 2 & -1 & -4 \\ 4 & -2 & -6 \end{bmatrix}$. Find B^{-1} and use it to find the solution of $B\vec{x} = \vec{b}$ where:

(a) $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (c) $\vec{b} = \begin{bmatrix} -2 \\ -6 \\ 2 \end{bmatrix}$

B18 Let $B = \begin{bmatrix} 3 & 4 & -3 \\ 3 & 8 & -5 \\ -3 & -4 & 4 \end{bmatrix}$. Find B^{-1} and use it to find the solution of $B\vec{x} = \vec{b}$ where:

(a) $\vec{b} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$ (b) $\vec{b} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ (c) $\vec{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

B19 Let $A = \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Find A^{-1} and B^{-1}.

(b) Calculate AB and $(AB)^{-1}$ and check that $(AB)^{-1} = B^{-1}A^{-1}$.

(c) Calculate $(2A)^{-1}$ and check that it equals $\frac{1}{2}A^{-1}$.

(d) Calculate $(A^T)^{-1}$ and check that $A^T(A^T)^{-1} = I$.

B20 The mappings in this problem are from $\mathbb{R}^2 \to \mathbb{R}^2$.

(a) Determine the matrix of the rotation $R_{\pi/3}$ and the matrix of $R_{\pi/3}^{-1}$.

(b) Determine the matrix of a contraction T with $t = 1/2$ and the matrix of T^{-1}.

(c) Determine the matrix of the reflection R in the x_1-axis and the matrix of R^{-1}.

(d) Determine the matrix of $(R \circ T)^{-1}$ and the matrix of $(T \circ R)^{-1}$ (without determining the matrices of $R \circ T$ and $T \circ R$).

For Problems B21–B26, use a geometrical argument to determine the inverse of the matrix. What do you notice about the matrix and its inverse?

B21 $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ **B22** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ **B23** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

B24 $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ **B25** $\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **B26** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

For Problems B27–B30, find the inverse of the linear mapping.

B27 $L(x_1, x_2) = (x_1 - 4x_2, 2x_1 + 2x_2)$

B28 $L(x_1, x_2) = (-2x_1 - x_2, 7x_1 + 4x_2)$

B29 $L(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, x_1 + 3x_2 + 3x_3, -2x_2 + x_3)$

B30 $L(x_1, x_2, x_3) = (x_2 + 2x_3, x_1 + 2x_2 + 5x_3, -x_1 - x_2 - 2x_3)$

For Problems B31–B38, solve the matrix equation for the matrix X.

B31 $\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ **B32** $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

B33 $X \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 2 & 4 \end{bmatrix}$ **B34** $X \begin{bmatrix} 6 & -1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

B35 $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$

B36 $X \begin{bmatrix} 2 & 1 & 4 \\ 4 & 3 & 12 \\ 4 & 4 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

B37 $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

B38 $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} X = X \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

Conceptual Problems

C1 Determine an expression in terms of A^{-1} and B^{-1} for $((AB)^T)^{-1}$.

C2 Suppose that $A \in M_{n\times n}(\mathbb{R})$ such that $A^3 = I$. Find an expression for A^{-1} in terms of A. (Hint: find X such that $AX = I$.)

C3 Suppose that B satisfies $B^5 + B^3 + B = I$. Find an expression for B^{-1} in terms of B.

C4 (a) Prove that if A and B are square matrices such that AB is invertible, then A and B are invertible.
(b) Find non-invertible matrices C and D such that CD is invertible.

For Problems C5–C12, recall the following definitions. Let $A \in M_{m\times n}(\mathbb{R})$. If $B \in M_{n\times m}(\mathbb{R})$ such that $AB = I$, then A is called a **left inverse** of B and B is called a **right inverse** of A.

C5 Give a method for either finding a right inverse B of A or showing that A does not have a right inverse.

C6 Prove if A has a right inverse and $m < n$, then A has infinitely many right inverses.

C7 Prove if $m > n$, then A cannot have a right inverse.

C8 Show that a non-square matrix cannot have both a left and a right inverse. [Hint: use Problem C7.]

C9 Find all right inverses of $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

C10 Find all right inverses of $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

C11 Find all left inverses of $B = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 1 \end{bmatrix}$.

C12 Find all left inverses of $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}$.

C13 Assume that $A \in M_{n\times n}(\mathbb{R})$ is invertible. Prove that if $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set in $\mathbb{R}^n$, then $\{A\vec{v}_1, \dots, A\vec{v}_k\}$ is also linearly independent.

C14 Assume that $A \in M_{n\times n}(\mathbb{R})$ is invertible. Prove that if $\{\vec{w}_1, \dots, \vec{w}_k\}$ spans $\mathbb{R}^n$, the $\{A\vec{w}_1, \dots, A\vec{w}_k\}$ also spans $\mathbb{R}^n$.

C15 (a) Find $A, B \in M_{n\times n}(\mathbb{R})$ such that $(AB)^{-1} \neq A^{-1}B^{-1}$.
(b) What condition is required on AB so that we do have $(AB)^{-1} = A^{-1}B^{-1}$.

C16 Prove the Invertible Matrix Theorem by proving the following statements in the given order.
(a) (1) if and only if (2)
(b) (2) implies (3)
(c) (3) implies (4)
(d) (4) implies (5)
(e) (5) implies (6)
(f) (6) implies (7)
(g) (7) implies (8)
(h) (8) implies (9)
(i) (9) implies (1)

For Problems C17–C26, let $A, B \in M_{n\times n}(\mathbb{R})$. Prove or disprove the statement.

C17 If P is an invertible matrix such that $P^{-1}AP = B$, then $A = B$.

C18 If $AA = A$ and A is not the zero matrix, then A is invertible.

C19 If AA is invertible, then A is invertible.

C20 If A and B satisfy $AB = O_{n,n}$, then B is not invertible.

C21 There exists a linearly independent set $\{A_1, A_2, A_3, A_4\}$ of 2×2 matrices such that each A_i is invertible.

C22 If A has a column of zeroes, then A is not invertible.

C23 If $A\vec{x} = \vec{0}$ has a unique solution, then $\mathrm{Col}(A) = \mathbb{R}^n$.

C24 If A satisfies $A^2 - 2A + I = O_{n,n}$, then A is invertible.

C25 If A is not invertible, then the columns of A form a linearly dependent set.

C26 If the columns of A form a linearly dependent set, then A is not invertible.

3.6 Elementary Matrices

In Sections 3.1 and 3.5, we saw connections between matrix-vector multiplication and systems of linear equations. In Section 3.2, we observed the connection between linear mappings and matrix-vector multiplication. Since matrix multiplication is an extension of matrix-vector multiplication, it should not be surprising that there is a connection between matrix multiplication, systems of linear equations, and linear mappings. We examine this connection through the use of elementary matrices.

Definition
Elementary Matrix

A matrix that can be obtained from the identity matrix by performing a single elementary row operation is called an **elementary matrix**.

Note that it follows from the definition that an elementary matrix must be square.

EXAMPLE 3.6.1

Find the 2×2 elementary matrix corresponding to each row operation by performing the row operation on the 2×2 identity matrix.

(a) $R_2 - 3R_1$ (b) $3R_2$ (c) $R_1 \updownarrow R_2$

Solution: For (a) we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Thus, the corresponding elementary matrix is $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

For (b) we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \\ 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Thus, the corresponding elementary matrix is $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

For (c) we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \\ R_1 \updownarrow R_2 \end{matrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus, the corresponding elementary matrix is $E_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

CONNECTION

Observe in Example 3.6.1 that the matrix E_1 is the standard matrix of a vertical shear by amount -3, E_2 is the standard matrix of a stretch in the x_2-direction by a factor 3, and E_3 is the standard matrix of a reflection over the line $x_1 = x_2$. It can be shown that every $n \times n$ elementary matrix is the standard matrix of a shear, a stretch, or a reflection.

EXAMPLE 3.6.2

Determine which of the following matrices are elementary. For each elementary matrix, indicate the associated elementary row operation.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution: For (a), the matrix is elementary as it can be obtained from the 3×3 identity matrix by performing the row operation $R_1 + 2R_3$.

For (b), the matrix is elementary as it can be obtained from the 3×3 identity matrix by performing the row operation $2R_2$.

For (c), the matrix is not elementary as it would require three elementary row operations to get this matrix from the 3×3 identity matrix.

It is clear that the RREF of every elementary matrix is I; we just need to perform the inverse (opposite) row operation to turn the elementary matrix E back to I. Thus, we not only have that every elementary matrix is invertible, but the inverse is the elementary matrix associated with the reverse row operation.

Theorem 3.6.1 If E is an elementary matrix, then E is invertible and E^{-1} is also an elementary matrix.

EXAMPLE 3.6.3

Find the inverse of each of the following elementary matrices. Check your answer by multiplying the matrices together.

(a) $E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Solution: The inverse matrix E_1^{-1} is the elementary matrix associated with the row operation required to bring E_1 back to I. That is, $R_1 - 2R_2$. Therefore,

$$E_1^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Checking, we get $E_1^{-1}E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The inverse matrix E_2^{-1} is the elementary matrix associated with the row operation required to bring E_2 back to I. That is, $R_1 \updownarrow R_3$. Therefore,

$$E_2^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Checking, we get $E_2^{-1}E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

EXAMPLE 3.6.3 (continued)

The inverse matrix E_3^{-1} is the elementary matrix associated with the row operation required to bring E_3 back to I. That is, $(-1/3)R_3$. Therefore,

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

Checking, we get $E_3^{-1}E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

If we look at our calculations in the example above, we see something interesting. Consider an elementary matrix E. We saw that the product EI is the matrix obtained from I by performing the row operations associated with E on I, and that $E^{-1}E$ is the matrix obtained from E by performing the row operation associated with E^{-1} on E. We demonstrate this further with another example.

EXAMPLE 3.6.4

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 5 & 6 \end{bmatrix}$. Calculate E_1A and E_2E_1A. Describe the products in terms of matrices obtained from A by elementary row operations.

Solution: By matrix multiplication, we get

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 8 \\ 0 & 5 & 6 \end{bmatrix}$$

Observe that E_1A is the matrix obtained from A by performing the row operation $R_2 + 2R_1$. That is, by performing the row operation associated with E_1.

$$E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 8 \\ 0 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 8 \\ 0 & 15 & 18 \end{bmatrix}$$

Thus, E_2E_1A is the matrix obtained from A by first performing the row operation $R_2 + 2R_1$, and then performing the row operation associated with E_2, namely $3R_3$.

We now state the general theorem.

Theorem 3.6.2 If $A \in M_{m\times n}(\mathbb{R})$ and E is the $m \times m$ elementary matrix corresponding to a certain elementary row operation, then the product EA is the matrix obtained from A by performing the same elementary row operation.

It would be tedious to write the proof of Theorem 3.6.2 in the general $n \times n$ case, so the proof is omitted.

Matrix Decomposition into Elementary Matrices

Consider the system of linear equations $A\vec{x} = \vec{b}$ where A is invertible. We compare our two methods of solving this system. First, we can solve this system using the approach shown in Chapter 2 to row reduce the augmented matrix:

$$\left[A \mid \vec{b}\right] \sim [I \mid \vec{x}]$$

Alternatively, we find A^{-1} using the method in the previous section. We row reduce

$$[A \mid I] \sim \left[I \mid A^{-1}\right]$$

and then solve the system by computing $\vec{x} = A^{-1}\vec{b}$.

Observe that both methods use exactly the same row operations to row reduce A to I. In the first method, we are applying those row operations directly to $\vec{b}$ to determine $\vec{x}$. In the second method, the row operations are being "stored" inside of A^{-1} and the matrix-vector product $A^{-1}\vec{b}$ is "performing" those row operations on $\vec{b}$ so that we get the same answer we obtained in the first method. From our work above, we conjecture that these elementary row operations are stored as elementary matrices.

Theorem 3.6.3 If $A \in M_{m\times n}(\mathbb{R})$ with reduced row echelon form R, then there exists a sequence of elementary matrices, $E_1, E_2, \ldots, E_k$, such that

$$E_k \cdots E_2E_1A = R$$

Proof: From our work in Chapter 2, we know that there is a sequence of elementary row operations to bring A into its reduced row echelon form. Call the elementary matrix corresponding to the first operation E_1, the elementary matrix corresponding to the second operation E_2, and so on, until the final elementary row operation corresponds to E_k. By Theorem 3.6.2 we get that E_1A is the matrix obtained by performing the first elementary row operation on A, E_2E_1A is the matrix obtained by performing the second elementary row operation on E_1A (that is, performing the first two elementary row operations on A), and $E_k \cdots E_2E_1A$ is the matrix obtained after performing all of the elementary row operations on A, in the specified order. ■

EXAMPLE 3.6.5 Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 4 \end{bmatrix}$. Find a sequence of elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1A$ is the reduced row echelon form of A.

Solution: We row reduce A keeping track of our elementary row operations:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first elementary row operation is $R_2 - 2R_1$, so $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

The second elementary row operation is $\frac{1}{2}R_2$, so $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$.

The third elementary row operation is $R_1 - R_2$, so $E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Thus, $E_3E_2E_1A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Remark

We know that the elementary matrices in Example 3.6.5 must be 2×2 for two reasons. First, we had only two rows in A to perform elementary row operations on, so this must be the same with the corresponding elementary matrices. Second, for the matrix multiplication E_1A to be defined, we know that the number of columns in E_1 must be equal to the number of rows in A. Also, E_1 is square since it is elementary.

EXERCISE 3.6.1

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 2 \end{bmatrix}$. Find a sequence of elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A$ is the reduced row echelon form of A.

In the special case where A is an invertible square matrix, the reduced row echelon form of A is I. Hence, by Theorem 3.6.3, there exists a sequence of elementary row operations such that

$$E_k \cdots E_1 A = I$$

Thus, the matrix

$$B = E_k \cdots E_1$$

satisfies $BA = I$, so B is the inverse of A. Observe that this result corresponds exactly to two facts we observed in Section 3.5. First, it demonstrates our procedure for finding the inverse of a matrix by row reducing $\left[\begin{array}{c|c} A & I \end{array}\right]$. Second, it shows that solving a system $A\vec{x} = \vec{b}$ by row reducing or by computing $\vec{x} = A^{-1}\vec{b}$ yields the same result.

Theorem 3.6.4 If A is invertible, then there exists a sequence of elementary matrices $E_1, \dots, E_k$ such that

$$A^{-1} = E_k \cdots E_1$$

and

$$A = E_1^{-1} E_2^{-2} \cdots E_k^{-1}$$

Proof: By Theorem 3.6.3, there exists a sequence of elementary row operations such that $E_k \cdots E_1 A = I$. Thus, by Theorem 3.5.2 we have that

$$A^{-1} = E_k \cdots E_1 \tag{3.8}$$

Taking the inverse of both sides of equation (3.8) gives

$$A = (E_K \cdots E_1)^{-1}$$

Since every elementary matrix is invertible, we can use Theorem 3.5.3 (2) to get

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

as required. ■

CONNECTION

Observe that writing A as a product of simpler matrices is kind of like factoring a polynomial (although it is definitely not the same). This is an example of a **matrix decomposition**. There are many very important matrix decompositions in linear algebra. We will look at a useful matrix decomposition in the next section and a couple more of them later in the book.

EXAMPLE 3.6.6

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & -1 \end{bmatrix}$. Write A and A^{-1} as products of elementary matrices.

Solution: Row reducing A to RREF gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \quad \tfrac{1}{2}R_2 \sim$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_2 \updownarrow R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \; E_5 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $E_5E_4E_3E_2E_1A = I$. Therefore,

$$A^{-1} = E_5E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PROBLEMS 3.6

Practice Problems

For Problems A1–A5, write the 3×3 elementary matrix that corresponds to the elementary row operation. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix}$ and verify that the product EA is the matrix obtained from A by performing the elementary row operation on A.

A1 Add -5 times the second row to the first row.
A2 Swap the second and third rows.
A3 Multiply the third row by -1.
A4 Multiply the second row by 6.
A5 Add 4 times the first row to the third row.

For Problems A6–A13, write the 4×4 elementary matrix that corresponds to the elementary row operation, and write the corresponding inverse elementary matrix.

A6 Add -3 times the third row to the fourth row.
A7 Swap the second and fourth rows.
A8 Multiply the third row by -3.
A9 Add 2 times the first row to the third row.
A10 Multiply the first row by 3.
A11 Swap the first and third rows.
A12 Add 1 times the second row to the first row.
A13 Add -3 times the fourth row to the first row.

For Problems A14–A22, either state that the matrix is elementary and state the corresponding elementary row operation, or explain why the matrix is not elementary.

A14 $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ **A15** $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ **A16** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

A17 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$ **A18** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **A19** $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A20 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **A21** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **A22** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For Problems A23–A28:

(a) Find a sequence of elementary matrices $E_k, \dots, E_1$ such that $E_k \cdots E_1 A = I$.
(b) Determine A^{-1} by computing $E_k \cdots E_1$.
(c) Write A as a product of elementary matrices.

A23 $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ **A24** $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

A25 $A = \begin{bmatrix} 2 & 0 & -4 \\ -2 & 1 & 4 \\ 3 & -1 & -5 \end{bmatrix}$ **A26** $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 6 & 2 \\ 1 & 3 & 2 \end{bmatrix}$

A27 $A = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 3 & -4 \\ 0 & 1 & 2 \end{bmatrix}$ **A28** $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & -2 \\ -4 & 1 & 4 \end{bmatrix}$

Homework Problems

For Problems B1–B6, write the 3×3 elementary matrix that corresponds to the elementary row operation. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix}$ and verify that the product EA is the matrix obtained from A by performing the elementary row operation on A.

B1 Swap the first and third rows.
B2 Add -2 times the first row to the second row.
B3 Multiply the third row by $1/2$.
B4 Multiply the first row by -1.
B5 Add 6 times the third row to the first row.
B6 Add 1 times the third row to the second row.

For Problems B7–B16, write the 3×3 elementary matrix that corresponds to the elementary row operation, and write the corresponding inverse elementary matrix.

B7 Add 3 times the third row to the first row.
B8 Swap the first and second rows.
B9 Swap the first and third rows.
B10 Add -4 times the first row to the second row.
B11 Add 2 times the second row to the third row.
B12 Multiply the third row by 2.
B13 Multiply the first row by $-1/3$.
B14 Add -1 times the first row to the third row.
B15 Swap the second and third rows.
B16 Multiply the second row by -3.

For Problems B17–B22, either state that the matrix is elementary and state the corresponding elementary row operation, or explain why the matrix is not elementary.

B17 $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B18 $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

B19 $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

B20 $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B21 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$

B22 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B23 Without using matrix-matrix multiplication, calculate

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

For Problems B24–B31:

(a) Find a sequence of elementary matrices $E_k, \dots, E_1$ such that $E_k \cdots E_1 A = I$.

(b) Determine A^{-1} by computing $E_k \cdots E_1$.

(c) Write A as a product of elementary matrices.

B24 $A = \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix}$

B25 $A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$

B26 $A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$

B27 $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

B28 $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

B29 $A = \begin{bmatrix} 4 & 0 & -4 \\ 1 & 1 & -1 \\ 6 & 2 & -5 \end{bmatrix}$

B30 $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 2 & 3 & 8 \end{bmatrix}$

B31 $A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 4 & 4 \\ -1 & 2 & 8 \end{bmatrix}$

Conceptual Problems

C1 (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the invertible linear operator with standard matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & -4 \end{bmatrix}$. By writing A as a product of elementary matrices, show that L can be written as a composition of shears, stretches, and reflections.

(b) Explain how we know that every invertible linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ can be written as a composition of shears, stretches, and reflections.

C2 Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(a) Determine elementary matrices E_1, E_2, and E_3 such that $E_3E_2E_1A = I$.

(b) Since A is invertible, we know that the system $A\vec{x} = \vec{b}$ has unique solution

$$\vec{x} = A^{-1}\vec{b} = E_3E_2E_1\vec{b}$$

Calculate the solution $\vec{x}$ without using matrix-matrix multiplication.

(c) Solve the system $A\vec{x} = \vec{b}$ by row reducing $\left[A \mid \vec{b}\right]$. Compare the operations that you use on the augmented part of the system with the operations in part (b).

C3 Let $A \in M_{m\times n}(\mathbb{R})$ with reduced row echelon form R.

(a) Use elementary matrices to prove that there exists an invertible matrix E such that $EA = R$.

(b) Is the matrix E in part (a) unique?

C4 We have seen that multiplying an $m \times n$ matrix A on the left by an elementary matrix has the same effect as performing the elementary row operation corresponding to the elementary matrix on A. If E is an $n \times n$ elementary matrix, then what effect does multiplying by A on the right by E have? Justify your answer.

For Problems C5 and C6, observe that we can now say that A is **row equivalent** to B if there exists a sequence of elementary matrices $E_1, \dots, E_K$ such that $E_k \cdots E_1A = B$.

C5 Prove that if A is row equivalent to a matrix B, then B is row equivalent to A.

C6 Prove that if A is row equivalent to a matrix B and C is also row equivalent to B, then A is row equivalent to C.

3.7 *LU*-Decomposition

One of the most basic and useful ideas in mathematics is the concept of a factorization of an object. You have already seen that it can be very useful to factor a number into primes or to factor a polynomial. Similarly, in many applications of linear algebra, we may want to decompose a matrix into factors that have certain properties.

In applied linear algebra, we often need to quickly solve multiple systems $A\vec{x} = \vec{b}$, where the coefficient matrix A remains the same but the vector $\vec{b}$ changes. The goal of this section is to derive a matrix factorization called the LU-decomposition, which is commonly used in computer algorithms to solve such problems.

We now start our look at the LU-decomposition by recalling the definition of upper triangular and lower triangular matrices.

Definition
Upper Triangular
Lower Triangular

A matrix $U \in M_{n\times n}(\mathbb{R})$ is said to be **upper triangular** if the entries beneath the main diagonal are all zero—that is, $(U)_{ij} = 0$ whenever $i > j$. A matrix $L \in M_{n\times n}(\mathbb{R})$ is said to be **lower triangular** if the entries above the main diagonal are all zero—in particular, $(L)_{ij} = 0$ whenever $i < j$.

Observe that for any system $A\vec{x} = \vec{b}$ with the same coefficient matrix A, we can use the same row operations to row reduce $\left[\begin{array}{c|c} A & \vec{b}\end{array}\right]$ to REF. Hence, the only difference between solving the systems $\left[\begin{array}{c|c} A & \vec{b}_1\end{array}\right]$ and $\left[\begin{array}{c|c} A & \vec{b}_2\end{array}\right]$ will then be the effect of the row operations on $\vec{b}_1$ and $\vec{b}_2$. In particular, we see that the two important pieces of information we require are an REF of A and the elementary row operations used.

For our purposes, we will assume that our $n\times n$ coefficient matrix A can be brought into REF using only elementary row operations of the form $R_i + sR_j$, where $i > j$. Since we can row reduce a matrix to a REF without multiplying a row by a non-zero constant, omitting this row operation is not a problem. However, omitting row interchanges may seem rather serious: without row interchanges, we cannot bring a matrix such as $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ into REF using only $R_i + sR_j$, where $i > j$. However, we omit row interchanges only because it is difficult to keep track of them by hand. A computer can keep track of row interchanges without physically moving entries from one location to another. At the end of the section, we will comment on the case where swapping rows is required.

Row operations of the form $R_i + sR_j$, where $i > j$, have a corresponding elementary matrix that is lower triangular and has ones along the main diagonal. So, under our assumption about A, there are elementary matrices $E_1, \dots, E_k$ that are all lower triangular such that

$$E_k \cdots E_1 A = U$$

where U is a REF of A. Since $E_k \cdots E_1$ is invertible, we can write $A = (E_k \cdots E_1)^{-1} U$ and define

$$L = (E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1}$$

We get that L is lower triangular because the inverse of a lower triangular elementary matrix is lower triangular, and a product of lower triangular matrices is lower triangular (see Problem A1). Additionally, by definition of REF, U is upper triangular. Consequently, we have a matrix decomposition $A = LU$, where U is upper triangular and L is lower triangular. Moreover, L contains the information about the row operations used to bring A to U.

Theorem 3.7.1 If $A \in M_{n\times n}(\mathbb{R})$ can be row reduced to REF without swapping rows, then there exists an upper triangular matrix U and lower triangular matrix L such that $A = LU$.

Definition
***LU*-Decomposition**

Writing $A \in M_{n\times n}(\mathbb{R})$ as a product LU, where L is lower triangular and U is upper triangular, is called an ***LU*-decomposition** of A.

Our derivation has given an algorithm for finding an LU-decomposition of a matrix that can be row reduced to REF using only operations of the form $R_i + sR_j$, where $i > j$.

EXAMPLE 3.7.1

Find an LU-decomposition of $A = \begin{bmatrix} 2 & -1 & 4 \\ 4 & -1 & 6 \\ -1 & -1 & 3 \end{bmatrix}$.

Solution: Row reducing and keeping track of our row operations gives

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & -1 & 6 \\ -1 & -1 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + \frac{1}{2}R_1 \end{matrix} \sim \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & -3/2 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 + \frac{3}{2}R_2 \end{matrix} \sim \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} = U$$

We have $E_3E_2E_1A = U$, so $A = E_1^{-1}E_2^{-1}E_3^{-1}U$, where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}$$

Hence, we let

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & -3/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/2 & -3/2 & 1 \end{bmatrix}$$

And we get $A = LU$.

Observe from this example that the entries in L that are beneath the main diagonal are just the negative of the multipliers used to put a zero in the corresponding entry of U. That is, if we use the operation $R_i + cR_j$ to put a 0 in the i, j-th entry of U, then the i, j-th entry of L is $-c$. To see why this is the case, observe that if $E_k \cdots E_1 A = U$, then

$$(E_k \cdots E_1)L = (E_k \cdots E_1)(E_k \cdots E_1)^{-1} = I$$

Hence, the same row operations that reduce A to U will reduce L to I.

Also note that the diagonal entries of L will all be 1s, and, since L is to be lower triangular, all the entries above the main diagonal are all 0s. These observations makes the LU-decomposition extremely easy to find.

EXAMPLE 3.7.2

Find an LU-decomposition of $B = \begin{bmatrix} 2 & 1 & -1 \\ -4 & 3 & 3 \\ 6 & 8 & -3 \end{bmatrix}$.

Solution: By row reducing, we get

$$\begin{bmatrix} 2 & 1 & -1 \\ -4 & 3 & 3 \\ 6 & 8 & -3 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 5 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

We used the multiplier 2 to get a 0 in the 2, 1-entry of U, so $(L)_{21} = -2$.
We used the multiplier -3 to get a 0 in the 3, 1-entry of U, so $(L)_{31} = 3$.
We used the multiplier -1 to get a 0 in the 3, 2-entry of U, we $(L)_{32} = 1$.
Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

We can easily verify that $B = LU$.

EXAMPLE 3.7.3

Find an LU-decomposition of $C = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ -4 & -2 & 1 \end{bmatrix}$.

Solution: By row reducing, we get

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ -4 & -2 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 9 \\ 0 & 6 & -11 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 9 \\ 0 & 0 & 16 \end{bmatrix} = U$$

From our elementary row operations, we get that

$$(L)_{21} = 2$$
$$(L)_{31} = -4$$
$$(L)_{32} = -3$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}$$

We then have $C = LU$.

It is important to note that the method we are using to find L only works when we row reduce the matrix by in order from the leftmost column to the rightmost, and from top to bottom within each column. We also must always be using the appropriate diagonal entry as the pivot.

EXERCISE 3.7.1

Find an LU-decomposition of $A = \begin{bmatrix} -1 & 1 & 2 \\ 4 & -1 & -3 \\ -3 & -3 & 1 \end{bmatrix}$.

Solving Systems with the *LU*-Decomposition

We now look at how to use the LU-decomposition to solve the system $A\vec{x} = \vec{b}$. If $A = LU$, the system can be written as

$$LU\vec{x} = \vec{b}$$

Letting $\vec{y} = U\vec{x}$, we can write $LU\vec{x} = \vec{b}$ as two systems:

$$L\vec{y} = \vec{b} \qquad \text{and} \qquad U\vec{x} = \vec{y}$$

which both have triangular coefficient matrices. This allows us to solve both systems immediately, using substitution. In particular, since L is lower triangular, we use forward-substitution to solve $\vec{y}$ and then solve $U\vec{x} = \vec{y}$ for $\vec{x}$ using back-substitution.

Remark

Observe that the first system is really calculating how performing the row operations on A would have affected $\vec{b}$.

EXAMPLE 3.7.4

Let $B = \begin{bmatrix} 2 & 1 & -1 \\ -4 & 3 & 3 \\ 6 & 8 & -3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ -13 \\ 4 \end{bmatrix}$. Use an LU-decomposition of B to solve $B\vec{x} = \vec{b}$.

Solution: In Example 3.7.2 we found an LU-decomposition of B. We write $B\vec{x} = \vec{b}$ as $LU\vec{x} = \vec{b}$ and take $\vec{y} = U\vec{x}$. Writing out the system $L\vec{y} = \vec{b}$, we get

$$\begin{aligned} y_1 &= 3 \\ -2y_1 + y_2 &= -13 \\ 3y_1 + y_2 + y_3 &= 4 \end{aligned}$$

Using forward-substitution, we find that $y_1 = 3$, so $y_2 = -13 + 2(3) = -7$ and $y_3 = 4 - 3(3) - (-7) = 2$. Hence, $\vec{y} = \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}$.

Thus, our system $U\vec{x} = \vec{y}$ is

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 3 \\ 5x_2 + x_3 &= -7 \\ -x_3 &= 2 \end{aligned}$$

Using back-substitution, we get $x_3 = -2$, $5x_2 = -7 - (-2) \Rightarrow x_2 = -1$ and $2x_1 = 3 - (-1) + (-2) \Rightarrow x_1 = 1$. Thus, the solution is $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

EXAMPLE 3.7.5

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ -2 & -4 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 6 \\ 12 \end{bmatrix}$. Use an LU-decomposition to solve $A\vec{x} = \vec{b}$.

Solution: We first find an LU-decomposition for A. Row reducing gives

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ -2 & -4 & 6 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & -2 & 8 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = U$$

From our elementary row operations, we find that $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$.

We let $\vec{y} = U\vec{x}$ and solve $L\vec{y} = \vec{b}$. This gives

$$\begin{aligned} y_1 &= 1 \\ -y_1 + y_2 &= 6 \\ -2y_1 + 2y_2 + y_3 &= 12 \end{aligned}$$

Hence, $y_1 = 1$, $y_2 = 6 + 1 = 7$, and $y_3 = 12 + 2 - 14 = 0$. Next we solve $U\vec{x} = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$.

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -x_2 + 4x_3 &= 7 \\ 0x_3 &= 0 \end{aligned}$$

This gives $x_3 = t \in \mathbb{R}$, $x_2 = -7 + 4t$, and $x_1 = 1 + (7 - 4t) - t = 8 - 5t$. Thus,

$$\vec{x} = \begin{bmatrix} 8 - 5t \\ -7 + 4t \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}, \qquad t \in \mathbb{R}$$

EXERCISE 3.7.2

Let $A = \begin{bmatrix} -1 & 1 & 2 \\ 4 & -1 & -3 \\ -3 & -3 & 1 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$. Use the LU-decomposition of A that you found in Exercise 3.7.1 to solve the system $A\vec{x}_i = \vec{b}_i$, for $i = 1, 2$.

A Comment About Swapping Rows

For any $A \in M_{n\times n}(\mathbb{R})$, we can first rearrange the rows of A to get a matrix that has an LU-factorization. In particular, for every matrix A, there exists a matrix P, called a permutation matrix, that can be obtained by only performing row swaps on the identity matrix such that

$$PA = LU$$

Then, we can solve $A\vec{x} = \vec{b}$ by finding the solutions of

$$PA\vec{x} = P\vec{b}$$

PROBLEMS 3.7

Practice Problems

A1 (a) Prove that the inverse of a lower triangular elementary matrix is lower triangular.
(b) Prove that a product of lower triangular matrices is lower triangular.

For Problems A2–A7, find an *LU*-decomposition for the matrix.

A2 $\begin{bmatrix} -2 & -1 & 5 \\ -4 & 0 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ **A3** $\begin{bmatrix} 1 & -2 & 4 \\ 3 & -2 & 4 \\ 2 & 2 & -5 \end{bmatrix}$

A4 $\begin{bmatrix} 2 & -4 & 5 \\ 2 & 5 & 2 \\ 2 & -1 & 5 \end{bmatrix}$ **A5** $\begin{bmatrix} 1 & 5 & 3 & 4 \\ -2 & -6 & -1 & 3 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A6 $\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 3 & -3 & 2 & -1 \\ 0 & 4 & -3 & 0 \end{bmatrix}$ **A7** $\begin{bmatrix} -2 & -1 & 2 & 0 \\ 4 & 3 & -2 & 2 \\ 3 & 3 & 4 & 3 \\ 2 & -1 & 2 & -4 \end{bmatrix}$

For Problems A8–A12, find an *LU*-decomposition for A and use it to solve $A\vec{x} = \vec{b}_i$, for $i = 1, 2$.

A8 $A = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

A9 $A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -3 \\ -1 & 4 & 5 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ -5 \\ -2 \end{bmatrix}$

A10 $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & -4 & 4 \\ 3 & -4 & -1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} -1 \\ -7 \\ -5 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

A11 $A = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -4 \\ 4 \\ -5 \end{bmatrix}$

A12 $A = \begin{bmatrix} -1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 3 & -8 & 3 & 2 \\ 1 & -2 & 3 & 1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} -6 \\ 7 \\ -4 \\ 5 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 5 \\ -3 \\ 3 \\ -5 \end{bmatrix}$

Homework Problems

For Problems B1–B12, find an *LU*-decomposition for the matrix.

B1 $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ **B2** $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 1 \\ 3 & 6 & 1 \end{bmatrix}$

B3 $\begin{bmatrix} -1 & 0 & -1 \\ 3 & 3 & 1 \\ -2 & 6 & 0 \end{bmatrix}$ **B4** $\begin{bmatrix} 2 & 8 & 5 \\ 2 & 7 & 6 \\ -2 & -6 & 5 \end{bmatrix}$

B5 $\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 1 \\ -8 & 2 & -3 \end{bmatrix}$ **B6** $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}$

B7 $\begin{bmatrix} 5 & 10 & 1 \\ 2 & 5 & 1 \\ -5 & -4 & 1 \end{bmatrix}$ **B8** $\begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

B9 $\begin{bmatrix} -1 & 3 & 6 \\ 3 & 1 & 0 \\ 5 & -5 & -3 \end{bmatrix}$ **B10** $\begin{bmatrix} 1 & 2 & 7 \\ 1 & -6 & 5 \\ 1 & -4 & -2 \end{bmatrix}$

B11 $\begin{bmatrix} 1 & -2 & 1 & 1 \\ -2 & 5 & 0 & 7 \\ 6 & -1 & 5 & -9 \\ 3 & 4 & 0 & 3 \end{bmatrix}$ **B12** $\begin{bmatrix} 2 & 3 & -1 & 3 \\ 4 & 8 & 2 & 1 \\ -2 & 1 & 4 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

For Problems B13–B18, find an *LU*-decomposition for A and use it to solve $A\vec{x} = \vec{b}_i$, for $i = 1, 2$.

B13 $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

B14 $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 1 \\ 3 & 6 & 1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} -9 \\ 2 \\ -3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -2 \\ -8 \\ 2 \end{bmatrix}$

B15 $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 5 \\ 1 & 1 & 4 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$

B16 $A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 3 & 15 \\ -4 & 6 & -10 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 2 \\ 15 \\ 10 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ 4 \\ -12 \end{bmatrix}$

B17 $A = \begin{bmatrix} -3 & 2 & -5 \\ 9 & -5 & 8 \\ -9 & 3 & 5 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 9 \\ -12 \\ -16 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -10 \\ 13 \\ 19 \end{bmatrix}$

B18 $A = \begin{bmatrix} 1 & -3 & 1 & 1 \\ -2 & 7 & -2 & -2 \\ 4 & -12 & 1 & 2 \\ 3 & -5 & 3 & -1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -5 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

CHAPTER REVIEW

Suggestions for Student Review

1 State the definition of addition and scalar multiplication for matrices in $M_{m\times n}(\mathbb{R})$ and list the ten properties in Theorem 3.1.1. (Section 3.1)

2 Give the definitions of the special types of matrices: square, upper triangular, lower triangular, and diagonal. List where we have used these so far and any special properties they have. (All Chapter 3)

3 State both definitions of matrix-vector multiplication, clearly stating the condition required for the product to be defined. How does each of these rules correspond to writing a system of linear equations? Why do we have two different definitions? List the properties of matrix-vector multiplication. (Section 3.1)

4 State the two methods for matrix-matrix multiplication. What condition must be satisfied by the sizes of A and B for AB to be defined? Create an example to show that matrix-matrix multiplication represents a composition of functions. List the properties of matrix-matrix multiplication. (Section 3.1)

5 State the definition and properties of the transpose of a vector, and the definition and properties of the transpose of a matrix. (Section 3.1)

6 State the definition of a linear mapping and explain the relationship between a linear mapping and the corresponding matrix mapping. Create some examples to show how to find and use a standard matrix of a linear mapping. (Section 3.2)

7 Give the standard matrix of a rotation, shear, stretch, and dilation in $\mathbb{R}^2$. (Section 3.3)

8 Give definitions of the two special subspaces of a linear mapping. Give a general procedure for finding a basis for each of these subspaces. (Section 3.4)

9 Give definitions of the four fundamental subspaces of a matrix and state the Fundamental Theorem of Linear Algebra. State the general procedure for finding a basis of each subspace, and state the dimension of each subspace. (Section 3.4)

10 (a) How many ways can you recognize the rank of a matrix? State them all.
(b) State the connection between the rank of a matrix A and the dimension of the solution space of $A\vec{x} = \vec{0}$.
(c) Illustrate your answers to (a) and (b) by constructing examples of 4×5 matrices in RREF of (i) rank 4; (ii) rank 3; and (iii) rank 2. In each case, actually determine the general solution of the system $A\vec{x} = \vec{0}$ and check that the solution space has the correct dimension. (Section 3.4)

11 (a) Outline the procedure for determining the inverse of a matrix. Indicate why it might not produce an inverse for a matrix A. Use the matrices of some geometric linear mappings to give two or three examples of matrices that have inverses and two examples of square matrices that do not have inverses. (Section 3.5)
(b) Pick a fairly simple 3×3 matrix (that does not contain too many zeros) and try to find its inverse. If it is not invertible, try another. When you have an inverse, check its correctness by multiplication. (Section 3.5)

12 State as many theorems and properties of invertible matrices as you can. (Section 3.5)

13 For 3×3 matrices, choose one elementary row operation of each of the three types; call these E_1, E_2, E_3. Choose an arbitrary 3×3 matrix A and check that E_iA is the matrix obtained from A by the appropriate elementary row operations. What is the relationship of elementary matrices with our geometrical mappings? (Section 3.6, 3.3)

14 Explain the algorithm for writing a matrix A as a product of elementary matrices. (Section 3.6)

15 Explain the procedure for finding an LU-decomposition of a matrix A, and how to use the LU-decomposition to solve a system $A\vec{x} = \vec{b}$. (Section 3.7)

Chapter Quiz

For Problems E1–E3, let $A = \begin{bmatrix} 2 & -5 & -3 \\ -3 & 4 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 2 \\ 1 & -1 & 5 \end{bmatrix}$. Compute the product or explain why it is not defined.

E1 AB **E2** BA **E3** BA^T

E4 (a) Let $A = \begin{bmatrix} -3 & 0 & 4 \\ 2 & -4 & -1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}$, and let f_A be the matrix mapping with matrix A. Determine $f_A(\vec{u})$ and $f_A(\vec{v})$.

(b) Use the result of part (a) to calculate $A\begin{bmatrix} 4 & 1 \\ -2 & 1 \\ -1 & -2 \end{bmatrix}$.

E5 Let R be the rotation through angle $\frac{\pi}{3}$ about the x_3-axis in $\mathbb{R}^3$. Let M be a reflection in $\mathbb{R}^3$ in the plane with equation $-x_1 - x_2 + 2x_3 = 0$. Determine

(a) The matrix of R

(b) The matrix of M

(c) The matrix of $[R \circ M]$

E6 Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 2 & 1 & 3 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 16 \\ 18 \end{bmatrix}$. Determine the solution set of $A\vec{x} = \vec{b}$ and the solution space of $A\vec{x} = \vec{0}$. Discuss the relationship between the two sets.

E7 Let $B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \\ 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 4 \\ -3 \\ 5 \\ 3 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -5 \\ 6 \\ -7 \\ -1 \end{bmatrix}$.

(a) Determine whether $\vec{u}$ or $\vec{v}$ is in Col(B).

(b) Determine from your calculation in part (a) a vector $\vec{x}$ such that $B\vec{x} = \vec{v}$.

E8 Find a basis for each of the four fundamental subspaces of $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 5 \\ 0 & 2 & -2 & 1 & 8 \\ 3 & 3 & 0 & 4 & 14 \end{bmatrix}$.

E9 Determine the inverse of $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

E10 Determine all values of p such that the matrix $\begin{bmatrix} 1 & 0 & p \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ is invertible and determine its inverse.

E11 Prove that the range of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of the codomain.

E12 Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a linearly independent set in $\mathbb{R}^n$ and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Prove that if Null$(L) = \{\vec{0}\}$, then $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ is a linearly independent set in $\mathbb{R}^m$.

E13 Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Determine a sequence of elementary matrices $E_1, \dots, E_k$, such that $E_k \cdots E_1 A = I$.

(b) By inverting the elementary matrices in part (a), write A as a product of elementary matrices.

For Problems E14–E19, either give an example or explain (in terms of theorems or definitions) why no such example can exist.

E14 A matrix K such that $KM = MK$ for all 3×3 matrices M.

E15 A matrix K such that $KM = MK$ for all 3×4 matrices M.

E16 The matrix of a linear mapping $L:\mathbb{R}^2 \to \mathbb{R}^3$ whose range is Span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ and whose nullspace is Span$\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$.

E17 The matrix of a linear mapping $L:\mathbb{R}^2 \to \mathbb{R}^3$ whose range is Span$\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$ and whose nullspace is Span$\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$.

E18 A linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that the range of L is all of $\mathbb{R}^3$ and the nullspace of L is Span$\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

E19 An invertible 4×4 matrix of rank 3.

Further Problems

These problems are intended to be challenging.

F1 Let $A \in M_{m\times n}(\mathbb{R})$ with $A \neq O_{m,n}$. Assume that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ such that $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is a basis for Null(A).

(a) Prove that $\{A\vec{v}_1, \dots, A\vec{v}_k\}$ is a basis for Col(A).

(b) Use the result of part (a) to prove the Rank-Nullity Theorem.

F2 Let $A = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_k\end{bmatrix}$ and $B = \begin{bmatrix}\vec{u}_1 & \cdots & \vec{u}_k\end{bmatrix}$ be $n\times k$ matrices. Prove that if

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \operatorname{Span}\{\vec{u}_1, \dots, \vec{u}_k\}$$

then there exists a matrix C such that $A = BC$.

F3 We say that a matrix C **commutes** with a matrix D if $CD = DC$. Show that the set of matrices that commute with $A = \begin{bmatrix}3 & 2\\ 0 & 1\end{bmatrix}$ is the set of matrices of the form $pI + qA$, where p and q are arbitrary scalars.

F4 Let A be some fixed $n \times n$ matrix. Show that the set $C(A)$ of matrices that commutes with A is closed under addition, scalar multiplication, and matrix multiplication.

F5 A square matrix A is said to be **nilpotent** if some power of A is equal to the zero matrix. Show that the matrix $\begin{bmatrix}0 & a_{12} & a_{13}\\ 0 & 0 & a_{23}\\ 0 & 0 & 0\end{bmatrix}$ is nilpotent. Generalize.

F6 (a) Suppose that ℓ is a line in $\mathbb{R}^2$ passing through the origin and making an angle θ with the positive x_1-axis. Let refl_θ denote a reflection in this line. Determine the matrix $[\text{refl}_\theta]$ in terms of functions of θ.

(b) Let refl_α denote a reflection in a second line, and by considering the matrix $[\text{refl}_\alpha \circ \text{refl}_\theta]$, show that the composition of two reflections in the plane is a rotation. Express the angle of the rotation in terms of α and θ.

F7 (Isometries) A linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ is an **isometry** of $\mathbb{R}^2$ if L preserves lengths (that is, if $\|L(\vec{x})\| = \|\vec{x}\|$ for every $\vec{x} \in \mathbb{R}^2$).

(a) Show that an isometry preserves the dot product (that is, $L(\vec{x}) \cdot L(\vec{y}) = \vec{x} \cdot \vec{y}$ for every $\vec{x}, \vec{y} \in \mathbb{R}^2$). (Hint: consider $L(\vec{x} + \vec{y})$.)

(b) Show that the columns of that matrix $[L]$ must be orthogonal to each other and of length 1. Deduce that any isometry of $\mathbb{R}^2$ must be the composition of a reflection and a rotation. (Hint: you may find it helpful to use the result of Problem F4 (a).)

F8 (a) Suppose that A and B are $n \times n$ matrices such that $A + B$ and $A - B$ are invertible and that C and D are arbitrary $n \times n$ matrices. Show that there are $n \times n$ matrices X and Y satisfying the system

$$AX + BY = C$$
$$BX + AY = D$$

(b) With the same assumptions as in part (a), give a careful explanation of why the matrix $\begin{bmatrix}A & B\\ B & A\end{bmatrix}$ must be invertible. Obtain an expression for its inverse in terms of $(A + B)^{-1}$ and $(A - B)^{-1}$.

CHAPTER 4

Vector Spaces

CHAPTER OUTLINE

In the first three chapters, we explored some of the most important concepts in linear algebra, including spanning, linear independence, and bases, in the context of vectors in $\mathbb{R}^n$ and matrices. In this chapter, we will examine these concepts, as well as other important ideas in linear algebra, in a more general setting. However, as we will see, although the setting is more abstract, we will use the same procedures and tools as we did with vectors in $\mathbb{R}^n$.

4.1 Spaces of Polynomials

We now compare sets of polynomials under standard addition and scalar multiplication to sets of vectors in $\mathbb{R}^n$ and sets of matrices.

Addition and Scalar Multiplication of Polynomials

Definition
$P_n(\mathbb{R})$
Addition of Polynomials
Scalar Multiplication of Polynomials

We let $P_n(\mathbb{R})$ denote the set of all polynomials $\mathbf{p}(x) = a_0 + a_1x + \cdots + a_nx^n$ of degree *at most n* where the coefficients a_i are real numbers.

The polynomials $\mathbf{p}(x) = a_0 + a_1x + \cdots + a_nx^n$, $\mathbf{q}(x) = b_0 + b_1x + \cdots + b_nx^n \in P_n(\mathbb{R})$ are said to be **equal** if $a_i = b_i$ for $0 \leq i \leq n$.

We define the **addition of** of $\mathbf{p}$ and $\mathbf{q}$, denoted $\mathbf{p} + \mathbf{q}$, by

$$(\mathbf{p} + \mathbf{q})(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

We define **scalar multiplication** of $\mathbf{p}$ by a scalar $t \in \mathbb{R}$ by

$$(t\mathbf{p})(x) = ta_0 + (ta_1)x + \cdots + (ta_n)x^n$$

As before, when we write $\mathbf{p} - \mathbf{q}$ we mean $\mathbf{p} + (-1)\mathbf{q}$.

EXAMPLE 4.1.1

Evaluate the following linear combinations of polynomials

(a) $(2 + 3x + 4x^2 + x^3) + (5 + x - 2x^2 + 7x^3)$

(b) $2(1 + 3x - x^3) + 3(4 + x^2 + 2x^3)$

Solution: For (a) we have

$$\begin{aligned}(2 + 3x + 4x^2 + x^3) + (5 + x - 2x^2 + 7x^3) &= 2 + 5 + (3 + 1)x + (4 - 2)x^2 + (1 + 7)x^3 \\ &= 7 + 4x + 2x^2 + 8x^3\end{aligned}$$

For (b) we have

$$\begin{aligned}2(1 + 3x - x^3) + 3(4 + x^2 + 2x^3) &= 2(1) + 2(3)x + 2(0)x^2 + 2(-1)x^3 \\ &\quad + 3(4) + 3(0)x + 3(1)x^2 + 3(2)x^3 \\ &= 2 + 12 + (6 + 0)x + (0 + 3)x^2 + (-2 + 6)x^3 \\ &= 14 + 6x + 3x^2 + 4x^3\end{aligned}$$

Properties of Polynomial Addition and Scalar Multiplication

Theorem 4.1.1 For all $\mathbf{p}, \mathbf{q}, \mathbf{r} \in P_n(\mathbb{R})$ and $s, t \in \mathbb{R}$ we have

(1) $\mathbf{p} + \mathbf{q} \in P_n(\mathbb{R})$

(2) $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$

(3) $(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$

(4) There exists a polynomial $\mathbf{0} \in P_n(\mathbb{R})$ such that $\mathbf{p} + \mathbf{0} = \mathbf{p}$ for all $\mathbf{p} \in P_n(\mathbb{R})$.

(5) For each polynomial $\mathbf{p} \in P_n(\mathbb{R})$, there exists a polynomial $(-\mathbf{p}) \in P_n(\mathbb{R})$ such that $\mathbf{p} + (-\mathbf{p}) = \mathbf{0}$

(6) $s\mathbf{p} \in P_n(\mathbb{R})$

(7) $s(t\mathbf{p}) = (st)\mathbf{p}$

(8) $(s + t)\mathbf{p} = s\mathbf{p} + t\mathbf{p}$

(9) $s(\mathbf{p} + \mathbf{q}) = s\mathbf{p} + s\mathbf{q}$

(10) $1\mathbf{p} = \mathbf{p}$

These properties follow easily from the definitions of addition and scalar multiplication and are very similar to those for vectors in $\mathbb{R}^n$. Thus, the proofs are left to the reader.

The polynomial $\mathbf{0}$, called the **zero polynomial**, is defined by

$$\mathbf{0}(x) = 0 = 0 + 0x + \cdots + 0x^n, \qquad \text{for all } x \in \mathbb{R}$$

The additive inverse $(-\mathbf{p})$ of a polynomial $\mathbf{p}(x) = a_0 + a_1x + \cdots + a_nx^n$ is

$$(-\mathbf{p})(x) = -a_0 - a_1x - \cdots - a_nx^n, \qquad \text{for all } x \in \mathbb{R}$$

It is important to recognize that these are the same ten properties we had for addition and scalar multiplication of vectors in $\mathbb{R}^n$ (Theorem 1.4.1) and of matrices (Theorem 3.1.1).

To match what we did with vectors in $\mathbb{R}^n$ and matrices in $M_{m\times n}(\mathbb{R})$, we now define the concepts of spanning and linear independence for polynomials.

Definition
Span

Let $\mathcal{B} = \{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ be a set of polynomials in $P_n(\mathbb{R})$. The **span** of $\mathcal{B}$ is defined as

$$\operatorname{Span}\mathcal{B} = \{t_1\mathbf{p}_1 + \cdots + t_k\mathbf{p}_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$$

Definition
Linearly Dependent

Linearly Independent

Let $\mathcal{B} = \{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ be a set of polynomials in $P_n(\mathbb{R})$. The set $\mathcal{B}$ is said to be **linearly dependent** if there exists real coefficients $t_1, \ldots, t_k$ not all zero such that

$$t_1\mathbf{p}_1 + \cdots + t_k\mathbf{p}_k = \mathbf{0}$$

The set $\mathcal{B}$ is said to be **linearly independent** if the only solution to

$$t_1\mathbf{p}_1 + \cdots + t_k\mathbf{p}_k = \mathbf{0}$$

is the trivial solution $t_1 = \cdots = t_k = 0$.

EXAMPLE 4.1.2

Determine whether $\mathbf{p}(x) = 1 + 2x + 3x^2 + 4x^3$ is in the span of $\mathcal{B} = \{1 + x, 1 + x^3, x + x^2, x + x^3\}$.

Solution: We want to determine if there are t_1, t_2, t_3, t_4 such that

$$t_1(1 + x) + t_2(1 + x^3) + t_3(x + x^2) + t_4(x + x^3) = 1 + 2x + 3x^2 + 4x^3$$

Performing the linear combination on the left-hand side gives

$$(t_1 + t_2) + (t_1 + t_3 + t_4)x + t_3x^2 + (t_2 + t_4)x^3 = 1 + 2x + 3x^2 + 4x^3$$

Comparing the coefficients of powers of x on both sides of the equation, we get the system of linear equations

$$\begin{aligned} t_1 + t_2 &= 1 \\ t_1 + t_3 + t_4 &= 2 \\ t_3 &= 3 \\ t_2 + t_4 &= 4 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

We see that the system is consistent; therefore, $1 + 2x + 3x^2 + 4x^3$ is in the span of $\mathcal{B}$. In particular, we have

$$(-2)(1 + x) + 3(1 + x^3) + 3(x + x^2) + (x + x^3) = 1 + 2x + 3x^2 + 4x^3$$

EXAMPLE 4.1.3

Determine whether the set

$$\mathcal{B} = \{1 + 2x + 2x^2 - x^3, 3 + 2x + x^2 + x^3, 2x^2 + 2x^3\}$$

is linearly independent in $P_3(\mathbb{R})$.

Solution: Consider

$$t_1(1 + 2x + 2x^2 - x^3) + t_2(3 + 2x + x^2 + x^3) + t_3(2x^2 + 2x^3) = 0$$

Performing the linear combination on the left-hand side gives

$$(t_1 + 3t_2) + (2t_1 + 2t_2)x + (2t_1 + t_2 + 2t_3)x^2 + (-t_1 + t_2 + 2t_3)x^3 = 0$$

Comparing coefficients of the powers of x, we get the homogeneous system of linear equations

$$\begin{aligned} t_1 + 3t_2 &= 0 \\ 2t_1 + 2t_2 &= 0 \\ 2t_1 + t_2 + 2t_3 &= 0 \\ -t_1 + t_2 + 2t_3 &= 0 \end{aligned}$$

Row reducing the associated coefficient matrix gives

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The only solution is $t_1 = t_2 = t_3 = 0$, so $\mathcal{B}$ is linearly independent.

EXERCISE 4.1.1

Determine whether

$$\mathcal{B} = \{1 + 2x + x^2 + x^3, 1 + x + 3x^2 + x^3, 3 + 5x + 5x^2 - 3x^3, -x - 2x^2\}$$

is linearly dependent or linearly independent.

Is $\mathbf{p}(x) = 1 + 5x - 5x^2 + x^3$ in the span of $\mathcal{B}$?

EXERCISE 4.1.2

Consider $\mathcal{B} = \{1, x, x^2, x^3\}$. Prove that $\mathcal{B}$ is linearly independent and show that $\operatorname{Span}\mathcal{B} = P_3(\mathbb{R})$.

PROBLEMS 4.1

Practice Problems

For Problems A1–A7, evaluate the expression.

A1 $(2 - 2x + 3x^2 + 4x^3) + (-3 - 4x + x^2 + 2x^3)$

A2 $(-3)(1 - 2x + 2x^2 + x^3 + 4x^4)$

A3 $(2 + 3x + x^2 - 2x^3) - 3(1 - 2x + 4x^2 + 5x^3)$

A4 $(2 + 3x + 4x^2) - (5 + x - 2x^2)$

A5 $-2(-5 + x + x^2) + 3(-1 - x^2)$

A6 $2\left(\frac{2}{3} - \frac{1}{3}x + 2x^2\right) + \frac{1}{3}\left(3 - 2x + x^2\right)$

A7 $\sqrt{2}(1 + x + x^2) + \pi(-1 + x^2)$

For Problems A8–A13, either express the polynomial as a linear combination of the polynomials in $\mathcal{B} = \{1 + x^2 + x^3, 2 + x + x^3, -1 + x + 2x^2 + x^3\}$ or show that it is not in Span $\mathcal{B}$.

A8 $\mathbf{p}(x) = 0$

A9 $\mathbf{p}(x) = 2 + 4x + 3x^2 + 4x^3$

A10 $\mathbf{p}(x) = -x + 2x^2 + x^3$

A11 $\mathbf{p}(x) = -4 - x + 3x^2$

A12 $\mathbf{p}(x) = -1 + 7x + 5x^2 + 4x^3$

A13 $\mathbf{p}(x) = 2 + x + 5x^3$

For Problems A14–A17, determine whether the set is linearly independent. If a set is linearly dependent, find all linear combinations of the polynomials that equal the zero polynomial.

A14 $\left\{1 + 2x + x^2 - x^3, 5x + x^2, 1 - 3x + 2x^2 + x^3\right\}$

A15 $\left\{1 + x + x^2, x, x^2 + x^3, 3 + 2x + 2x^2 - x^3\right\}$

A16 $\left\{3 + x + x^2, 4 + x - x^2, 1 + 2x + x^2 + 2x^3, -1 + 5x^2 + x^3\right\}$

A17 $\left\{1 + x + x^3 + x^4, 2 + x - x^2 + x^3 + x^4, x + x^2 + x^3 + x^4\right\}$

A18 Prove that the set $\mathcal{B} = \{1, x - 1, (x - 1)^2\}$ is linearly independent and show that Span $\mathcal{B} = P_2(\mathbb{R})$.

Homework Problems

For Problems B1–B7, evaluate the expression.

B1 $(1 + 2x - 3x^2 + 4x^3) - (2 - x^2 + 2x^3)$

B2 $(-2)(1 + 2x - 3x^2 + 3x^3 + 5x^4)$

B3 $2(1 + 2x + 7x^2 - 3x^3) - 3(1 + 6x - 2x^2)$

B4 $(-4)(1 + x^2 - 2x^3) + (1 - x^2 + x^3)$

B5 $0(1 + 2x^2 + 3x^4)$

B6 $\frac{1}{2}\left(2 + \frac{1}{3}x + x^2\right) + \frac{1}{4}\left(4 - 2x + 2x^2\right)$

B7 $\left(1 - \sqrt{2}\right)\left(1 + \sqrt{2} + (\sqrt{2} + 1)x^2\right) - \frac{1}{2}\left(-2 + 2x^2\right)$

For Problems B8–B11, $\mathcal{B} = \{1 + x + x^3, 1 + 2x + x^2, x + x^3\}$. Either express the polynomial as a linear combination of the polynomials in $\mathcal{B}$ or show that it is not in Span $\mathcal{B}$.

B8 $\mathbf{p}(x) = 2$

B9 $\mathbf{p}(x) = 1 - x + x^3$

B10 $\mathbf{p}(x) = 2 + 2x - x^2 + 4x^3$

B11 $\mathbf{p}(x) = 6 + 4x + 2x^2$

For Problems B12–B18, determine whether the set is linearly independent. If a set is linearly dependent, find all linear combinations of the polynomials that equal the zero polynomial.

B12 $\left\{0, 1 + x^2, x + x^2 - x^3\right\}$

B13 $\left\{1 + x, 1 - x^2, x + x^3, 1 + 2x^2 + x^3\right\}$

B14 $\left\{1 + 2x + 5x^2, x + 3x^2, 2 + 3x + 8x^2\right\}$

B15 $\{4x - 3x^2, 1 - 3x + 2x^2, 2 - 2x + x^2\}$

B16 $\{1 + 3x + x^2, 8 + 9x + 2x^2, -3 - 3x - x^2\}$

B17 $\{4 + x - 2x^3, 5 + 2x + x^3, -2 + x + 8x^3\}$

B18 $\{1 - x + x^2, 1 + 2x^2, -1 + 2x + x^2\}$

B19 Prove that $\mathcal{B} = \{1, 1 - x, (1 - x)^2\}$ is linearly independent and show that Span $\mathcal{B} = P_2(\mathbb{R})$.

Conceptual Problems

For Problems C1–C3, let $\mathcal{B} = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ be a set of polynomials in $P_n(\mathbb{R})$.

C1 Prove if $k < n + 1$, then there exists a polynomial $\mathbf{q} \in P_n(\mathbb{R})$ such that $\mathbf{q} \notin$ Span $\mathcal{B}$.

C2 Prove if $k > n + 1$, then $\mathcal{B}$ must be linearly dependent.

C3 Prove if $k = n + 1$ and $\mathcal{B}$ is linearly independent, then Span $\mathcal{B} = P_n(\mathbb{R})$.

4.2 Vector Spaces

We have now seen that addition and scalar multiplication of vectors in $\mathbb{R}^n$, matrices in $M_{m\times n}(\mathbb{R})$, and polynomials in $P_n(\mathbb{R})$ satisfy the same ten properties. Moreover, we can show that addition and scalar multiplication of linear mappings also satisfy these same ten properties. In fact, many other mathematical objects also have these important properties. Instead of analyzing each of these objects separately, it is useful to define one abstract concept that encompasses all of them.

Vector Spaces

Definition
Vector Space over $\mathbb{R}$

A **vector space over** $\mathbb{R}$ is a set $\mathbb{V}$ together with an operation of **addition**, usually denoted $\mathbf{x}+\mathbf{y}$ for any $\mathbf{x},\mathbf{y}\in\mathbb{V}$, and an operation of **scalar multiplication**, usually denoted $s\mathbf{x}$ for any $\mathbf{x}\in\mathbb{V}$ and $s\in\mathbb{R}$, such that for any $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{V}$ and $s,t\in\mathbb{R}$ we have all of the following properties:

V1 $\mathbf{x}+\mathbf{y}\in\mathbb{V}$
V2 $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$
V3 $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$
V4 There exists $\mathbf{0}\in\mathbb{V}$, called the **zero vector**, such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for all $\mathbf{x}\in\mathbb{V}$
V5 For each $\mathbf{x}\in\mathbb{V}$, there exists $(-\mathbf{x})\in\mathbb{V}$ such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$
V6 $s\mathbf{x}\in\mathbb{V}$
V7 $s(t\mathbf{x})=(st)\mathbf{x}$
V8 $(s+t)\mathbf{x}=s\mathbf{x}+t\mathbf{x}$
V9 $s(\mathbf{x}+\mathbf{y})=s\mathbf{x}+s\mathbf{y}$
V10 $1\mathbf{x}=\mathbf{x}$

Remarks

1. The elements of a vector space are called **vectors**. Note that these can be very different objects than vectors in $\mathbb{R}^n$. Thus, we will always denote, as in the definition above, a vector in a general vector space in boldface (for example, $\mathbf{x}$). However, for vector spaces such as $\mathbb{R}^n$ or $M_{m\times n}(\mathbb{R})$ we will often use the notation we introduced earlier.

2. As usual, we call a sum of scalar multiples of vectors a **linear combination**, and when we write $\mathbf{x}-\mathbf{y}$ we mean $\mathbf{x}+(-1)\mathbf{y}$.

3. Some people denote the operations of addition and scalar multiplication in general vector spaces by $\oplus$ and $\odot$, respectively, to stress the fact that these do not need to be "standard" addition and scalar multiplication.

4. Since every vector space contains a zero vector by V4, the empty set cannot be a vector space.

5. When working with multiple vector spaces, we sometimes use a subscript to denote the vector space to which the zero vector belongs. For example, $\mathbf{0}_{\mathbb{V}}$ would represent the zero vector in the vector space $\mathbb{V}$.

6. Vector spaces can be defined using other number systems as the scalars. For example, the definition makes perfect sense if rational numbers are used instead of the real numbers. Vector spaces over the complex numbers are discussed in Chapter 9. Until Chapter 9, "vector space" means "vector space over $\mathbb{R}$."

7. We have defined vector spaces to have the same structure as $\mathbb{R}^n$. The study of vector spaces is the study of this common structure. However, it is possible that vectors in individual vector spaces have other aspects not common to all vector spaces, such as matrix multiplication or factorization of polynomials.

EXAMPLE 4.2.1 $\mathbb{R}^n$ is a vector space with addition and scalar multiplication defined in the usual way. We call these standard addition and scalar multiplication of vectors in $\mathbb{R}^n$.

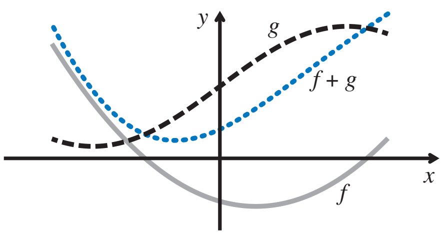
EXAMPLE 4.2.2 $P_n(\mathbb{R})$, the set of all polynomials of degree at most n with real coefficients, is a vector space with standard addition and scalar multiplication of polynomials.

EXAMPLE 4.2.3 $M_{m\times n}(\mathbb{R})$, the set of all $m \times n$ matrices with real entries, is a vector space with standard addition and scalar multiplication of matrices.

EXAMPLE 4.2.4 Consider the set of polynomials of degree n with real coefficients. Is this a vector space with standard addition and scalar multiplication? No, it does not contain the zero polynomial since the zero polynomial is not of degree n. Note also that the sum of two polynomials of degree n may not be of degree n. For example, $(1 + x^n) + (1 - x^n) = 2$, which is of degree 0. Thus, the set is also not closed under addition.

EXAMPLE 4.2.5 Let $\mathcal{F}(a, b)$ denote the set of all functions $f : (a, b) \to \mathbb{R}$. If $f, g \in \mathcal{F}(a, b)$, then the sum is defined by $(f + g)(x) = f(x) + g(x)$, and multiplication by a scalar $t \in \mathbb{R}$ is defined by $(tf)(x) = tf(x)$. With these definitions, $\mathcal{F}(a, b)$ is a vector space.

EXAMPLE 4.2.6 Let $C(a, b)$ denote the set of all functions that are continuous on the interval (a, b). Since the sum of continuous functions is continuous and a scalar multiple of a continuous function is continuous, $C(a, b)$ is a vector space.



EXAMPLE 4.2.7 Let $\mathbb{T}$ be the set of all solutions to $x_1 + 2x_2 = 1$, $2x_1 + 3x_2 = 0$. Is $\mathbb{T}$ a vector space with standard addition and scalar multiplication?

Solution: No. This set with these operations does not satisfy many of the vector space axioms. For example, V6 does not hold since $\begin{bmatrix} -3 \\ 2 \end{bmatrix} \in \mathbb{T}$ is a solution of this system, but $2\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$ is not a solution of the system and hence is not in $\mathbb{T}$.

EXAMPLE 4.2.8

Consider $\mathbb{V} = \{(x, y) \mid x, y \in \mathbb{R}\}$ with addition and scalar multiplication defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$k(x, y) = (ky, kx)$$

Is $\mathbb{V}$ a vector space?

Solution: No, since $1(2, 3) = (3, 2) \neq (2, 3)$, it does not satisfy V10. Note that it also does not satisfy V7.

EXAMPLE 4.2.9

Is $\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$ a vector space with standard addition and scalar multiplication in $\mathbb{R}^3$?

Solution: Yes! To prove it we need to verify that all ten axioms hold. However, since axioms V2, V3, V7, V8, V9, and V10 refer only to the operations of addition and scalar multiplication, we know by Theorem 1.4.1 that these all hold. We now prove that the remaining axioms also hold.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_1 + y_2 \end{bmatrix}$ be vectors in $\mathbb{S}$.

V1: We have

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_1 + y_1 + x_2 + y_2 \end{bmatrix}$$

If we let $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$, then $z_1 + z_2 = x_1 + y_1 + x_2 + y_2$ and hence

$$\vec{x} + \vec{y} = \begin{bmatrix} z_1 \\ z_2 \\ z_1 + z_2 \end{bmatrix} \in \mathbb{S}$$

V6: For any $t \in \mathbb{R}$, $t\vec{x} = t \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_1 + tx_2 \end{bmatrix} \in \mathbb{S}$.

V4: Since V6 holds, we have that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\vec{x} \in \mathbb{S}$.

V5: Since V6 holds, we have that $\begin{bmatrix} -x_1 \\ -x_2 \\ -x_1 + (-x_2) \end{bmatrix} = (-1)\vec{x} \in \mathbb{S}$.

Thus, $\mathbb{S}$ with these operators is a vector space as it satisfies all ten axioms.

EXERCISE 4.2.1

Prove that the set $\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{Z} \right\}$ is not a vector space using standard addition and scalar multiplication of vectors in $\mathbb{R}^2$.

EXERCISE 4.2.2

Let $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\}$. Prove that $\mathbb{S}$ is a vector space using standard addition and scalar multiplication of matrices. This is the vector space of 2×2 diagonal matrices.

EXERCISE 4.2.3

Let $\mathbb{T} = \left\{ a_1 + a_2x^3 \mid a_1, a_2 \in \mathbb{R} \right\}$. Prove that $\mathbb{T}$ is a vector space using standard addition and scalar multiplication of polynomials in $P_3(\mathbb{R})$. How does this vector space compare with the vector space in Exercise 4.2.2?

Again, one advantage of having the abstract concept of a vector space is that when we prove a result about a general vector space, it instantly applies to all of the examples of vector spaces. To demonstrate this, we give three additional properties that follow from the vector space axioms.

Theorem 4.2.1

If $\mathbb{V}$ is a vector space, then
(1) $0\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{V}$
(2) $(-1)\mathbf{x} = -\mathbf{x}$ for all $\mathbf{x} \in \mathbb{V}$
(3) $t\mathbf{0} = \mathbf{0}$ for all $t \in \mathbb{R}$

Proof: We will prove (1). You are asked to prove (2) in Problem C1 and to prove (3) in Problem C2.

For any $\mathbf{x} \in \mathbb{V}$ we have

$$\begin{aligned}
0\mathbf{x} &= 0\mathbf{x} + \mathbf{0} && \text{by V4} \\
&= 0\mathbf{x} + [\mathbf{x} + (-\mathbf{x})] && \text{by V5} \\
&= 0\mathbf{x} + [1\mathbf{x} + (-\mathbf{x})] && \text{by V10} \\
&= [0\mathbf{x} + 1\mathbf{x}] + (-\mathbf{x}) && \text{by V3} \\
&= (0 + 1)\mathbf{x} + (-\mathbf{x}) && \text{by V8} \\
&= 1\mathbf{x} + (-\mathbf{x}) && \text{operation of numbers in } \mathbb{R} \\
&= \mathbf{x} + (-\mathbf{x}) && \text{by V10} \\
&= \mathbf{0} && \text{by V5}
\end{aligned}$$

■

Thus, if we know that $\mathbb{V}$ is a vector space, we can determine the zero vector of $\mathbb{V}$ by finding $0\mathbf{x}$ for any $\mathbf{x} \in \mathbb{V}$. Similarly, we can determine the additive inverse of any vector $\mathbf{x} \in \mathbb{V}$ by computing $(-1)\mathbf{x}$.

EXAMPLE 4.2.10 Let $\mathbb{V} = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$ and define addition by

$$(a, b) \oplus (c, d) = (ad + bc, bd)$$

and define scalar multiplication by

$$t \odot (a, b) = (tab^{t-1}, b^t)$$

Use Theorem 4.2.1 to show that axioms V4 and V5 hold for $\mathbb{V}$ with these operations. (Note that we are using $\oplus$ and $\odot$ to represent the operations of addition and scalar multiplication in the vector space to help distinguish the difference between these and the operations of addition and multiplication of real numbers.)

Solution: We do not know if $\mathbb{V}$ is a vector space, but if it is, then by Theorem 4.2.1 we must have

$$\mathbf{0} = 0 \odot (a, b) = (0ab^{-1}, b^0) = (0, 1)$$

Observe that $(0, 1) \in \mathbb{V}$ and for any $(a, b) \in \mathbb{V}$ we have

$$(a, b) \oplus (0, 1) = (a(1) + b(0), b(1)) = (a, b)$$

So, $\mathbb{V}$ satisfies V4 using $\mathbf{0} = (0, 1)$.

Similarly, if $\mathbb{V}$ is a vector space, then by Theorem 4.2.1 for any $\mathbf{x} = (a, b) \in \mathbb{V}$ we must have

$$(-\mathbf{x}) = (-1) \odot (a, b) = (-ab^{-2}, b^{-1})$$

Observe that for any $(a, b) \in \mathbb{V}$ we have $(-ab^{-2}, b^{-1}) \in \mathbb{V}$ since $b^{-1} > 0$ whenever $b > 0$. Also,

$$(a, b) \oplus (-ab^{-2}, b^{-1}) = (ab^{-1} + b(-ab^{-2}), bb^{-1}) = (ab^{-1} - ab^{-1}, 1) = (0, 1)$$

So, $\mathbb{V}$ satisfies V5 using $-(a, b) = (-ab^{-2}, b^{-1})$.

You are asked to complete the proof that $\mathbb{V}$ is indeed a vector space in Problem C5.

Subspaces

In Example 4.2.9 we showed that $\mathbb{S}$ is a vector space that is contained inside the vector space $\mathbb{R}^3$. Upon inspection, we see that our steps in the solution of the example also show that $\mathbb{S}$ is a subspace of $\mathbb{R}^3$.

Definition
Subspace

Suppose that $\mathbb{V}$ is a vector space. A non-empty subset $\mathbb{S}$ of $\mathbb{V}$ is called a **subspace** of $\mathbb{V}$ if for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}$ and $s, t \in \mathbb{R}$ we have

$$s\mathbf{x} + t\mathbf{y} \in \mathbb{S} \tag{4.1}$$

Equivalently, if $\mathbb{S}$ is a subset of a vector space $\mathbb{V}$ and $\mathbb{S}$ is also a vector space using the same operations as $\mathbb{V}$, then $\mathbb{S}$ is a **subspace** of $\mathbb{V}$.

To prove that both definitions are equivalent, we first observe that if $\mathbb{S}$ is a vector space, then it satisfies vector space axioms V1 and V6. Combining V1 and V6 proves that $s\mathbf{x} + t\mathbf{y} \in \mathbb{S}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}$ and $s, t \in \mathbb{R}$.
On the other hand, as in Example 4.2.9, we know the operations must satisfy axioms V2, V3, V7, V8, V9, and V10 since $\mathbb{V}$ is a vector space. For the remaining axioms we have

V1: Taking $s = t = 1$ in equation (4.1) gives $\mathbf{x} + \mathbf{y} \in \mathbb{S}$.

V4: Taking $s = t = 0$ in equation (4.1) gives $0\mathbf{x} + 0\mathbf{y} \in \mathbb{S}$. But, $0\mathbf{x} + 0\mathbf{y} = \mathbf{0}$ by Theorem 4.2.1. So, $\mathbf{0} \in \mathbb{S}$.

V5: By Theorem 4.2.1 we know that $(-\mathbf{x}) = (-1)\mathbf{x}$. So, taking $s = -1$, $t = 0$ in equation (4.1) gives

$$(-\mathbf{x}) = (-1)\mathbf{x} + \mathbf{0} = (-1)\mathbf{x} + 0\mathbf{y} \in \mathbb{S}$$

V6: Taking $t = 0$ in equation (4.1) gives $s\mathbf{x} = s\mathbf{x} + \mathbf{0} = s\mathbf{x} + 0\mathbf{y} \in \mathbb{S}$.

Hence, all 10 axioms are satisfied. Therefore, $\mathbb{S}$ is also a vector space under the operations of $\mathbb{V}$.

Remarks

1. When proving that a set $\mathbb{S}$ is a subspace of a vector space $\mathbb{V}$, it is important not to forget to show that $\mathbb{S}$ is actually a subset of $\mathbb{V}$.
2. As with subspaces of $\mathbb{R}^n$ in Section 1.4, we typically show that the subset is non-empty by showing that it contains the zero vector of $\mathbb{V}$.

EXAMPLE 4.2.11

In Exercise 4.2.2 you proved that $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\}$ is a vector space. Thus, since $\mathbb{S}$ is a subset of $M_{2\times 2}(\mathbb{R})$, it is a subspace of $M_{2\times 2}(\mathbb{R})$.

EXAMPLE 4.2.12

Let $\mathbb{U} = \{\mathbf{p} \in P_3(\mathbb{R}) \mid \mathbf{p}(3) = 0\}$. Show that $\mathbb{U}$ is a subspace of $P_3(\mathbb{R})$.

Solution: By definition, $\mathbb{U}$ is a subset of $P_3(\mathbb{R})$. The zero vector $\mathbf{0} \in P_3(\mathbb{R})$ maps x to 0 for all x; hence it maps 3 to 0. Therefore, the zero vector of $P_3(\mathbb{R})$ is in $\mathbb{U}$, and hence $\mathbb{U}$ is non-empty.

Let $\mathbf{p}, \mathbf{q} \in \mathbb{U}$. Then, by definition of $\mathbb{U}$, polynomials $\mathbf{p}$ and $\mathbf{q}$ satisfy $\mathbf{p}(3) = 0$ and $\mathbf{q}(3) = 0$. Now, for any $s, t \in \mathbb{R}$, we need to show that the polynomial $s\mathbf{p} + t\mathbf{q}$ also satisfies the condition on $\mathbb{U}$. We have that

$$(s\mathbf{p} + t\mathbf{q})(3) = s\mathbf{p}(3) + t\mathbf{q}(3) = s(0) + t(0) = 0$$

Hence, $\mathbb{U}$ is a subspace of $P_3(\mathbb{R})$. This implies that $\mathbb{U}$ is itself a vector space using the operations of $P_3(\mathbb{R})$.

EXAMPLE 4.2.13

Define the **trace** of a 2×2 matrix by $\operatorname{tr}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} + a_{22}$. Prove that $\mathbb{S} = \{A \in M_{2\times2}(\mathbb{R}) \mid \operatorname{tr}(A) = 0\}$ is a subspace of $M_{2\times2}(\mathbb{R})$.

Solution: By definition, $\mathbb{S}$ is a subset of $M_{2\times2}(\mathbb{R})$. The zero vector of $M_{2\times2}(\mathbb{R})$ is $O_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We have $\operatorname{tr}(O_{2,2}) = 0 + 0 = 0$, so $O_{2,2} \in \mathbb{S}$.

Let $A, B \in \mathbb{S}$ and $s, t \in \mathbb{R}$. Then $a_{11} + a_{22} = \operatorname{tr}(A) = 0$ and $b_{11} + b_{22} = \operatorname{tr}(B) = 0$. Hence,

$$\begin{aligned} \operatorname{tr}(sA + tB) &= \operatorname{tr}\left(\begin{bmatrix} sa_{11} + tb_{11} & sa_{12} + tb_{12} \\ sa_{21} + tb_{21} & sa_{22} + tb_{22} \end{bmatrix}\right) \\ &= sa_{11} + tb_{11} + sa_{22} + tb_{22} \\ &= s(a_{11} + a_{22}) + t(b_{11} + b_{22}) \\ &= s(0) + t(0) = 0 \end{aligned}$$

Hence, $sA + tB \in \mathbb{S}$ and so $\mathbb{S}$ is a subspace of $M_{2\times2}(\mathbb{R})$.

EXAMPLE 4.2.14

The vector space $\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$, since $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. That is, if we take any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, this is not a vector in $\mathbb{R}^3$, since a vector in $\mathbb{R}^3$ has three components.

EXERCISE 4.2.4

Prove that $\mathbb{U} = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid b + c = a\}$ is a subspace of $P_2(\mathbb{R})$.

EXERCISE 4.2.5

Let $\mathbb{V}$ be a vector space. Prove that $\{\mathbf{0}\}$ is also a vector space, called the **trivial vector space**, under the same operations as $\mathbb{V}$ by proving it is a subspace of $\mathbb{V}$.

In Exercise 4.2.5 you proved that $\{\mathbf{0}\}$ is a subspace of any vector space $\mathbb{V}$. Furthermore, by definition, $\mathbb{V}$ is a subspace of itself. As in Chapter 1, the set of all possible linear combinations of a set of vectors in a vector space $\mathbb{V}$ is also a subspace.

Theorem 4.2.2

If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space $\mathbb{V}$ and $\mathbb{S}$ is the set of all possible linear combinations of these vectors,

$$\mathbb{S} = \{t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k \mid t_1, \dots, t_k \in \mathbb{R}\}$$

then $\mathbb{S}$ is a subspace of $\mathbb{V}$.

The proof of Theorem 4.2.2 is identical to the proof of Theorem 1.4.2 and hence is omitted.

PROBLEMS 4.2

Practice Problems

For Problems A1–A8, determine, with proof, whether the set is a subspace of the given vector space.

A1 $\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 = 0\right\}$ of $\mathbb{R}^4$

A2 $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a_1 + 2a_2 = 0\right\}$ of $M_{2\times 2}(\mathbb{R})$

A3 $\{a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{R}) \mid a_0 + 2a_1 = 0\}$ of $P_3(\mathbb{R})$

A4 $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1, a_2, a_3, a_4 \in \mathbb{Z}\right\}$ of $M_{2\times 2}(\mathbb{R})$

A5 $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a_1a_4 - a_2a_3 = 0\right\}$ of $M_{2\times 2}(\mathbb{R})$

A6 $\left\{\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a_1 = a_2\right\}$ of $M_{2\times 2}(\mathbb{R})$

A7 $\{\mathbf{p} \in P_3(\mathbb{R}) \mid \mathbf{p}(1) = 1\}$ of $P_3(\mathbb{R})$

A8 $\{\mathbf{p} \in P_3(\mathbb{R}) \mid \mathbf{p}(2) = 0 \in \mathbb{R}\}$ of $P_3(\mathbb{R})$

For Problems A9–A11, determine, with proof, whether the subset of $M_{n\times n}(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.

A9 The subset of matrices that are in row echelon form.

A10 The subset of upper triangular matrices.

A11 The subset of matrices such that $A^T = A$.

For Problems A12–A16, determine, with proof, whether the subset of $P_5(\mathbb{R})$ is a subspace of $P_5(\mathbb{R})$.

A12 $\{\mathbf{p} \in P_5(\mathbb{R}) \mid \mathbf{p}(-x) = \mathbf{p}(x) \text{ for all } x \in \mathbb{R}\}$

A13 $\{(1 + x^2)\mathbf{p} \mid \mathbf{p} \in P_3(\mathbb{R})\}$

A14 $\{a_0 + a_1x + \cdots + a_4x^4 \in P_5(\mathbb{R}) \mid a_0 = a_4, a_1 = a_3\}$

A15 $\{\mathbf{p} \in P_5(\mathbb{R}) \mid p(0) = 1\}$

A16 $\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$

For Problems A17–A20, let $\mathcal{F}$ be the vector space of all real-valued functions of a real variable. Determine, with proof, whether the subset of $\mathcal{F}$ is a subspace of $\mathcal{F}$.

A17 $\{f \in \mathcal{F} \mid f(3) = 0\}$

A18 $\{f \in \mathcal{F} \mid f(3) = 1\}$

A19 $\{f \in \mathcal{F} \mid f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$

A20 $\{f \in \mathcal{F} \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$

Homework Problems

For Problems B1–B9, determine, with proof, whether the set is a subspace of the given vector space.

B1 $\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1x_2 = x_3\right\}$ of $\mathbb{R}^3$

B2 $\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = x_2 + 2x_3\right\}$ of $\mathbb{R}^3$

B3 $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a_1 = a_2 = a_4\right\}$ of $M_{2\times 2}(\mathbb{R})$

B4 $\{a_0 + a_1x + a_2x^2 \in P_2(\mathbb{R}) \mid a_0 + a_1 = 1\}$ of $P_2(\mathbb{R})$

B5 $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_2^2 = a_3^2\right\}$ of $M_{2\times 2}(\mathbb{R})$

B6 $\{\mathbf{p} \in P_2(\mathbb{R}) \mid \mathbf{p}(-1) = 0 \in \mathbb{R}\}$ of $P_2(\mathbb{R})$

B7 $\left\{\begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a_1 = -a_2\right\}$ of $M_{2\times 2}(\mathbb{R})$

B8 $\left\{\begin{bmatrix} a_1 & a_2 \\ 1 & 1 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid 2a_1 = 3a_2\right\}$ of $M_{2\times 2}(\mathbb{R})$

B9 $\{\mathbf{p} \in P_2(\mathbb{R}) \mid \mathbf{p}(1) = 0 \text{ and } \mathbf{p}(2) = 0\}$ of $P_2(\mathbb{R})$

For Problems B10–B17, determine, with proof, whether the subset of $M_{2\times 2}(\mathbb{R})$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

B10 The subset of matrices A such that $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

B11 The subset of matrices A such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

B12 The subset of matrices A such that
$$A\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

B13 The subset of matrices A such that
$$A\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}A$$

B14 The subset of elementary matrices.

B15 The subset of non-invertible matrices.

B16 The subset of matrices such that $A^T = -A$.

B17 The subset of matrices such that $A^2 = O_{2,2}$.

For Problems B18–B22, determine, with proof, whether the subset of $P_2(\mathbb{R})$ is a subspace of $P_2(\mathbb{R})$.

B18 $\{\mathbf{p} \mid \mathbf{p}(-x) = -\mathbf{p}(x) \text{ for all } x \in \mathbb{R}\}$

B19 $\{x\mathbf{p} \mid \mathbf{p} \in P_1(\mathbb{R})\}$

B20 The subset of polynomials $\mathbf{p}$ that have derivative equal to 0.

B21 The subset of polynomials $\mathbf{p}$ that have all real roots.

B22 The subset of polynomials $\mathbf{p}$ that have degree 1.

For Problems B23–B27, let $\mathcal{F}$ be the vector space of all real-valued functions of a real variable. Determine, with proof, whether the subset of $\mathcal{F}$ is a subspace of $\mathcal{F}$.

B23 $\{f \in \mathcal{F} \mid f(1) = f(-1)\}$

B24 $\{f \in \mathcal{F} \mid f(2) + f(3) = 0\}$

B25 $\{f \in \mathcal{F} \mid f(1) + f(2) = 1\}$

B26 $\{f \in \mathcal{F} \mid |f(x)| \leq 1\}$

B27 $\{f \in \mathcal{F} \mid f \text{ is non decreasing on } \mathbb{R}\}$

Conceptual Problems

For Problems C1–C4, let $\mathbb{V}$ be a vector space. Prove the statement, clearly justifying each step.

C1 $-\mathbf{x} = (-1)\mathbf{x}$ for every $\mathbf{x} \in \mathbb{V}$.

C2 $t\mathbf{0} = \mathbf{0}$ for every $t \in \mathbb{R}$.

C3 If $t\mathbf{x} = \mathbf{0}$, then either $t = 0$ or $\mathbf{x} = \mathbf{0}$.

C4 If $\mathbf{x} + \mathbf{y} = \mathbf{z} + \mathbf{x}$, then $\mathbf{y} = \mathbf{z}$.

C5 Let $\mathbb{V} = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$ and define addition by $(a, b) \oplus (c, d) = (ad + bc, bd)$ and scalar multiplication by $t \odot (a, b) = (tab^{t-1}, b^t)$ for any $t \in \mathbb{R}$. Prove that $\mathbb{V}$ is a vector space with these operations.

C6 Let $\mathbb{V} = \{x \in \mathbb{R} \mid x > 0\}$ and define addition by $x \oplus y = xy$ and scalar multiplication by $t \odot x = x^t$ for any $t \in \mathbb{R}$. Prove that $\mathbb{V}$ is a vector space with these operations.

C7 Let $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ be the set of complex numbers. Prove that $\mathbb{C}$ is a vector space with addition and scalar multiplication defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$t(a + bi) = (ta) + (tb)i, \qquad \text{for all } t \in \mathbb{R}$$

C8 Let $\mathbb{L}$ denote the set of all linear operators $L : \mathbb{R}^n \to \mathbb{R}^n$ with standard addition and scalar multiplication of linear mappings. Prove that $\mathbb{L}$ is a vector space under these operations.

C9 Categorize all subspaces of $\mathbb{R}^2$.

C10 Suppose that $\mathbb{U}$ and $\mathbb{V}$ are vector spaces over $\mathbb{R}$. The **Cartesian product** of $\mathbb{U}$ and $\mathbb{V}$ is defined to be

$$\mathbb{U} \times \mathbb{V} = \{(\mathbf{u}, \mathbf{v}) \mid \mathbf{u} \in \mathbb{U}, \mathbf{v} \in \mathbb{V}\}$$

(a) In $\mathbb{U} \times \mathbb{V}$ define addition and scalar multiplication by

$$(\mathbf{u}_1, \mathbf{v}_1) \oplus (\mathbf{u}_2, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2)$$
$$t \odot (\mathbf{u}_1, \mathbf{v}_1) = (t\mathbf{u}_1, t\mathbf{v}_1)$$

Verify that with these operations that $\mathbb{U} \times \mathbb{V}$ is a vector space.

(b) Verify that $\mathbb{U} \times \{\mathbf{0}_{\mathbb{V}}\}$ is a subspace of $\mathbb{U} \times \mathbb{V}$.

(c) Suppose instead that scalar multiplication is defined by $t \odot (\mathbf{u}, \mathbf{v}) = (t\mathbf{u}, \mathbf{v})$, while addition is defined as in part (a). Is $\mathbb{U} \times \mathbb{V}$ a vector space with these operations?

For Problems C11–C13, let $\mathbb{V} = \{(x, y) \mid x, y \in \mathbb{R}\}$. Show that $\mathbb{V}$ is not a vector space for the given operations of addition and scalar multiplication by listing all the properties that fail to hold.

C11
$$(x_1, y_1) + (x_2, y_2) = (x_1 + y_2, x_2 + y_1)$$
$$t(x_1, x_2) = (tx_2, tx_1)$$

C12
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 0)$$
$$t(x_1, x_2) = (tx_1, 0)$$

C13
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$$
$$t(x_1, x_2) = (tx_1, tx_2)$$

4.3 Bases and Dimensions

We have seen the importance and usefulness of bases in $\mathbb{R}^n$. A basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$ provides an explicit formula for every vector in $\mathbb{S}$. Moreover, we can geometrically view this explicit formula as a coordinate system. That is, a basis of a subspace of $\mathbb{R}^n$ forms a set of coordinate axes for that subspace. Thus, we desire to extend the concept of bases to general vector spaces.

We begin by defining the concepts of spanning and linear independence in general vector spaces.

Definition
Span
Spanning Set

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. The **span** of $\mathcal{B}$ is defined as

$$\operatorname{Span}\mathcal{B} = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

The subspace $\mathbb{S} = \operatorname{Span}\mathcal{B}$ of $\mathbb{V}$ is said to be **spanned** by $\mathcal{B}$, and $\mathcal{B}$ is called a **spanning set** for $\mathbb{S}$.

Definition
Linearly Dependent
Linearly Independent

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. The set $\mathcal{B}$ is said to be **linearly dependent** if there exists real coefficients $t_1, \ldots, t_k$ not all zero such that

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

The set $\mathcal{B}$ is said to be **linearly independent** if the only solution to

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

is the trivial solution $t_1 = \cdots = t_k = 0$.

In a general vector space, the procedure for determining if a vector is in a span, if a set spans a subspace, or if a set is linearly independent is exactly the same as we saw for $\mathbb{R}^n$, $M_{m\times n}(\mathbb{R})$, and $P_n(\mathbb{R})$ in Sections 1.4, 2.3, 3.1, and 4.1.

Bases

Definition
Basis

A set $\mathcal{B}$ in a vector space $\mathbb{V}$ is called a **basis** for $\mathbb{V}$ if
(1) $\mathcal{B}$ is linearly independent, and
(2) $\operatorname{Span}\mathcal{B} = \mathbb{V}$.
A basis for the trivial vector space $\{\mathbf{0}\}$ is defined to be the empty set $\emptyset = \{\,\}$.

It is clear that we should want a basis $\mathcal{B}$ for a vector space $\mathbb{V}$ to be a spanning set so that every vector in $\mathbb{V}$ can be written as a linear combination of the vectors in $\mathcal{B}$. Why would it be important that the set $\mathcal{B}$ be linearly independent? The following theorem answers this question.

Theorem 4.3.1 Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for a vector space $\mathbb{V}$. Every vector in $\mathbb{V}$ can be expressed in a *unique* way as a linear combination of the vectors of $\mathcal{B}$ if and only if the set $\mathcal{B}$ is linearly independent.

Proof: Let $\mathbf{x}$ be any vector in $\mathbb{V}$. Since Span $\mathcal{B} = \mathbb{V}$, we have that $\mathbf{x}$ can be written as a linear combination of the vectors in $\mathcal{B}$. Assume that there are linear combinations

$$\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{x} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

This gives

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{x} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

which implies

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) - (b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n) = (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n$$

If $\mathcal{B}$ is linearly independent, then we must have $a_i - b_i = 0$, so $a_i = b_i$ for $1 \leq i \leq n$. Hence, $\mathbf{x}$ has a unique representation.

On the other hand, if $\mathcal{B}$ is linearly dependent, then

$$\mathbf{0} = t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n$$

has a solution where at least one of the coefficients is non-zero. But

$$\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$$

Hence, $\mathbf{0}$ can be expressed as a linear combination of the vectors in $\mathcal{B}$ in multiple ways. ■

Thus, if $\mathcal{B}$ is a basis for a vector space $\mathbb{V}$, then every vector in $\mathbb{V}$ can be written as a unique linear combination of the vectors in $\mathcal{B}$.

EXAMPLE 4.3.1

The set of vectors $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ in Exercise 3.1.2 is a basis for $M_{2\times 2}(\mathbb{R})$. It is called the **standard basis** for $M_{2\times 2}(\mathbb{R})$.

EXAMPLE 4.3.2

The set of vectors $\{1, x, x^2, x^3\}$ in Exercise 4.1.2 is a basis for $P_3(\mathbb{R})$. In particular, the set $\{1, x, \ldots, x^n\}$ is called the **standard basis** for $P_n(\mathbb{R})$.

Not surprisingly, the method for proving a set $\mathcal{B}$ is a basis for a vector space $\mathbb{V}$ is the same as the procedure we used in Section 2.3 for subspaces of $\mathbb{R}^n$.

EXAMPLE 4.3.3

Is the set $C = \{3 + 2x + 2x^2, 1 + x^2, 1 + x + x^2\}$ a basis for $P_2(\mathbb{R})$?

Solution: We need to determine whether C is linearly independent and if it spans $P_2(\mathbb{R})$. Consider the equation

$$\begin{aligned} a_0 + a_1x + a_2x^2 &= t_1(3 + 2x + 2x^2) + t_2(1 + x^2) + t_3(1 + x + x^2) \\ &= (3t_1 + t_2 + t_3) + (2t_1 + t_3)x + (2t_1 + t_2 + t_3)x^2 \end{aligned}$$

Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By the System-Rank Theorem (3), the system is consistent for all $a_0, a_1, a_2 \in \mathbb{R}$. Thus, C spans $P_2(\mathbb{R})$. By the System-Rank Theorem (2), the system has a unique solution for all $a_0, a_1, a_2 \in \mathbb{R}$ (including $a_0 = a_1 = a_2 = 0$). Thus, C is also linearly independent and hence is a basis for $P_2(\mathbb{R})$.

EXAMPLE 4.3.4

Determine whether the set $C = \left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$.

Solution: Consider

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = t_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + t_2\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + t_3\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 + t_3 & 2t_2 + 3t_3 \\ t_1 + t_2 & t_2 + t_3 \end{bmatrix}$$

This gives a system of 4 equation in the 3 variables t_1, t_2, t_3. The maximum rank of the coefficient matrix is then 3. Therefore, by the System-Rank Theorem (3), the system is not consistent for all $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and hence C cannot span $M_{2\times 2}(\mathbb{R})$.

EXAMPLE 4.3.5

Determine whether the set $\mathcal{B} = \left\{\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\right\}$ is a basis for the subspace Span $\mathcal{B}$ of $M_{2\times 2}(\mathbb{R})$.

Solution: Since $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$, we just need to check if the vectors in $\mathcal{B}$ are linearly independent. Consider the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + t_2\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} + t_3\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_3 & 2t_1 + t_2 + 5t_3 \\ -t_1 + 3t_2 + t_3 & t_1 + t_2 + 3t_3 \end{bmatrix}$$

Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By the System-Rank Theorem (2), the system has infinitely many solutions. Thus, $\mathcal{B}$ is linearly dependent and hence is not a basis.

Obtaining a Basis from a Finite Spanning Set

Many times throughout the rest of this book, we will need to determine a basis for a vector space. One standard way of doing this is to first determine a spanning set for the vector space and then to remove vectors from the spanning set until we have a basis. We will need the following theorem.

Theorem 4.3.2 Basis Reduction Theorem

If $\mathcal{T} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a spanning set for a non-trivial vector space $\mathbb{V}$, then some subset of $\mathcal{T}$ is a basis for $\mathbb{V}$.

Proof: If $\mathcal{T}$ is linearly independent, then $\mathcal{T}$ is a basis for $\mathbb{V}$, and we are done. If $\mathcal{T}$ is linearly dependent, then $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ has a solution where at least one of the coefficients is non-zero, say $t_i \neq 0$. Then, we can solve the equation for $\mathbf{v}_i$ to get

$$\mathbf{v}_i = -\frac{1}{t_i}(t_1\mathbf{v}_1 + \cdots + t_{i-1}\mathbf{v}_{i-1} + t_{i+1}\mathbf{v}_{i+1} + \cdots + t_k\mathbf{v}_k)$$

So, for any $\mathbf{x} \in \mathbb{V}$ we have

$$\begin{aligned}
\mathbf{x} =& a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + a_i\mathbf{v}_i + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_k\mathbf{v}_k \\
=& a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + a_i\left[-\frac{1}{t_i}(t_1\mathbf{v}_1 + \cdots + t_{i-1}\mathbf{v}_{i-1} + t_{i+1}\mathbf{v}_{i+1} + \cdots + t_k\mathbf{v}_k)\right] \\
&+ a_{i+1}\mathbf{v}_{i+1} + \cdots + a_k\mathbf{v}_k \\
=&(a_1 - \frac{a_it_1}{t_i})\mathbf{v}_1 + \cdots + (a_{i-1} - \frac{a_it_{i-1}}{t_i})\mathbf{v}_{i-1} + (a_{i+1} - \frac{a_it_{i+1}}{t_i})\mathbf{v}_{i+1} + \cdots + (a_n - \frac{a_it_n}{t_i})\mathbf{v}_n
\end{aligned}$$

Thus, $\mathbf{x} \in \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$. That is, $\mathcal{T}_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{V}$. If $\mathcal{T}_1$ is linearly independent, it is a basis for $\mathbb{V}$, and the procedure is finished. Otherwise, we repeat the procedure to omit a second vector, say $\mathbf{v}_j$, and get another subset $\mathcal{T}_2$, which still spans $\mathbb{V}$. In this fashion, we must eventually get a linearly independent set. (Certainly, if there is only one non-zero vector left, it forms a linearly independent set.) Thus, we obtain a subset of $\mathcal{T}$ that is a basis for $\mathbb{V}$. ■

EXAMPLE 4.3.6 Determine a basis for the subspace $\mathbb{S} = \{\mathbf{p} \in P_2(\mathbb{R}) \mid \mathbf{p}(1) = 0\}$ of $P_2(\mathbb{R})$.

Solution: We first find a spanning set for $\mathbb{S}$. By the Factor Theorem, if $\mathbf{p}(1) = 0$, then $(x - 1)$ is a factor of $\mathbf{p}$. That is, every polynomial $\mathbf{p} \in \mathbb{S}$ can be written in the form

$$\mathbf{p}(x) = (x - 1)(ax + b) = a(x^2 - x) + b(x - 1)$$

Thus, we see that $\mathcal{T} = \{x^2 - x, x - 1\}$ spans $\mathbb{S}$.

Next, we need to determine whether $\mathcal{T}$ is linearly independent. Consider

$$0 = t_1(x^2 - x) + t_2(x - 1) = t_1x^2 + (-t_1 + t_2)x - t_2$$

The only solution is $t_1 = t_2 = 0$. Hence, $\mathcal{T}$ is linearly independent. Thus, $\mathcal{T}$ is a basis for $\mathbb{S}$.

EXAMPLE 4.3.7

Consider the subspace of $\mathbb{R}^3$ spanned by $\mathcal{T} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$. Determine a subset of $\mathcal{T}$ that is a basis for Span $\mathcal{T}$.

Solution: Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t_4 \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 + t_3 + t_4 \\ t_1 - t_2 - 2t_3 + 5t_4 \\ -2t_1 + t_2 + 3t_3 + 3t_4 \end{bmatrix}$$

We row reduce the corresponding coefficent matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & -2 & 5 \\ -2 & 1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The general solution is $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $s \in \mathbb{R}$. Taking $s = 1$, we get $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, which gives

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus, we can omit $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ from $\mathcal{T}$ and consider $\mathcal{T}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$.

Now consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

The coefficient matrix corresponding to this system is the same as above except that the third column is omitted, so the same row operations give

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 5 \\ -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the only solution is $t_1 = t_2 = t_3 = 0$ and we conclude that $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$ is linearly independent and thus a basis for Span $\mathcal{T}$.

CONNECTION

Compare this to how we solved Example 3.4.7 on page 196 and our procedure for finding a basis for the column space of a matrix (Theorem 3.4.5).

EXERCISE 4.3.1

Consider the subspace of $P_2(\mathbb{R})$ spanned by $\mathcal{B} = \{1 - x, 2 + 2x + x^2, x + x^2, 1 + x^2\}$. Determine a subset of $\mathcal{B}$ that is a basis for Span $\mathcal{B}$.

Dimension

We saw in Section 2.3 that every basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$ contains the same number of vectors. We now prove that this result holds for general vector spaces. Observe that the proof is identical to that in Section 2.3.

Theorem 4.3.3 Suppose that $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is a basis for a non-trivial vector space $\mathbb{V}$ and that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a set in $\mathbb{V}$. If $k > \ell$, then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly dependent.

Proof: Since each $\mathbf{u}_i$, $1 \leq i \leq k$, is a vector in $\mathbb{V}$ and $\mathcal{B}$ is a basis for $\mathbb{V}$, by the Unique Representation Theorem, each $\mathbf{u}_i$ can be written as a unique linear combination of the vectors in $\mathcal{B}$. We get

$$\begin{aligned}
\mathbf{u}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{\ell 1}\mathbf{v}_\ell \\
\mathbf{u}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{\ell 2}\mathbf{v}_\ell \\
&\ \ \vdots \\
\mathbf{u}_k &= a_{1k}\mathbf{v}_1 + a_{2k}\mathbf{v}_2 + \cdots + a_{\ell k}\mathbf{v}_\ell
\end{aligned}$$

Consider the equation

$$\begin{aligned}
\mathbf{0} &= t_1\mathbf{u}_1 + \cdots + t_k\mathbf{u}_k \qquad (4.2)\\
&= t_1(a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{\ell 1}\mathbf{v}_\ell) + \cdots + t_k(a_{1k}\mathbf{v}_1 + a_{2k}\mathbf{v}_2 + \cdots + a_{\ell k}\mathbf{v}_\ell) \\
&= (a_{11}t_1 + \cdots + a_{1k}t_k)\mathbf{v}_1 + \cdots + (a_{\ell 1}t_1 + \cdots + a_{\ell k}t_k)\mathbf{v}_\ell
\end{aligned}$$

But, $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is linearly independent, so the only solution to this equation is

$$\begin{aligned}
a_{11}t_1 + \cdots + a_{1k}t_k &= 0 \\
\vdots \qquad\qquad & \ \ \vdots \\
a_{\ell 1}t_1 + \cdots + a_{\ell k}t_k &= 0
\end{aligned}$$

The rank of the coefficient matrix of this homogeneous system is at most ℓ because $\ell < k$. Hence, by the System-Rank Theorem (2), the solution space has at least $k - \ell > 0$ parameters. Therefore, there are infinitely many possible $t_1, \dots, t_k$ and so $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly dependent since equation (4.2) has infinitely many solutions. ■

Theorem 4.3.4

Dimension Theorem

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ are both bases of a vector space $\mathbb{V}$, then $k = \ell$.

Proof: Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ is a basis for S and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent, by Theorem 4.3.3, we must have $k \leq \ell$. Similarly, since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis and $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ is linearly independent, we must also have $\ell \leq k$. Therefore, $\ell = k$. ■

As in Section 2.3, this theorem justifies the following definition of the dimension of a vector space.

Definition

Dimension

If a vector space $\mathbb{V}$ has a basis with n vectors, then we say that the **dimension** of $\mathbb{V}$ is n and write

$$\dim \mathbb{V} = n$$

If a vector space $\mathbb{V}$ does not have a basis with finitely many elements, then $\mathbb{V}$ is called **infinite-dimensional**.

Remarks

1. By definition, the trivial vector space $\{\mathbf{0}\}$ is zero dimensional since we defined a basis for this vector space to be the empty set.
2. Properties of infinite-dimensional spaces are beyond the scope of this book.

EXAMPLE 4.3.8

(a) $\mathbb{R}^n$ is n-dimensional because the standard basis contains n vectors.

(b) The vector space $M_{m\times n}(\mathbb{R})$ is $(m \times n)$-dimensional since the standard basis has $m \times n$ vectors.

(c) The vector space $P_n(\mathbb{R})$ is $(n + 1)$-dimensional as it has the standard basis $\{1, x, x^2, \ldots, x^n\}$.

(d) The vector space $C(a, b)$ is infinite-dimensional as it contains all polynomials (along with many other types of functions). Most function spaces are infinite-dimensional.

EXAMPLE 4.3.9

Let $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}-1\\-2\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\3\\4\end{bmatrix}\right\}$. Show that $\dim \mathbb{S} = 2$.

Solution: Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = t_1\begin{bmatrix}1\\3\\2\end{bmatrix} + t_2\begin{bmatrix}-1\\-2\\-1\end{bmatrix} + t_3\begin{bmatrix}-1\\2\\3\end{bmatrix} + t_4\begin{bmatrix}-1\\3\\4\end{bmatrix} = \begin{bmatrix}t_1 - t_2 - t_3 - t_4\\3t_1 - 2t_2 + 2t_3 + 3t_4\\2t_1 - t_2 + 3t_3 + 4t_4\end{bmatrix}$$

We row reduce the corresponding coefficient matrix

$$\begin{bmatrix}1 & -1 & -1 & -1\\3 & -2 & 2 & 3\\2 & -1 & 3 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 4 & 5\\0 & 1 & 5 & 6\\0 & 0 & 0 & 0\end{bmatrix}$$

This implies that $\begin{bmatrix}-1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\3\\4\end{bmatrix}$ can be written as linear combinations of the first two vectors. Thus, $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}-1\\-2\\-1\end{bmatrix}\right\}$. Moreover, $\mathcal{B} = \left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}-1\\-2\\-1\end{bmatrix}\right\}$ is clearly linearly independent since neither vector is a scalar multiple of the other, hence $\mathcal{B}$ is a basis for $\mathbb{S}$. Thus, $\dim \mathbb{S} = 2$.

EXAMPLE 4.3.10

Let $\mathbb{S} = \left\{\begin{bmatrix}a & b\\c & d\end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a + b = d\right\}$. Determine the dimension of $\mathbb{S}$.

Solution: Since $d = a + b$, observe that every matrix in $\mathbb{S}$ has the form

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}a & b\\c & a+b\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

Thus,

$$\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$$

It is easy to show that $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$ is also linearly independent and hence is a basis for $\mathbb{S}$. Thus, $\dim \mathbb{S} = 3$.

EXERCISE 4.3.2

Find the dimension of $\mathbb{S} = \{a + bx + cx^2 + dx^3 \in P_3(\mathbb{R}) \mid a + b + c + d = 0\}$.

Extending a Linearly Independent Subset to a Basis

Sometimes a linearly independent set $\mathcal{T} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is given in an n-dimensional vector space $\mathbb{V}$, and it is necessary to include the vectors in $\mathcal{T}$ in a basis for $\mathbb{V}$.

Theorem 4.3.5 Basis Extension Theorem

If $\mathbb{V}$ is an n-dimensional vector space and $\mathcal{T} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set in $\mathbb{V}$ with $k < n$, then there exist vectors $\mathbf{w}_{k+1}, \dots, \mathbf{w}_n$ such that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_{k+1}, \dots, \mathbf{w}_n\}$ is a basis for $\mathbb{V}$.

Proof: Since $k < n$, the set $\mathcal{T} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ cannot be a basis for $\mathbb{V}$ as that would contradict the Dimension Theorem. Thus, since the set is linearly independent, it must not span $\mathbb{V}$. Therefore, there exists a vector $\mathbf{w}_{k+1}$ that is in $\mathbb{V}$ but not in Span $\mathcal{T}$. Now consider

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{w}_{k+1} = \mathbf{0} \tag{4.3}$$

If $t_{k+1} \neq 0$, then we have

$$\mathbf{w}_{k+1} = -\frac{t_1}{t_{k+1}}\mathbf{v}_1 - \cdots - \frac{t_k}{t_{k+1}}\mathbf{v}_k$$

and so $\mathbf{w}_{k+1}$ can be written as a linear combination of the vectors in $\mathcal{T}$, which cannot be since $\mathbf{w}_{k+1} \notin \operatorname{Span}\mathcal{T}$. Therefore, we must have $t_{k+1} = 0$. In this case, equation (4.3) becomes

$$\mathbf{0} = t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$$

But, $\mathcal{T}$ is linearly independent, which implies that $t_1 = \cdots = t_k = 0$. Thus, the only solution to equation (4.3) is $t_1 = \cdots = t_{k+1} = 0$, and hence $\mathcal{T}_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_{k+1}\}$ is linearly independent.

Now, if $\operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_{k+1}\} = \mathbb{V}$, then $\mathcal{T}_1$ is a basis for $\mathbb{V}$. If not, we repeat the procedure to add another vector $\mathbf{w}_{k+2}$ to get $\mathcal{T}_2 = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_{k+1}, \mathbf{w}_{k+2}\}$, which is linearly independent. In this fashion, we must eventually get a basis, since according to Theorem 4.3.3, there cannot be more than n linearly independent vectors in an n-dimensional vector space. ■

The Basis Extension Theorem proves that every n-dimensional vector space $\mathbb{V}$ has a basis. In particular, $\{\mathbf{0}\}$ has a basis by definition. If $n \geq 1$, then we can pick any non-zero vector $\mathbf{v} \in \mathbb{V}$ and then extend $\{\mathbf{v}\}$ to a basis for $\mathbb{V}$.

It also gives us the following useful result.

Theorem 4.3.6 If $\mathbb{S}$ is a subspace of an n-dimensional vector space $\mathbb{V}$, then

$$\dim \mathbb{S} \leq n$$

EXAMPLE 4.3.11

Let $\mathcal{T} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \right\}$. Extend $\mathcal{T}$ to a basis for $M_{2\times 2}(\mathbb{R})$.

Solution: We first want to determine whether $\mathcal{T}$ is a spanning set for $M_{2\times 2}(\mathbb{R})$. Consider

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} t_1 - 2t_2 & t_1 - t_2 \\ t_2 & t_1 + t_2 \end{bmatrix}$$

Row reducing the augmented matrix of the associated system gives

$$\left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 1 & -1 & b_2 \\ 0 & 1 & b_3 \\ 1 & 1 & b_4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_1 - b_2 + b_3 \\ 0 & 0 & 2b_1 - 3b_2 + b_4 \end{array}\right]$$

Observe that if $b_1 - b_2 + b_3 \neq 0$ (or $2b_1 - 3b_2 + b_4 \neq 0$), then the system is inconsistent. Therefore, $\mathcal{T}$ is not a spanning set of $M_{2\times 2}(\mathbb{R})$ since any matrix $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ with $b_1 - b_2 + b_3 \neq 0$ is not in Span $\mathcal{T}$. In particular, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not in the span of $\mathcal{T}$. Following the steps in the proof, we should add this matrix to $\mathcal{T}$. We let

$$\mathcal{T}_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

and repeat the procedure. Consider

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} t_1 - 2t_2 & t_1 - t_2 \\ t_2 + t_3 & t_1 + t_2 \end{bmatrix}$$

Row reducing the augmented matrix of the associated system gives

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & b_1 \\ 1 & -1 & 0 & b_2 \\ 0 & 1 & 1 & b_3 \\ 1 & 1 & 0 & b_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & b_1 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 1 & b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 2b_1 - 3b_2 + b_4 \end{array}\right]$$

So, any matrix $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ with $2b_1 - 3b_2 + b_4 \neq 0$ is not in Span $\mathcal{T}_1$. For example, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not in the span of $\mathcal{T}_1$ and thus $\mathcal{T}_1$ is not a basis for $M_{2\times 2}(\mathbb{R})$. Adding $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ to $\mathcal{T}_1$ we get

$$\mathcal{T}_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

By construction $\mathcal{T}_2$ is linearly independent. Moreover, we can show that it spans $M_{2\times 2}(\mathbb{R})$. Thus, it is a basis for $M_{2\times 2}(\mathbb{R})$.

EXERCISE 4.3.3

Extend the set $\mathcal{T} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ to a basis for $\mathbb{R}^3$.

Knowing the dimension of a finite dimensional vector space $\mathbb{V}$ is very useful when trying to construct a basis for $\mathbb{V}$, as the next theorem demonstrates.

Theorem 4.3.7 Basis Theorem

If $\mathbb{V}$ is an n-dimensional vector space, then

(1) A set of more than n vectors in $\mathbb{V}$ must be linearly dependent.

(2) A set of fewer than n vectors cannot span $\mathbb{V}$.

(3) If $\mathcal{B}$ contains n elements of $\mathbb{V}$ and Span $\mathcal{B} = \mathbb{V}$, then $\mathcal{B}$ is a basis for $\mathbb{V}$.

(4) If $\mathcal{B}$ contains n elements of $\mathbb{V}$ and $\mathcal{B}$ is linearly independent, then $\mathcal{B}$ is a basis for $\mathbb{V}$.

Proof: (1) This is Theorem 4.3.3 above.

(2) Suppose that $\mathbb{V}$ can be spanned by a set $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ where $k < n$. Then, by the Basis Reduction Theorem, some subset of $\mathcal{B}$ is a basis for $\mathbb{V}$. Hence, there is a basis of $\mathbb{V}$ that contains less than n vectors which contradicts the Dimension Theorem.

(3) If $\mathcal{B}$ is a spanning set for $\mathbb{V}$ that is not linearly independent, then by the Basis Reduction Theorem, there is a proper subset of $\mathcal{B}$ that is a basis for $\mathbb{V}$. But, this would contradict the Dimension Theorem.

(4) If $\mathcal{B}$ is a linearly independent set of n vectors that does not span $\mathbb{V}$, then it can be extended to a basis for $\mathbb{V}$ by the Basis Extension Theorem. But, this would contradict the Dimension Theorem. ■

Remarks

1. The Basis Theorem categorizes a basis as a **maximally linearly independent set**. That is, if we have a linearly independent set $\mathcal{B}$ in an n-dimensional vector space $\mathbb{V}$ such that adding any other vector in $\mathbb{V}$ to $\mathcal{B}$ makes $\mathcal{B}$ linearly dependent, then $\mathcal{B}$ is a basis for $\mathbb{V}$.

2. The Basis Theorem also categorizes a basis as a **minimal spanning set**. That is, if we have a spanning set $\mathcal{C}$ in an n-dimensional vector space $\mathbb{V}$ such that removing any vector from $\mathcal{C}$ would result in a set that does not span $\mathbb{V}$, then $\mathcal{C}$ is a basis for $\mathbb{V}$.

EXAMPLE 4.3.12

Consider the plane $\mathcal{P}$ in $\mathbb{R}^3$ with equation $x_1 + 2x_2 - x_3 = 0$.

(a) Produce a basis $\mathcal{B}$ for $\mathcal{P}$.

(b) Extend the basis $\mathcal{B}$ to obtain a basis C for $\mathbb{R}^3$.

Solution: (a) By definition, a plane in $\mathbb{R}^3$ has dimension 2. So, by the Basis Theorem, we just need to pick two linearly independent vectors that lie in the plane.

Observe that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ both satisfy the equation of the plane and hence $\vec{v}_1, \vec{v}_2 \in \mathcal{P}$.

Since neither $\vec{v}_1$ nor $\vec{v}_2$ is a scalar multiple of the other, we have that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

Therefore, $\mathcal{B}$ is a linearly independent set of two vectors in a 2-dimensional vector space. Hence, by the Basis Theorem, it is a basis for the plane $\mathcal{P}$.

(b) From the proof of the Basis Extension Theorem, we just need to add a vector that is not in the span of $\{\vec{v}_1, \vec{v}_2\}$. But, Span$\{\vec{v}_1, \vec{v}_2\}$ is the plane, so we need to pick any vector not in the plane. We pick a normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Thus, $\{\vec{v}_1, \vec{v}_2, \vec{n}\}$ is a linearly independent set of three vectors in $\mathbb{R}^3$ and therefore, by the Basis Theorem, is a basis for $\mathbb{R}^3$.

CONNECTION

To extend the basis $\mathcal{B}$ in Example 4.3.12, we just need to pick a vector that does not satisfy the equation of the plane. However, we will see in Section 4.6 that there can be a benefit to picking a normal vector for the plane over other vectors.

EXERCISE 4.3.4

Produce a basis for the hyperplane in $\mathbb{R}^4$ with equation $x_1 - x_2 + x_3 - 2x_4 = 0$ and extend the basis to obtain a basis for $\mathbb{R}^4$.

PROBLEMS 4.3

Practice Problems

For Problems A1–A4, determine whether the set is a basis for $\mathbb{R}^3$.

A1 $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix},\begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$

A2 $\left\{\begin{bmatrix}-2\\2\\1\end{bmatrix},\begin{bmatrix}3\\-1\\2\end{bmatrix}\right\}$

A3 $\left\{\begin{bmatrix}-1\\3\\5\end{bmatrix},\begin{bmatrix}2\\4\\0\end{bmatrix},\begin{bmatrix}1\\4\\2\end{bmatrix}\right\}$

A4 $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}-1\\2\\1\end{bmatrix},\begin{bmatrix}1\\3\\5\end{bmatrix},\begin{bmatrix}2\\-1\\-4\end{bmatrix}\right\}$

For Problems A5–A7, determine whether the set is a basis for $P_2(\mathbb{R})$.

A5 $\{1 + x + 2x^2, 1 - x - x^2, 2 + x + x^2\}$

A6 $\{-2 + 2x + x^2, 3 - x + 2x^2\}$

A7 $\{1 - x + x^2, 1 + 2x - x^2, 3 + x^2\}$

A8 Determine whether $\mathcal{B} = \left\{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix},\begin{bmatrix}1 & 2\\0 & -1\end{bmatrix},\begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}\right\}$ is a basis of the subspace $\mathbb{U} = \left\{\begin{bmatrix}a & b\\0 & c\end{bmatrix} \mid a, b, c \in \mathbb{R}\right\}$ of $M_{2\times2}(\mathbb{R})$.

For Problems A9 and A10, determine the dimension of the subspace of $\mathbb{R}^3$ spanned by $\mathcal{B}$.

A9 $\mathcal{B} = \left\{\begin{bmatrix}1\\-2\\1\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix},\begin{bmatrix}2\\0\\10\end{bmatrix},\begin{bmatrix}1\\1\\7\end{bmatrix}\right\}$

A10 $\mathcal{B} = \left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}-2\\-6\\-4\end{bmatrix},\begin{bmatrix}-1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\4\\8\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

For Problems A11–A13, determine the dimension of the subspace of $M_{2\times2}(\mathbb{R})$ spanned by $\mathcal{B}$.

A11 $\mathcal{B} = \left\{\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\3 & -1\end{bmatrix},\begin{bmatrix}1 & -1\\2 & -3\end{bmatrix},\begin{bmatrix}2 & 1\\4 & -3\end{bmatrix}\right\}$

A12 $\mathcal{B} = \left\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix},\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix},\begin{bmatrix}0 & 2\\1 & 1\end{bmatrix},\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix},\begin{bmatrix}3 & 1\\2 & 2\end{bmatrix}\right\}$

A13 $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix},\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right\}$

For Problems A14–A16, determine the dimension of the subspace of $P_3(\mathbb{R})$ spanned by $\mathcal{B}$.

A14 $\mathcal{B} = \{1 + x, 1 + x + x^2, 1 + x^3\}$

A15 $\mathcal{B} = \{1 + x, 1 - x, 1 + x^3, 1 - x^3\}$

A16 $\mathcal{B} = \{1 + x + x^2, 1 - x^3, 1 - 2x + 2x^2 - x^3, 1 - x^2 + 2x^3, x^2 + x^3\}$

A17 (a) Using the method in Example 4.3.12, determine a basis for the plane in $\mathbb{R}^3$ with equation $2x_1 - x_2 - x_3 = 0$.

(b) Extend the basis of part (a) to obtain a basis for $\mathbb{R}^3$.

A18 (a) Using the method in Example 4.3.12, determine a basis for the hyperplane in $\mathbb{R}^4$ with equation $x_1 - x_2 + x_3 - x_4 = 0$.

(b) Extend the basis of part (a) to obtain a basis for $\mathbb{R}^4$.

A19 Find a basis for $P_3(\mathbb{R})$ that includes the vectors $\vec{v}_1 = 1 + x + x^3$ and $\vec{v}_2 = 1 + x^2$.

A20 Find a basis for $M_{2\times2}(\mathbb{R})$ that includes the vectors $\vec{v}_1 = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$.

A21 State the standard basis for $M_{2\times3}(\mathbb{R})$.

For Problems A22–A27, obtain a basis for the vector space and determine its dimension.

A22 $\mathbb{S} = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid a = -c\}$

A23 $\mathbb{S} = \{a + bx + cx^2 + dx^3 \in P_3(\mathbb{R}) \mid a - 2b = d\}$

A24 $\mathbb{S} = \left\{\begin{bmatrix}a & b\\0 & c\end{bmatrix} \in M_{2\times2}(\mathbb{R}) \mid a, b, c \in \mathbb{R}\right\}$

A25 $\mathbb{S} = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3 \mid \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = 0\right\}$

A26 $\mathbb{S} = \{\mathbf{p} \in P_2(\mathbb{R}) \mid \mathbf{p}(2) = 0 \text{ and } \mathbf{p}(3) = 0\}$

A27 $\mathbb{S} = \left\{\begin{bmatrix}a & b\\c & d\end{bmatrix} \in M_{2\times2}(\mathbb{R}) \mid a = -c \text{ and } b = -c\right\}$

Homework Problems

For Problems B1–B6, determine whether the set is a basis for $\mathbb{R}^3$.

B1 $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\3\\-2\end{bmatrix}\right\}$

B2 $\left\{\begin{bmatrix}3\\0\\3\end{bmatrix},\begin{bmatrix}5\\2\\0\end{bmatrix},\begin{bmatrix}4\\-1\\3\end{bmatrix}\right\}$

B3 $\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}2\\-6\\3\end{bmatrix},\begin{bmatrix}6\\-2\\-5\end{bmatrix}\right\}$

B4 $\left\{\begin{bmatrix}2\\-1\\-4\end{bmatrix},\begin{bmatrix}5\\7\\-5\end{bmatrix}\right\}$

B5 $\left\{\begin{bmatrix}3\\2\\-2\end{bmatrix},\begin{bmatrix}4\\-1\\1\end{bmatrix},\begin{bmatrix}6\\7\\-4\end{bmatrix}\right\}$

B6 $\left\{\begin{bmatrix}3\\1\\4\end{bmatrix},\begin{bmatrix}1\\5\\9\end{bmatrix},\begin{bmatrix}2\\6\\5\end{bmatrix},\begin{bmatrix}3\\5\\8\end{bmatrix}\right\}$

For Problems B7–B10, determine whether the set is a basis for $P_2(\mathbb{R})$.

B7 $\{1+2x, 3+5x+x^2, 4+x+2x^2\}$

B8 $\{2+x+3x^2, 3+x+5x^2, 1+x+x^2\}$

B9 $\{1-x, 1+x+x^2, 1+2x^2, x+3x^2\}$

B10 $\{1+2x+3x^2, -1-4x+3x^2\}$

For Problems B11–B13, determine whether the set is a basis for the subspace $\mathbb{S}$ of $M_{2\times2}(\mathbb{R})$ defined by

$$\mathbb{S} = \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \mid a = 2b+c+d \in \mathbb{R}\right\}$$

B11 $\mathcal{B}_1 = \left\{\begin{bmatrix}2 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix},\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}$

B12 $\mathcal{B}_2 = \left\{\begin{bmatrix}4 & 1\\1 & 1\end{bmatrix},\begin{bmatrix}5 & 2\\0 & 1\end{bmatrix}\right\}$

B13 $\mathcal{B}_3 = \left\{\begin{bmatrix}0 & 0\\1 & -1\end{bmatrix},\begin{bmatrix}0 & 1\\0 & -2\end{bmatrix},\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}\right\}$

For Problems B14–B17, determine the dimension of the subspace of $\mathbb{R}^3$ spanned by $\mathcal{B}$.

B14 $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}3\\1\\7\end{bmatrix},\begin{bmatrix}0\\-1\\-1\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$

B15 $\mathcal{B} = \left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}3\\5\\7\end{bmatrix}\right\}$

B16 $\mathcal{B} = \left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}3\\9\\6\end{bmatrix},\begin{bmatrix}4\\1\\2\end{bmatrix},\begin{bmatrix}-5\\7\\2\end{bmatrix},\begin{bmatrix}5\\4\\4\end{bmatrix}\right\}$

B17 $\mathcal{B} = \left\{\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}9\\-1\\-4\end{bmatrix},\begin{bmatrix}-8\\2\\4\end{bmatrix}\right\}$

For Problems B18–B20, determine the dimension of the subspace of $M_{2\times2}(\mathbb{R})$ spanned by $\mathcal{B}$.

B18 $\mathcal{B} = \left\{\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix},\begin{bmatrix}2 & 4\\1 & 2\end{bmatrix},\begin{bmatrix}3 & 1\\2 & 1\end{bmatrix},\begin{bmatrix}-1 & 5\\-6 & 3\end{bmatrix}\right\}$

B19 $\mathcal{B} = \left\{\begin{bmatrix}3 & 2\\1 & 2\end{bmatrix},\begin{bmatrix}4 & 1\\1 & 1\end{bmatrix},\begin{bmatrix}-6 & 2\\-2 & 3\end{bmatrix},\begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix}\right\}$

B20 $\mathcal{B} = \left\{\begin{bmatrix}5 & 3\\3 & 4\end{bmatrix},\begin{bmatrix}4 & 6\\2 & 3\end{bmatrix},\begin{bmatrix}-7 & 3\\-5 & -6\end{bmatrix},\begin{bmatrix}3 & 9\\1 & 2\end{bmatrix}\right\}$

For Problems B21–B23, determine the dimension of the subspace of $P_2(\mathbb{R})$ spanned by $\mathcal{B}$.

B21 $\mathcal{B} = \{1, 1+2x, 1+4x+4x^2\}$

B22 $\mathcal{B} = \{1+2x+5x^2, -2+3x-8x^2, 1+9x+7x^2, 4+x\}$

B23 $\mathcal{B} = \{2+x^2, 1+x+x^2, 8+2x+5x^2, -2+4x+x^2\}$

For Problems B24–B27:

(a) Determine a basis $\mathcal{B}$ for the given plane in $\mathbb{R}^3$.

(b) Extend $\mathcal{B}$ to obtain a basis for $\mathbb{R}^3$.

B24 $x_1 - 2x_2 + x_3 = 0$

B25 $3x_1 + 5x_2 - x_3 = 0$

B26 $2x_1 + 4x_2 + 3x_3 = 0$

B27 $3x_1 - 2x_3 = 0$

For Problems B28 and B29:

(a) Determine a basis $\mathcal{B}$ for the given hyperplane in $\mathbb{R}^4$.

(b) Extend $\mathcal{B}$ to obtain a basis for $\mathbb{R}^4$.

B28 $x_1 + 3x_2 - x_3 + 2x_4 = 0$

B29 $-x_1 + 2x_2 + 3x_3 + x_4 = 0$

B30 Find a basis for $P_3(\mathbb{R})$ that includes the vectors $\vec{v}_1 = 1 - x^2 + x^3$ and $\vec{v}_2 = 1 + x + 2x^2 + x^3$.

B31 Find a basis for $M_{2\times2}(\mathbb{R})$ that includes the vectors $\vec{v}_1 = \begin{bmatrix}1 & -1\\3 & 2\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}2 & -3\\3 & 1\end{bmatrix}$.

For Problems B32–B37, obtain a basis for the vector space and determine its dimension.

B32 $\mathbb{S} = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3 \mid 3x_2 = 5x_3 \text{ and } x_1 = -2x_3\right\}$

B33 $\mathbb{S} = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid a + b = c\}$

B34 $\mathbb{S} = \{a + bx + cx^2 + dx^3 \in P_3(\mathbb{R}) \mid a + b = d\}$

B35 $\mathbb{S} = \left\{\begin{bmatrix}a & b\\0 & c\end{bmatrix} \in M_{2\times2}(\mathbb{R}) \mid a - 2c = 3b\right\}$

B36 $\mathbb{S} = \{\mathbf{p} \in P_2(\mathbb{R}) \mid \mathbf{p}(0) = 0 \text{ and } \mathbf{p}(1) = 0\}$

B37 $\mathbb{S} = \left\{\begin{bmatrix}a & b\\c & d\end{bmatrix} \in M_{2\times2}(\mathbb{R}) \mid a + 2b = c - d\right\}$

Conceptual Problems

C1 Let $\mathbb{V}$ be an n-dimensional vector space. Prove that if $\mathbb{S}$ is a subspace of $\mathbb{V}$ and $\dim \mathbb{S} = n$, then $\mathbb{S} = \mathbb{V}$.

C2 Show that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for a vector space $\mathbb{V}$, then for any real number t, $\{\mathbf{v}_1, \mathbf{v}_2 + t\mathbf{v}_1\}$ is also a basis for $\mathbb{V}$.

C3 Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for a vector space $\mathbb{V}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + t\mathbf{v}_1 + s\mathbf{v}_2\}$ is also a basis for $\mathbb{V}$ for any $s, t \in \mathbb{R}$.

For Problems C4–C11, prove or disprove the statement.

C4 If $\mathbb{V}$ is an n-dimensional vector space and $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set in $\mathbb{V}$, then $k \leq n$.

C5 Every basis for $P_2(\mathbb{R})$ has exactly two vectors in it.

C6 If $\{\vec{v}_1, \vec{v}_2\}$ is a basis for a 2-dimensional vector space $\mathbb{V}$, then $\{a\vec{v}_1 + b\vec{v}_2, c\vec{v}_1 + d\vec{v}_2\}$ is also a basis for $\mathbb{V}$ for any non-zero real numbers a, b, c, d.

C7 If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ spans a vector space $\mathbb{V}$, then some subset of $\mathcal{B}$ is a basis for $\mathbb{V}$.

C8 If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set in a vector space $\mathbb{V}$, then $\dim \mathbb{V} \geq k$.

C9 Every set of four non-zero matrices in $M_{2\times 2}(\mathbb{R})$ is a basis for $M_{2\times 2}(\mathbb{R})$.

C10 If $\mathbb{U}$ is a subspace of a finite dimensional vector space $\mathbb{V}$ and $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{V}$, then some subset of $\mathcal{B}$ is a basis for $\mathbb{U}$.

C11 If $\mathcal{B} = \{\vec{x}, \vec{y}, \vec{z}\}$ is a basis for a vector space $\mathbb{V}$, then $C = \{\vec{x} - \vec{y}, \vec{x} + \vec{y}, 2\vec{z}\}$ is also a basis for $\mathbb{V}$.

C12 Prove that if $\mathbb{S}$ and $\mathbb{T}$ are both 3-dimensional subspaces of a 5-dimensional vector space $\mathbb{V}$, then $\mathbb{S} \cap \mathbb{T} \neq \{\vec{0}\}$.

C13 In Section 4.2 Problem C5 you proved that

$$\mathbb{V} = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$$

with addition and scalar multiplication defined by

$$\begin{aligned} (a, b) \oplus (c, d) &= (ad + bc, bd) \\ t \odot (a, b) &= (tab^{t-1}, b^t) \quad \text{for all } t \in \mathbb{R} \end{aligned}$$

is a vector space. Find, with justification, a basis for $\mathbb{V}$ and hence determine the dimension of $\mathbb{V}$.

C14 In Section 4.2 Problem C6 you proved that $\mathbb{V} = \{x \in \mathbb{R} \mid x > 0\}$ with addition and scalar multiplication defined by

$$\begin{aligned} x \oplus y &= xy \\ t \odot x &= x^t \quad \text{for all } t \in \mathbb{R} \end{aligned}$$

is a vector space. Find, with justification, a basis for $\mathbb{V}$ and hence determine the dimension of $\mathbb{V}$.

C15 In Section 4.2 Problem C7 you proved that $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ with addition and scalar multiplication defined by

$$\begin{aligned} (a + bi) + (c + di) &= (a + c) + (b + d)i \\ t(a + bi) &= (ta) + (tb)i, \quad \text{for all } t \in \mathbb{R} \end{aligned}$$

is a vector space. Find, with justification, a basis for $\mathbb{V}$ and hence determine the dimension of $\mathbb{V}$.

C16 Consider the vector space $\mathbb{V} = \{(a, 1 + a) \mid a \in \mathbb{R}\}$ with addition and scalar multiplication defined by

$$\begin{aligned} (a, 1 + a) \oplus (b, 1 + b) &= (a + b, 1 + a + b) \\ k \odot (a, 1 + a) &= (ka, 1 + ka), \quad k \in \mathbb{R} \end{aligned}$$

Find, with justification, a basis for $\mathbb{V}$ and hence determine the dimension of $\mathbb{V}$.

4.4 Coordinates

Recall that when we write $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ *in* $\mathbb{R}^n$ *we really mean* $\vec{x} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$ *where* $\{\vec{e}_1, \ldots, \vec{e}_n\}$ *is the standard basis for* $\mathbb{R}^n$*. For example, when you originally learned to plot the point* $(1, 2)$ *in the xy-plane, you were taught that this means you move 1 in the x-direction* ($\vec{e}_1$) *and* 2 *in the y-direction* ($\vec{e}_2$)*. We can extend this idea to bases for general vector spaces.*

Definition
Coordinates
Coordinate Vector

Suppose that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for the vector space $\mathbb{V}$. If $\mathbf{x} \in \mathbb{V}$ with $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, then $c_1, c_2, \ldots, c_n$ are called the **coordinates** of $\mathbf{x}$ with respect to $\mathcal{B}$ (or the $\mathcal{B}$**-coordinates**) and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector** of $\mathbf{x}$ with respect to the basis $\mathcal{B}$ (or the $\mathcal{B}$**-coordinate vector**).

Remarks

1. This definition makes sense because of the Unique Representation Theorem.
2. It is important to observe that $[\mathbf{x}]_{\mathcal{B}}$ is a vector in $\mathbb{R}^n$.
3. Observe that the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ depends on the order in which the basis vectors appear. In this book, "basis" always means **ordered basis**; that is, it is always assumed that a basis is specified in the order in which the basis vectors are listed.

EXAMPLE 4.4.1

Find the coordinate vector of $\mathbf{p}(x) = 4 + x$ with respect to the basis $\mathcal{B} = \{-1 + 2x, 1 + x\}$ of $P_1(\mathbb{R})$.

Solution: By definition, we need to write $\mathbf{p}$ as a linear combination of the vectors in $\mathcal{B}$. We consider

$$4 + x = c_1(-1 + 2x) + c_2(1 + x) = (-c_1 + c_2) + (2c_1 + c_2)x$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|c} -1 & 1 & 4 \\ 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right]$$

Thus,

$$4 + x = (-1)(-1 + 2x) + 3(1 + x)$$

Hence,

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Compare Example 4.4.1 to Example 1.2.10 on page 25. The similarity is not surprising upon realizing that the coordinate vectors of $4 + x$, $-1 + 2x$, and $1 + x$ with respect to the standard basis $\{1, x\}$ for $P_1(\mathbb{R})$ are $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. In fact, we can draw the same diagram where the x_1- and x_2-axes are now in terms of the standard basis $\{1, x\}$ for $P_1(\mathbb{R})$.

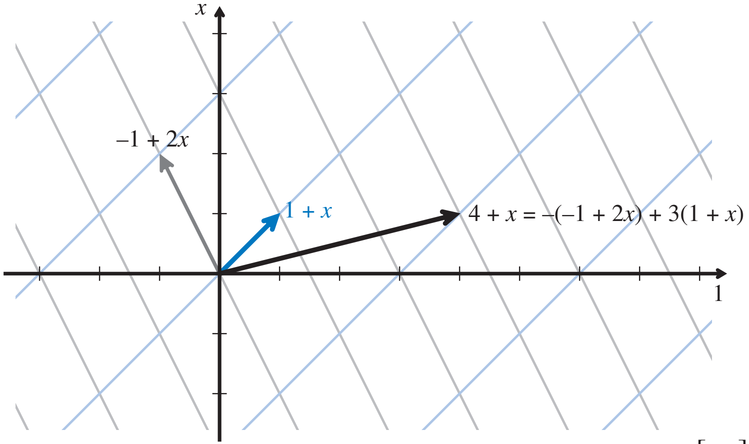


Figure 4.4.1 The basis $\mathcal{B} = \{-1 + 2x, 1 + x\}$ in $P_1(\mathbb{R})$; $[4 + x]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

In particular, finding the coordinate vector of a vector $\vec{v}$ with respect to a basis $\mathcal{B}$ of an n-dimensional vector space $\mathbb{V}$ is a way of transforming the vector $\vec{v}$ into a vector in $\mathbb{R}^n$. Therefore, no matter how complicated a vector space looks, as long as we have a basis for the vector space, we can convert its vectors into vectors in $\mathbb{R}^n$. We will look more at this in Section 4.7.

EXAMPLE 4.4.2

Given that $\mathcal{B} = \left\{ \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for the subspace Span $\mathcal{B}$ of $M_{2\times 2}(\mathbb{R})$, find the coordinate vector of $A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$.

Solution: We are required to determine whether there are numbers c_1, c_2, c_3 such that

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} = c_1 \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 + c_3 & 2c_1 + c_3 \\ 2c_1 + c_2 + c_3 & 2c_1 + c_2 \end{bmatrix}$$

Row reducing the augmented matrix gives

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 2 & 0 & 1 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus,

$$[A]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

Note that there are only three $\mathcal{B}$-coordinates because the basis $\mathcal{B}$ has only three vectors.

EXERCISE 4.4.1

Find the coordinate vector of $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \right\}$ of Span $\mathcal{B}$.

EXAMPLE 4.4.3

Suppose that you have written a computer program to perform certain operations with polynomials in $P_2(\mathbb{R})$. You need to include some method of inputting the polynomial you are considering. If you use the standard basis $\{1, x, x^2\}$ for $P_2(\mathbb{R})$ to input the polynomial $3 - 5x + 2x^2$, you would surely write your program in such a way that you would type "3, −5, 2," the standard coordinates, as the input.

On the other hand, for some problems in differential equations, you might prefer the basis $\mathcal{B} = \{1 - x^2, x, 1 + x^2\}$. To find the $\mathcal{B}$-coordinates of $3 - 5x + 2x^2$, we must find t_1, t_2 and t_3 such that

$$3 - 5x + 2x^2 = t_1(1 - x^2) + t_2x + t_3(1 + x^2) = (t_1 + t_3) + t_2x + (t_3 - t_1)x^2$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -5 \\ -1 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 5/2 \end{array}\right]$$

It follows that the coordinates of $3 - 5x + 2x^2$ with respect to $\mathcal{B}$ are $1/2, -5, 5/2$. Thus, if your computer program is written to work in the basis $\mathcal{B}$, then you would input "0.5, −5, 2.5."

EXERCISE 4.4.2

Prove that if $\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$ is the standard basis for $\mathbb{R}^n$, then $[\vec{x}]_{\mathcal{S}} = \vec{x}$ for any $\vec{x} \in \mathbb{R}^n$.

Change of Coordinates

We might need to input several polynomials into the computer program written to work in the basis $\mathcal{B}$. In this case, we would want a much faster way of converting standard coordinates to coordinates with respect to the basis $\mathcal{B}$. We now develop a method for doing this. We will require the following theorem.

Theorem 4.4.1

If $\mathcal{B}$ is a basis for a finite dimensional vector space $\mathbb{V}$, then for any $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $s, t \in \mathbb{R}$ we have

$$[s\mathbf{x} + t\mathbf{y}]_{\mathcal{B}} = s[\mathbf{x}]_{\mathcal{B}} + t[\mathbf{y}]_{\mathcal{B}}$$

Proof: Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, $\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$, and $\mathbf{y} = y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n$ in $\mathbb{V}$. Then, we have

$$s\mathbf{x} + t\mathbf{y} = (sx_1 + ty_1)\mathbf{v}_1 + \cdots + (sx_n + ty_n)\mathbf{v}_n$$

Thus,

$$[s\mathbf{x} + t\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} sx_1 + ty_1 \\ \vdots \\ sx_n + ty_n \end{bmatrix} = s\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + t\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = s[\mathbf{x}]_{\mathcal{B}} + t[\mathbf{y}]_{\mathcal{B}}$$

■

Let $\mathcal{B}$ be a basis for an n-dimensional vector space $\mathbb{V}$ and let $C = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be another basis for $\mathbb{V}$. Consider $\mathbf{x} \in \mathbb{V}$. Writing $\mathbf{x}$ as a linear combination of the vectors in C gives

$$\mathbf{x} = x_1\mathbf{w}_1 + \cdots + x_n\mathbf{w}_n$$

Taking $\mathcal{B}$-coordinates gives

$$\begin{aligned}
[\mathbf{x}]_B &= [x_1\mathbf{w}_1 + \cdots + x_n\mathbf{w}_n]_{\mathcal{B}} \\
&= x_1[\mathbf{w}_1]_{\mathcal{B}} + \cdots + x_n[\mathbf{w}_n]_{\mathcal{B}} && \text{by Theorem 4.4.1} \\
&= \begin{bmatrix} [\mathbf{w}_1]_{\mathcal{B}} & \cdots & [\mathbf{w}_n]_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} && \text{by definition of matrix-vector multiplication}
\end{aligned}$$

Since $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\mathbf{x}]_C$, we see that this equation gives a formula for calculating the $\mathcal{B}$-coordinates of $\mathbf{x}$ from the C-coordinates of $\mathbf{x}$ simply using matrix-vector multiplication. We call this equation the **change of coordinates equation** and make the following definition.

Definition
Change of Coordinates Matrix

Let $\mathcal{B}$ and $C = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ both be bases for a vector space $\mathbb{V}$. The matrix

$$P = \begin{bmatrix} [\mathbf{w}_1]_{\mathcal{B}} & \cdots & [\mathbf{w}_n]_{\mathcal{B}} \end{bmatrix}$$

is called the **change of coordinates matrix** from C-coordinates to $\mathcal{B}$-coordinates and satisfies

$$[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_C$$

Of course, we could exchange the roles of $\mathcal{B}$ and C to find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to C-coordinates.

Theorem 4.4.2

Let $\mathcal{B}$ and C both be bases for a finite-dimensional vector space $\mathbb{V}$. If P is the change of coordinates matrix from C-coordinates to $\mathcal{B}$-coordinates, then P is invertible and P^{-1} is the change of coordinates matrix from $\mathcal{B}$-coordinates to C-coordinates.

The proof of Theorem 4.4.2 is left to Problem C4.

EXAMPLE 4.4.4

Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$ and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}$. Find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates. Find the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates. Verify that $PQ = I$.

Solution: To find the change of coordinates matrix Q, we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to the standard basis $\mathcal{S}$. We get

$$Q = \begin{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix}$$

To find the change of coordinates matrix P, we need to find the coordinates of the standard basis vectors with respect to the basis $\mathcal{B}$. To do this, we need to solve the systems

$$a_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$b_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$c_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This gives three systems of linear equations with the same coefficient matrix. To make this easier, we row reduce the corresponding triple-augmented matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/5 & -1/5 & -1 \\ 0 & 1 & 0 & 7/5 & -4/5 & -1 \\ 0 & 0 & 1 & -4/5 & 3/5 & 1 \end{array}\right]$$

Thus,

$$P = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 & -1 \\ 7/5 & -4/5 & -1 \\ -4/5 & 3/5 & 1 \end{bmatrix}$$

We now see that

$$\begin{bmatrix} 3/5 & -1/5 & -1 \\ 7/5 & -4/5 & -1 \\ -4/5 & 3/5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 4.4.5

Find the change of coordinates matrix P from the standard basis $\mathcal{S} = \{1, x, x^2\}$ to the basis $\mathcal{B} = \{1 - x^2, x, 1 + x^2\}$ in $P_2(\mathbb{R})$. Use P to find $[a + bx + cx^2]_{\mathcal{B}}$.

Solution: We need to find the coordinates of the standard basis vectors with respect to the basis $\mathcal{B}$. To do this, we solve the three systems of linear equations given by

$$\begin{aligned} t_{11}(1 - x^2) + t_{12}(x) + t_{13}(1 + x^2) &= 1 \\ t_{21}(1 - x^2) + t_{22}(x) + t_{23}(1 + x^2) &= x \\ t_{31}(1 - x^2) + t_{32}(x) + t_{33}(1 + x^2) &= x^2 \end{aligned}$$

We row reduce the triple-augmented matrix to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right]$$

Hence,

$$P = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

To calculate $[a + bx + cx^2]_{\mathcal{B}}$, we first observe that

$$[a + bx + cx^2]_{\mathcal{S}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Thus,

$$\begin{aligned} [a + bx + cx^2]_{\mathcal{B}} &= P[a + bx + cx^2]_{\mathcal{S}} \\ &= \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}a - \frac{1}{2}c \\ b \\ \frac{1}{2}a + \frac{1}{2}c \end{bmatrix} \end{aligned}$$

EXERCISE 4.4.3

Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$ and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$. Find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates. Find the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates. Verify that $PQ = I$.

PROBLEMS 4.4

Practice Problems

A1 (a) Verify that $\mathcal{B} = \left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\-1\end{bmatrix}\right\}$ is a basis for the plane with equation $2x_1 - x_2 - 2x_3 = 0$.

(b) Determine whether $\vec{x}$ lies in the plane of part (a). If it does, find $[\vec{x}]_{\mathcal{B}}$.

(i) $\vec{x} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ (ii) $\vec{x} = \begin{bmatrix}3\\2\\2\end{bmatrix}$ (iii) $\vec{x} = \begin{bmatrix}5\\2\\3\end{bmatrix}$

A2 Consider the basis $\mathcal{B} = \{1+x^2, 1-x+2x^2, -1-x+x^2\}$ for $P_2(\mathbb{R})$.

(a) Find the $\mathcal{B}$-coordinates of each polynomials.

(i) $\mathbf{q}(x) = 4-2x+7x^2$ (ii) $\mathbf{r}(x) = -2-2x+3x^2$

(b) Determine $[2-4x+10x^2]_{\mathcal{B}}$ and use your answers to part (a) to check that

$$\begin{aligned}&[4-2x+7x^2]_{\mathcal{B}} + [-2-2x+3x^2]_{\mathcal{B}}\\ &= [(4-2)+(-2-2)x+(7+3)x^2]_{\mathcal{B}}\end{aligned}$$

For Problems A3–A13, determine the coordinates of the vectors with respect to the given basis $\mathcal{B}$.

A3 $\mathcal{B} = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\-5\end{bmatrix}\right\}$, $\mathbf{x} = \begin{bmatrix}5\\-7\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}-7\\8\end{bmatrix}$

A4 $\mathcal{B} = \left\{\begin{bmatrix}2\\-5\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right\}$, $\mathbf{x} = \begin{bmatrix}5\\-7\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}9\\-5\end{bmatrix}$

A5 $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$, $\mathbf{x} = \begin{bmatrix}8\\-7\\3\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}3\\-5\\2\end{bmatrix}$

A6 $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}\right\}$, $\mathbf{x} = \begin{bmatrix}5\\-2\\5\\3\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}-1\\3\\-1\\2\end{bmatrix}$

A7 $\mathcal{B} = \{1+3x, 2-5x\}$, $\mathbf{p}(x) = 5-7x$, $\mathbf{q}(x) = 1$

A8 $\mathcal{B} = \{1+x+x^2, 1+3x+2x^2, 4+x^2\}$, $\mathbf{p}(x) = -2+8x+5x^2$, $\mathbf{q}(x) = -4+8x+4x^2$

A9 $\mathcal{B} = \{1+x^2, 1+x+2x^2+x^3, x-x^2+x^3\}$, $\mathbf{p}(x) = 2+x-5x^2+x^3$, $\mathbf{q}(x) = 1+x+4x^2+x^3$

A10 $\mathcal{B} = \left\{\begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}2&0\\0&-1\end{bmatrix}\right\}$, $A = \begin{bmatrix}0&1\\1&2\end{bmatrix}$, $B = \begin{bmatrix}-4&1\\1&4\end{bmatrix}$

A11 $\mathcal{B} = \left\{\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}, \begin{bmatrix}0&2&-1\\1&3&-1\end{bmatrix}\right\}$, $A = \begin{bmatrix}1&3&-1\\1&4&0\end{bmatrix}$, $B = \begin{bmatrix}3&-1&2\\-2&-3&5\end{bmatrix}$

A12 $\mathcal{B} = \left\{\begin{bmatrix}1&2\\1&3\end{bmatrix}, \begin{bmatrix}2&1\\-1&2\end{bmatrix}, \begin{bmatrix}-2&2\\4&10\end{bmatrix}\right\}$, $A = \begin{bmatrix}-1&1\\2&7\end{bmatrix}$, $B = \begin{bmatrix}6&3\\-3&2\end{bmatrix}$

A13 $\mathcal{B} = \left\{\begin{bmatrix}1&3\\2&3\end{bmatrix}, \begin{bmatrix}1&-2\\1&2\end{bmatrix}, \begin{bmatrix}0&1\\-1&1\end{bmatrix}\right\}$, $A = \begin{bmatrix}3&-5\\8&3\end{bmatrix}$, $B = \begin{bmatrix}0&3\\3&-1\end{bmatrix}$

For Problems A14–A20, find the change of coordinates matrix to and from the basis $\mathcal{B}$ and the standard basis of the given vector space.

A14 $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}$ for $\mathbb{R}^2$.

A15 $\mathcal{B} = \left\{\begin{bmatrix}3\\4\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-3\\3\end{bmatrix}\right\}$ for $\mathbb{R}^3$

A16 $\mathcal{B} = \{1, 1-2x, 1-4x+4x^2\}$ for $P_2(\mathbb{R})$

A17 $\mathcal{B} = \{1+2x+x^2, x+x^2, 1+3x\}$ for $P_2(\mathbb{R})$

A18 $\mathcal{B} = \{1-2x+5x^2, 1-2x^2, x+x^2\}$ for $P_2(\mathbb{R})$

A19 $\mathcal{B} = \{x^2, x^3, x, 1\}$ for $P_3(\mathbb{R})$

A20 $\mathcal{B} = \left\{\begin{bmatrix}1&-1\\0&-1\end{bmatrix}, \begin{bmatrix}0&-4\\0&-1\end{bmatrix}, \begin{bmatrix}2&1\\0&1\end{bmatrix}\right\}$ for the subspace of $M_{2\times 2}(\mathbb{R})$ of upper-triangular matrices

For Problems A21–A23, find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates and the change of coordinates matrix P from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates. Verify that $PQ = I$.

A21 In $\mathbb{R}^2$: $\mathcal{B} = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}5\\3\end{bmatrix}\right\}$, $\mathcal{C} = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}\right\}$

A22 In $\mathbb{R}^2$: $\mathcal{B} = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$, $\mathcal{C} = \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}5\\-4\end{bmatrix}\right\}$

A23 In $P_2(\mathbb{R})$: $\mathcal{B} = \{1, -1+x, (-1+x)^2\}$, $\mathcal{C} = \{1+x+x^2, 1+3x-x^2, 1-x-x^2\}$

Homework Problems

B1 (a) Verify that $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\5\end{bmatrix} \right\}$ is a basis for the plane with equation $3x_1 + 2x_2 - x_3 = 0$.

(b) For each of the following vectors, determine whether it lies in the plane of part (a). If it does, find the vector's $\mathcal{B}$-coordinates.

(i) $\vec{x}_1 = \begin{bmatrix}1\\3\\9\end{bmatrix}$ (ii) $\vec{x}_2 = \begin{bmatrix}1\\2\\4\end{bmatrix}$ (iii) $\vec{x}_3 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

B2 (a) Verify that $\mathcal{B} = \left\{ \begin{bmatrix}0\\3\\2\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix} \right\}$ is a basis for the plane with equation $x_1 - 4x_2 + 6x_3 = 0$.

(b) For each of the following vectors, determine whether it lies in the plane of part (a). If it does, find the vector's $\mathcal{B}$-coordinates.

(i) $\vec{x}_1 = \begin{bmatrix}2\\8\\5\end{bmatrix}$ (ii) $\vec{x}_2 = \begin{bmatrix}-4\\11\\8\end{bmatrix}$ (iii) $\vec{x}_3 = \begin{bmatrix}-2\\0\\1/3\end{bmatrix}$

B3 Consider the basis $\mathcal{B} = \{1+2x+x^2, 1+x+2x^2, x-2x^2\}$ for $P_2(\mathbb{R})$.

(a) Determine the $\mathcal{B}$-coordinates of the following polynomials.

(i) $\mathbf{p}(x) = 1$

(ii) $\mathbf{q}(x) = 4 - 2x + 7x^2$

(iii) $\mathbf{r}(x) = -2 - 2x + 3x^2$

(b) Determine $[2 - 4x + 10x^2]_{\mathcal{B}}$ and use your answers to part (a) to check that

$$[4 - 2x + 7x^2]_{\mathcal{B}} + [-2 - 2x + 3x^2]_{\mathcal{B}} = [(4-2) + (-2-2)x + (7+3)x^2]_{\mathcal{B}}$$

For Problems B4–B16, determine the coordinates of the vectors with respect to the given basis $\mathcal{B}$.

B4 $\mathcal{B} = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\3\\1\end{bmatrix} \right\}$, $\mathbf{x} = \begin{bmatrix}1\\7\\3\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}-8\\9\\2\end{bmatrix}$

B5 $\mathcal{B} = \left\{ \begin{bmatrix}3\\2\\2\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}, \begin{bmatrix}5\\5\\2\end{bmatrix} \right\}$, $\mathbf{x} = \begin{bmatrix}2\\0\\6\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}3\\0\\18\end{bmatrix}$

B6 $\mathcal{B} = \{1 + 2x^2, 2 + x + 3x^2, 4 - x - 2x^2\}$, $\mathbf{p}(x) = 7 - 5x - 3x^2$, $\mathbf{q}(x) = 7 - 3x - 5x^2$

B7 $\mathcal{B} = \{1 - x - x^2, 2 - x + 5x^2, 1 + x + 5x^2\}$, $\mathbf{p}(x) = 4 - 3x + 7x^2$, $\mathbf{q}(x) = 4 + 3x - 3x^2$

B8 $\mathcal{B} = \left\{ \begin{bmatrix}0&1\\2&1\end{bmatrix}, \begin{bmatrix}1&0\\3&1\end{bmatrix}, \begin{bmatrix}2&1\\2&1\end{bmatrix} \right\}$, $A = \begin{bmatrix}7&-7\\-5&-4\end{bmatrix}$, $B = \begin{bmatrix}-1&6\\3&3\end{bmatrix}$

B9 $\mathcal{B} = \left\{ \begin{bmatrix}1&0\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix} \right\}$, $A = \begin{bmatrix}5&4\\2&5\end{bmatrix}$, $B = \begin{bmatrix}3&2\\2&6\end{bmatrix}$

B10 $\mathcal{B} = \left\{ \begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}, \begin{bmatrix}0&2&-1\\1&3&-1\end{bmatrix} \right\}$, $A = \begin{bmatrix}-4&2&-3\\3&5&-7\end{bmatrix}$, $B = \begin{bmatrix}3&5&-1\\1&6&2\end{bmatrix}$

B11 $\mathcal{B} = \left\{ \begin{bmatrix}1&0\\2&3\\1&4\end{bmatrix}, \begin{bmatrix}0&1\\1&4\\2&3\end{bmatrix}, \begin{bmatrix}0&0\\1&1\\1&1\end{bmatrix} \right\}$, $A = \begin{bmatrix}5&-4\\5&-2\\-4&7\end{bmatrix}$, $B = \begin{bmatrix}3&-5\\7&-5\\-1&3\end{bmatrix}$

B12 $\mathcal{B} = \{1 + x^2 + x^3, 3 + 2x + x^3, 2x + x^2\}$, $\mathbf{p}(x) = 3 + 2x + 2x^2 + 2x^3$, $\mathbf{q}(x) = 7 + 8x + x^2 + 2x^3$

B13 $\mathcal{B} = \left\{ \begin{bmatrix}2&3\\2&3\end{bmatrix}, \begin{bmatrix}4&1\\-1&-2\end{bmatrix} \right\}$ $A = \begin{bmatrix}14&6\\-1&-3\end{bmatrix}$, $B = \begin{bmatrix}2&-7\\-8&-13\end{bmatrix}$

B14 $\mathcal{B} = \left\{ \begin{bmatrix}1&0\\2&3\end{bmatrix}, \begin{bmatrix}1&1\\4&3\end{bmatrix}, \begin{bmatrix}3&1\\2&1\end{bmatrix} \right\}$ $A = \begin{bmatrix}-6&-1\\4&6\end{bmatrix}$, $B = \begin{bmatrix}1&1\\1&-1\end{bmatrix}$

B15 $\mathcal{B} = \left\{ \begin{bmatrix}3&5\\1&4\end{bmatrix}, \begin{bmatrix}7&16\\3&9\end{bmatrix}, \begin{bmatrix}-1&5\\1&2\end{bmatrix} \right\}$, $A = \begin{bmatrix}-4&9\\2&5\end{bmatrix}$, $B = \begin{bmatrix}0&9\\2&7\end{bmatrix}$

B16 $\mathcal{B} = \left\{ \begin{bmatrix}2&2\\4&2\end{bmatrix}, \begin{bmatrix}1&3\\1&2\end{bmatrix}, \begin{bmatrix}4&2\\3&4\end{bmatrix} \right\}$, $A = \begin{bmatrix}-4&6\\5&-2\end{bmatrix}$, $B = \begin{bmatrix}7&9\\7&9\end{bmatrix}$

For Problems B17–B22, find the change of coordinates matrix to and from the basis $\mathcal{B}$ and the standard basis of the given vector space.

B17 $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\}$ for $\mathbb{R}^2$

B18 $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

B19 $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$

B20 $\mathcal{B} = \{1, -2 + x, 4 - 4x + x^2\}$ for $P_2(\mathbb{R})$

B21 $\mathcal{B} = \{1 + x + x^2, 1 - 2x^2, 2 + 2x + x^2\}$ for $P_2(\mathbb{R})$

B22 $\mathcal{B} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 0 \\ 0 & 5 \end{bmatrix} \right\}$ for the subspace of $M_{2\times 2}(\mathbb{R})$ of diagonal matrices

For Problems B23–B25, find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to C-coordinates and the change of coordinates matrix P from C-coordinates to $\mathcal{B}$-coordinates. Verify that $PQ = I$.

B23 In $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \end{bmatrix} \right\}$

B24 In $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

B25 In $P_2(\mathbb{R})$: $\mathcal{B} = \{1, 3 + 2x, (3 + 2x)^2\}$,
$C = \{1 - 2x - 4x^2, 2 + 2x, -2x - 4x^2\}$

Conceptual Problems

C1 Suppose that $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for a vector space $\mathbb{V}$ and that $C = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ is another basis for $\mathbb{V}$ and that for every $\mathbf{x} \in \mathbb{V}$, $[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_C$. Must it be true that $\mathbf{v}_i = \mathbf{w}_i$ for each $1 \le i \le k$? Explain or prove your conclusion.

C2 Suppose $\mathbb{V}$ is a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Then $C = \{\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_1\}$ is also a basis of $\mathbb{V}$. Find a matrix P such that $P[\vec{x}]_{\mathcal{B}} = [\vec{x}]_C$.

C3 Let $B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping such that

$$[\vec{x}]_B = [L(\vec{x})]_C$$

(a) Find $L\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right)$.

(b) Find $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

C4 Let $\mathbb{V}$ be a finite dimensional vector space, and let $\mathcal{B}$ and C both be bases for $\mathbb{V}$. Prove if P is the change of coordinates matrix from C-coordinates to $\mathcal{B}$-coordinates, then P is invertible and P^{-1} is the change of coordinates matrix from $\mathcal{B}$-coordinates to C-coordinates.

C5 If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{V}$, show that $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_n]_{\mathcal{B}}\}$ is a basis for $\mathbb{R}^n$.

For Problems C6–C9, let $\mathbb{V}$ be an n-dimensional vector space and let $\mathcal{B}$ be a basis for $\mathbb{V}$. Determine whether the statement is true or false. Justify your answer.

C6 $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ for any $\mathbf{v} \in \mathbb{V}$.

C7 If C is another basis for $\mathbb{V}$, then $[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_C$ for any $\mathbf{v} \in \mathbb{V}$.

C8 If $\mathbf{v}, \mathbf{w} \in \mathbb{V}$ such that $[\mathbf{v}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$, then $\mathbf{v} = \mathbf{w}$.

C9 If $\vec{x} \in \mathbb{R}^n$, then there exists $\mathbf{v} \in \mathbb{V}$ such that $[\mathbf{v}]_{\mathcal{B}} = \vec{x}$.

C10 Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$.

(a) Find the change of coordinates matrix P from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates.

(b) Show that $[\vec{x}]_{\mathcal{B}} = P^{-1}\vec{x}$.

(c) Let A be an $n \times n$ matrix and define

$$B = \begin{bmatrix} [A\vec{v}_1]_{\mathcal{B}} & \cdots & [A\vec{v}_n]_{\mathcal{B}} \end{bmatrix}$$

Prove that $B = P^{-1}AP$.

4.5 General Linear Mappings

In Chapter 3 we looked at linear mappings $\mathrm{L} : \mathbb{R}^n \to \mathbb{R}^m$ *and found that they can be useful in solving some problems. Since vector spaces encompass the essential properties of* $\mathbb{R}^n$*, it makes sense that we can also define linear mappings whose domain and codomain are other vector spaces. This also turns out to be extremely useful and important in many real-world applications.*

Definition
Linear Mapping
Linear Transformation
Linear Operator

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. A function $L : \mathbb{V} \to \mathbb{W}$ is called a **linear mapping** (or **linear transformation**) if for every $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $s, t \in \mathbb{R}$ it satisfies

$$L(s\mathbf{x} + t\mathbf{y}) = sL(\mathbf{x}) + tL(\mathbf{y})$$

If $\mathbb{W} = \mathbb{V}$, then L may be called a **linear operator**.

As before, two linear mappings $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ are said to be **equal** if

$$L(\mathbf{v}) = M(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{V}$.

EXAMPLE 4.5.1

Let $L : M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = d + (b + d)x + ax^2$. Prove that L is a linear mapping.

Solution: For any $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{2\times2}(\mathbb{R})$ and $s, t \in \mathbb{R}$, we have

$$\begin{aligned} L\left(s\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + t\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} sa_1 + ta_2 & sb_1 + tb_2 \\ sc_1 + tc_2 & sd_1 + td_2 \end{bmatrix}\right) \\ &= (sd_1 + td_2) + (sb_1 + tb_2 + sd_1 + td_2)x + (sa_1 + ta_2)x^2 \\ &= s(d_1 + (b_1 + d_1)x + a_1x^2) + t(d_2 + (b_2 + d_2)x + a_2x^2) \\ &= sL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + tL\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \end{aligned}$$

So, L is linear.

EXAMPLE 4.5.2

Let $M : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $M(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$. Prove that M is a linear operator.

Solution: Let $\mathbf{p}(x) = a_0 + a_1x + a_2x^2$, $\mathbf{q}(x) = b_0 + b_1x + b_2x^2$, and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned} M(s\mathbf{p} + t\mathbf{q}) &= M((sa_0 + tb_0) + (sa_1 + tb_1)x + (sa_2 + tb_2)x^2) \\ &= (sa_1 + tb_1) + 2(sa_2 + tb_2)x \\ &= s(a_1 + 2a_2x) + t(b_1 + 2b_2x) \\ &= sM(\mathbf{p}) + tM(\mathbf{q}) \end{aligned}$$

Hence, M is linear.

EXERCISE 4.5.1

Let $L : \mathbb{R}^3 \to M_{2\times2}(\mathbb{R})$ be defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_1 + x_2 + x_3 \\ 0 & x_2 \end{bmatrix}$.

Prove that L is linear.

EXERCISE 4.5.2

Let $\mathcal{D} : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be the differential operator defined by $\mathcal{D}(\mathbf{p}) = \mathbf{p}'$. Prove that $\mathcal{D}$ is linear.

Definition
Zero Mapping

The **zero mapping** is the linear mapping $Z : \mathbb{V} \to \mathbb{W}$ defined by

$$Z(\vec{v}) = \vec{0}_{\mathbb{W}}, \qquad \text{for all } \vec{v} \in \mathbb{V}$$

Definition
Identity Mapping

The **identity mapping** is the linear operator $\mathrm{Id} : \mathbb{V} \to \mathbb{V}$ defined by

$$\mathrm{Id}(\vec{v}) = \vec{v}, \qquad \text{for all } \vec{v} \in \mathbb{V}$$

Range and Nullspace

Definition
Range
Nullspace

The **range** of a linear mapping $L : \mathbb{V} \to \mathbb{W}$ is defined to be the set

$$\mathrm{Range}(L) = \{L(\mathbf{x}) \in \mathbb{W} \mid \mathbf{x} \in \mathbb{V}\}$$

The **nullspace** of L is the set of all vectors in $\mathbb{V}$ whose image under L is the zero vector $\mathbf{0}_{\mathbb{W}}$. We write

$$\mathrm{Null}(L) = \{\mathbf{x} \in \mathbb{V} \mid L(\mathbf{x}) = \mathbf{0}_{\mathbb{W}}\}$$

EXAMPLE 4.5.3

Let $A = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ and let $L : M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear mapping defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = c + (b + d)x + ax^2$. Determine whether A is in the nullspace of L.

Solution: We have

$$L\left(\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right) = 0 + (-1 + 1)x + 0x^2 = 0 = \mathbf{0}_{P_2(\mathbb{R})}$$

Hence, $A \in \mathrm{Null}(L)$.

EXAMPLE 4.5.4

Let $L : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear mapping defined by $L(a + bx + cx^2) = \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix}$.

Determine whether $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the range of L.

Solution: We want to find a, b, and c such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = L(a + bx + cx^2) = \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix}$$

This gives us the system of linear equations $a - b = 1$, $b - c = 1$, and $c - a = 1$. Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Hence, the system is inconsistent, so $\vec{x}$ is not in the range of L.

Theorem 4.5.1

If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then

(1) $L(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$

(2) $\text{Null}(L)$ is a subspace of $\mathbb{V}$

(3) $\text{Range}(L)$ is a subspace of $\mathbb{W}$

The proof of Theorem 4.5.1 is left as Problems C1, C2, and C3.

EXAMPLE 4.5.5

Determine a basis for the range and a basis for the nullspace of the linear mapping $L : P_1(\mathbb{R}) \to \mathbb{R}^3$ defined by $L(a + bx) = \begin{bmatrix} a \\ 0 \\ a-2b \end{bmatrix}$.

Solution: If $a + bx \in \text{Null}(L)$, then we have $\begin{bmatrix} a \\ 0 \\ a-2b \end{bmatrix} = L(a + bx) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence, $a = 0$ and $a - 2b = 0$, which implies that $b = 0$. Thus, the only polynomial in the nullspace of L is the zero polynomial. That is, $\text{Null}(L) = \{0\}$, and so a basis for $\text{Null}(L)$ is the empty set. Any vector $\vec{y}$ in the range of L has the form

$$\vec{y} = \begin{bmatrix} a \\ 0 \\ a-2b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

Thus, $\text{Range}(L) = \text{Span}\,\mathcal{C}$, where $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\right\}$. Moreover, $\mathcal{C}$ is clearly linearly independent. Consequently, $\mathcal{C}$ is a basis for the range of L.

EXAMPLE 4.5.6

Determine a basis for the range and a basis for the nullspace of the linear mapping $L : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b + c) + (c - d)x^2$.

Solution: If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Null}(L)$, then

$$0 + 0x + 0x^2 = L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b + c) + (c - d)x^2$$

So, $b + c = 0$ and $c - d = 0$. Thus, $b = -c$ and $d = c$, so every matrix in the nullspace of L has the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & -c \\ c & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right\}$ spans Null(L) and is clearly linearly independent. Consequently, $\mathcal{B}$ is a basis for Null(L).

Any polynomial $\mathbf{p} \in \text{Range}(L)$ has the form

$$\mathbf{p}(x) = L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b + c) + (c - d)x^2 = b(1) + c(1 + x^2) - dx^2$$

Hence $\{1, 1 + x^2, -x^2\}$ spans Range(L). But this is linearly dependent. Clearly, we have $1 + (-1)(-x^2) = 1 + x^2$. Thus, $\mathcal{C} = \{1, -x^2\}$ also spans Range(L) and is linearly independent. Therefore, $\mathcal{C}$ is a basis for Range(L).

EXERCISE 4.5.3

Determine a basis for the range and a basis for the nullspace of the linear mapping $L : \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_2 + x_3 \\ x_2 + x_3 & x_1 \end{bmatrix}$.

Observe that in each of these examples, the dimension of the range of L plus the dimension of the nullspace of L equals the dimension of the domain of L. This matches the Rank-Nullity Theorem (Theorem 3.4.9). Before we extend this to general linear mappings, we make some definitions.

Definition
Rank of a Linear Mapping

The **rank of a linear mapping** $L : \mathbb{V} \to \mathbb{W}$ is the dimension of the range of L:

$$\text{rank}(L) = \dim\big(\text{Range}(L)\big)$$

Definition
Nullity of a Linear Mapping

The **nullity of a linear mapping** $L : \mathbb{V} \to \mathbb{W}$ is the dimension of the nullspace of L:

$$\text{nullity}(L) = \dim\big(\text{Null}(L)\big)$$

Theorem 4.5.2 Rank-Nullity Theorem

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces with $\dim \mathbb{V} = n$. If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then

$$\text{rank}(L) + \text{nullity}(L) = n$$

Proof: The idea of the proof is to assume that a basis for the nullspace of L contains k vectors and show that we can then construct a basis for the range of L that contains $n - k$ vectors.

Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for Null(L), so that nullity$(L) = k$. By the Basis Extension Theorem, there exist vectors $\mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ such that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis for $\mathbb{V}$.

Now consider any vector $\mathbf{w}$ in the range of L. Then $\mathbf{w} = L(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{V}$. But any $\mathbf{x} \in \mathbb{V}$ can be written as a linear combination of the vectors in the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$, so there exists $t_1, \dots, t_n$ such that $\mathbf{x} = t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{u}_{k+1} + \cdots + t_n\mathbf{u}_n$. Then,

$$\begin{aligned}\mathbf{w} &= L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{u}_{k+1} + \cdots + t_n\mathbf{u}_n)\\ &= t_1L(\mathbf{v}_1) + \cdots + t_kL(\mathbf{v}_k) + t_{k+1}L(\mathbf{u}_{k+1}) + \cdots + t_nL(\mathbf{u}_n)\end{aligned}$$

But each $\mathbf{v}_i$ is in the nullspace of L, so $L(\mathbf{v}_i) = \mathbf{0}$, and thus we have

$$\mathbf{w} = t_{k+1}L(\mathbf{u}_{k+1}) + \cdots + t_nL(\mathbf{u}_n)$$

Therefore, any $\mathbf{w} \in$ Range(L) can be expressed as a linear combination of the vectors in the set $C = \{L(\mathbf{u}_{k+1}), \dots, L(\mathbf{u}_n)\}$. Thus, C is a spanning set for Range(L). Is it linearly independent? We consider

$$t_{k+1}L(\mathbf{u}_{k+1}) + \cdots + t_nL(\mathbf{u}_n) = \mathbf{0}_{\mathbb{W}}$$

By the linearity of L, this is equivalent to

$$L(t_{k+1}\mathbf{u}_{k+1} + \cdots + t_n\mathbf{u}_n) = \mathbf{0}_{\mathbb{W}}$$

If this is true, then $t_{k+1}\mathbf{u}_{k+1} + \cdots + t_n\mathbf{u}_n$ is a vector in the nullspace of L. Hence, for some $d_1, \dots, d_k$, we have

$$t_{k+1}\mathbf{u}_{k+1} + \cdots + t_n\mathbf{u}_n = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k$$

But this is impossible unless all t_i and d_i are zero, because $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis for $\mathbb{V}$ and hence linearly independent.

It follows that C is a linearly independent spanning set for Range(L). Hence, it is a basis for Range(L) containing $n - k$ vectors. Thus, rank$(L) = n - k$ and

$$\text{rank}(L) + \text{nullity}(L) = (n - k) + k = n$$

as required. ■

EXAMPLE 4.5.7

Determine the dimension of the range and the dimension of the kernel of the linear mapping $L : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$L(a + bx + cx^2) = \begin{bmatrix} a+c & b+c \\ 0 & a-b \end{bmatrix}$$

Solution: If $a + bx + cx^2 \in \operatorname{Null}(L)$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L(a + bx + cx^2) = \begin{bmatrix} a+c & b+c \\ 0 & a-b \end{bmatrix}$$

This gives the system of equations $a + c = 0$, $b + c = 0$, $a - b = 0$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

That is, $a = -c$, $b = -c$, for any $c \in \mathbb{R}$. Hence, every vector in $\operatorname{Null}(L)$ has the form

$$-c - cx + cx^2 = c(-1 - x + x^2)$$

Thus, $\{-1 - x + x^2\}$ is a basis for $\operatorname{Null}(L)$ and so $\operatorname{nullity}(L) = 1$.

Every matrix $A \in \operatorname{Range}(L)$ has the form

$$A = L(a + bx + cx^2) = \begin{bmatrix} a+c & b+c \\ 0 & a-b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ spans $\operatorname{Range}(L)$. To determine if it is linearly independent, we consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

We observe that this gives the same coefficient matrix as above. Our row reduction above shows us that the third matrix is the sum of the first two. Hence,

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Range}(L)$ as it is linearly independent and spans $\operatorname{Range}(L)$. Therefore, $\operatorname{rank}(L) = 2$.

Then, as predicted by the Rank-Nullity Theorem, we have

$$\operatorname{rank}(L) + \operatorname{nullity}(L) = 2 + 1 = 3 = \dim P_2(\mathbb{R})$$

Inverse Linear Mappings

Definition
Composition of Linear Mappings

Let $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$ be vector spaces. If $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ are linear mappings, then the **composition** $M \circ L : \mathbb{V} \to \mathbb{U}$ is defined by

$$(M \circ L)(\mathbf{x}) = M(L(\mathbf{x}))$$

for all $\mathbf{x} \in \mathbb{V}$.

EXAMPLE 4.5.8

If $L : P_2(\mathbb{R}) \to \mathbb{R}^3$ is defined by $L(a + bx + cx^2) = \begin{bmatrix} a+b \\ c \\ 0 \end{bmatrix}$ and $M : \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ is defined by $M\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 & 0 \\ x_2 + x_3 & 0 \end{bmatrix}$, then $M \circ L$ is the mapping defined by

$$(M \circ L)(a + bx + cx^2) = M(L(a + bx + cx^2)) = M\left(\begin{bmatrix} a+b \\ c \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a+b & 0 \\ c & 0 \end{bmatrix}$$

Observe that $M \circ L$ is in fact a linear mapping from $P_2(\mathbb{R})$ to $M_{2\times 2}(\mathbb{R})$.

Theorem 4.5.3

If $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ are linear mappings, then $M \circ L : \mathbb{V} \to \mathbb{U}$ is also a linear mapping.

The proof of Theorem 4.5.3 is left as Problem C4.

Definition
Inverse Mapping

If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping and there exists another linear mapping $M : \mathbb{W} \to \mathbb{V}$ such that

$$M \circ L = \mathrm{Id}$$
$$L \circ M = \mathrm{Id}$$

then L is said to be **invertible**, and M is called the **inverse** of L, denoted L^{-1}.

Remarks

1. By definition, if M is the inverse of L, then L is the inverse of M.
2. Note that the domain and codomain of an invertible linear mapping need not be the same vector space. It is important to ask what condition on $\mathbb{V}$ and $\mathbb{W}$ is required so that it is possible for $L : \mathbb{V} \to \mathbb{W}$ to be invertible.

EXAMPLE 4.5.9 Let $L : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear mapping defined by

$$L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a+b+c & -a+b+c+d \\ a+c-d & a+b+d \end{bmatrix}$$

and $M : M_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear mapping defined by

$$M\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (-a+c+d)+(4a-b-3c-2d)x+(-2a+b+2c+d)x^2+(-3a+b+2c+2d)x^3$$

Prove that L and M are inverses of each other.

Solution: Observe that

$$\begin{aligned}(M \circ L)(a + bx + cx^2 + dx^3) &= M\left(\begin{bmatrix} a+b+c & -a+b+c+d \\ a+c-d & a+b+d \end{bmatrix}\right) \\ &= a + bx + cx^2 + dx^3 \\ (L \circ M)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= L((-a+c+d) + (4a-b-3c-2d)x \\ &\quad + (-2a+b+2c+d)x^2 + (-3a+b+2c+2d)x^3) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\end{aligned}$$

Thus, L and M are inverses of each other.

Theorem 4.5.4 If $L : \mathbb{V} \to \mathbb{W}$ is an invertible linear mapping, then L^{-1} is unique.

The proof of Theorem 4.5.4 is left as Problem C5.

Theorem 4.5.5 If $L : \mathbb{V} \to \mathbb{W}$ is invertible, then $\text{Null}(L) = \{\mathbf{0}_{\mathbb{V}}\}$ and $\text{Range}(L) = \mathbb{W}$.

Proof: Let $\mathbf{v} \in \text{Null}(L)$. Using Theorem 4.5.1 (1), we get

$$\mathbf{0} = L^{-1}(\mathbf{0}) = L^{-1}(L(\mathbf{v})) = \text{Id}(\mathbf{v}) = \mathbf{v}$$

Hence, $\text{Null}(L) = \{\mathbf{0}_{\mathbb{V}}\}$.

Let $\mathbf{y} \in \mathbb{W}$. Let $\mathbf{x} \in \mathbb{V}$ be the vector such that $L^{-1}(\mathbf{y}) = \mathbf{x}$. Then,

$$L(\mathbf{x}) = L\left(L^{-1}(\mathbf{y})\right) = \text{Id}(\mathbf{y}) = \mathbf{y}$$

Hence, $\mathbf{y} \in \text{Range}(L)$ and so $\text{Range}(L) = \mathbb{W}$. ■

PROBLEMS 4.5

Practice Problems

For Problems A1–A4, prove that the mapping is linear.

A1 $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_2 + x_3)$

A2 $L : \mathbb{R}^3 \to P_1(\mathbb{R})$ defined by $L(a, b, c) = (a + b) + (a + b + c)x$

A3 $\mathrm{tr} : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $\mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$

A4 $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

For Problems A5–A8, determine whether the mapping is linear.

A5 $D : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $D(A) = \det A$

A6 $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $L(a + bx + cx^2) = (a - b) + (b + c)x^2$

A7 $T : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ defined by $T(x_1, x_2) = \begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix}$

A8 $M : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $M\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

For Problems A9–A12, determine whether the given vector $\mathbf{y}$ is in the range of the given linear mapping $L : \mathbb{V} \to \mathbb{W}$. If it is, find a vector $\mathbf{x} \in \mathbb{V}$ such that $L(\mathbf{x}) = \mathbf{y}$.

A9 $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$L(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_3 \\ 0 \\ x_2 + x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

A10 $L : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by
$$L(a + bx + cx^2) = \begin{bmatrix} a + c & 0 \\ 0 & b + c \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

A11 $L : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $L(a + bx + cx^2) = (b + c) + (-b - c)x$, $\mathbf{y} = 1 + x$

A12 $L : \mathbb{R}^4 \to M_{2\times 2}(\mathbb{R})$ defined by
$$L(\vec{x}) = \begin{bmatrix} -2x_2 - 2x_3 - 2x_4 & x_1 + x_4 \\ -2x_1 - x_2 - x_4 & 2x_1 - 2x_2 - x_3 + 2x_4 \end{bmatrix},$$
$$\mathbf{y} = \begin{bmatrix} -1 & -1 \\ -2 & 2 \end{bmatrix}$$

For Problems A13–A18, find a basis for the range and a basis for the nullspace of the linear mapping and verify the Rank-Nullity Theorem.

A13 $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_2 + x_3)$

A14 $L : \mathbb{R}^3 \to P_1(\mathbb{R})$ defined by $L(a, b, c) = (a + b) + (a + b + c)x$

A15 $L : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $L(a + bx) = a - 2b + (a - 2b)x^2$

A16 $L : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $L\left(a + bx + cx^2\right) = (a - b, 2a + b, a - b + c)$

A17 $\mathrm{tr} : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $\mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$

A18 $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

For Problems A19 and A20, determine whether L and M are inverses.

A19 $L : \mathbb{R}^2 \to P_1(\mathbb{R})$ defined by $L(y_1, y_2) = (y_1 + y_2) + (2y_1 + 3y_2)x$
$M : P_1(\mathbb{R}) \to \mathbb{R}^2$ defined by $M(a + bx) = (3a - b, -a + b)$

A20 $L : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $L(a + bx + cx^2) = (a - b + c, b + 2c, c)$
$M : \mathbb{R}^3 \to P_2(\mathbb{R})$ defined by $M(y_1, y_2, y_3) = (y_1 + y_2 - 3y_3) + (y_2 - 2y_3)x + y_3x^2$

For Problems A21–A23, construct a linear mapping $L : \mathbb{V} \to \mathbb{W}$ that satisfies the given properties.

A21 $\mathbb{V} = \mathbb{R}^3$, $\mathbb{W} = P_2(\mathbb{R})$; $L(1, 0, 0) = x^2$, $L(0, 1, 0) = 2x$, $L(0, 0, 1) = 1 + x + x^2$

A22 $\mathbb{V} = P_2(\mathbb{R})$, $\mathbb{W} = M_{2\times 2}(\mathbb{R})$; $\mathrm{Null}(L) = \{0\}$ and $\mathrm{Range}(L) = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

A23 $\mathbb{V} = M_{2\times 2}(\mathbb{R})$, $\mathbb{W} = \mathbb{R}^4$; $\mathrm{nullity}(L) = 2$, $\mathrm{rank}(L) = 2$, and $L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Homework Problems

For Problems B1–B7, prove that the mapping is linear.

B1 $L : \mathbb{R}^3 \to P_2(\mathbb{R})$ defined by $L(a, b, c) = a + bx + cx^2$

B2 $L : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $L(a + bx) = a + b - 2ax$

B3 $L : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

B4 Let $\mathbb{D}$ be the subspace of $M_{2\times2}(\mathbb{R})$ of diagonal matrices; $L : P_1(\mathbb{R}) \to \mathbb{D}$ defined by
$$L(a + bx) = \begin{bmatrix} a - 2b & 0 \\ 0 & a - 2b \end{bmatrix}$$

B5 $L : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by
$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - b & 0 \\ a - d & -a + d \end{bmatrix}$$

B6 $L : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^3$ defined by
$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + b \\ 0 \\ c + d \end{bmatrix}$$

B7 $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(A) = A^T$

For Problems B8–B12, determine whether the mapping is linear.

B8 $L : P_2(\mathbb{R}) \to \mathbb{R}$ defined by $L(a + bx + cx^2) = \left\| \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\|$

B9 $M : P_1(\mathbb{R}) \to \mathbb{R}$ defined by $M(a + bx) = b - a$

B10 $N : \mathbb{R}^3 \to M_{2\times2}(\mathbb{R})$ defined by
$$N(x_1, x_2, x_3) = \begin{bmatrix} x_1 - x_3 & 1 \\ x_3 & x_1 \end{bmatrix}$$

B11 $L : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by
$$L(A) = A\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

B12 $T : M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (a + d)x + (b + c)x^2$$

For Problems B13–B18, determine whether the given vector $\mathbf{y}$ is in the range of the given linear mapping $L : \mathbb{V} \to \mathbb{W}$. If it is, find a vector $\mathbf{x} \in \mathbb{V}$ such that $L(\mathbf{x}) = \mathbf{y}$.

B13 $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 - 2x_2 \\ 2x_1 + x_3 \\ -2x_1 + x_2 - 2x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

B14 $L : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $L(a + bx) = (a + 2b) + (2a + 3b)x$, $\mathbf{y} = 3 - 4x$

B15 $L : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by
$$L(a + bx + cx^2) = \begin{bmatrix} a + b \\ b + c \\ a - c \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

B16 $L : \mathbb{R}^2 \to M_{2\times2}(\mathbb{R})$ defined by
$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 & x_1 - x_2 \\ 0 & 2x_1 + x_2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$$

B17 Let $\mathbb{T}$ denote the subspace of 2×2 upper-triangular matrices; $L : \mathbb{T} \to P_2(\mathbb{R})$ defined by $L\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = (-a - c) + (a - 2b)x^2$, $\mathbf{y} = 2 + x^2$

B18 $L : P_1(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by
$$L(a + bx) = \begin{bmatrix} -a & 2b \\ a - b & -2b \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

For Problems B19–B28, find a basis for the range and a basis for the nullspace of the linear mapping and verify the Rank-Nullity Theorem.

B19 $L : \mathbb{R}^3 \to P_2(\mathbb{R})$ defined by $L(a, b, c) = a + bx + cx^2$

B20 $L : P_1(\mathbb{R}) \to \mathbb{R}^3$ defined by $L(a + bx) = \begin{bmatrix} a - 2b \\ a + b \\ a - b \end{bmatrix}$

B21 $L : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $L\left(a + bx + cx^2\right) = b + 2ax$

B22 $L : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

B23 $L : \mathbb{R}^3 \to M_{2\times2}(\mathbb{R})$ defined by
$$L(x_1, x_2, x_3) = \begin{bmatrix} x_1 + 2x_2 & -x_1 + 2x_3 \\ x_2 + x_3 & x_1 + 2x_2 \end{bmatrix}$$

B24 Let $\mathbb{D}$ be the subspace of $M_{2\times2}(\mathbb{R})$ of diagonal matrices; $L : \mathbb{D} \to P_2(\mathbb{R})$ defined by
$$L\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = a + (a + b)x + bx^2$$

B25 $L : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by
$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - b & b - c \\ c - d & d - a \end{bmatrix}$$

B26 $L : M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b) + (c + d)x^2$$

B27 $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(A) = A^T$

B28 $L : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by
$$L(a + bx + cx^2) = \begin{bmatrix} -a - 2c & 2b - c \\ -2a + 2c & -2b - c \end{bmatrix}$$

For Problems B29–B31, determine whether L and M are inverses.

B29 $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$L(x_1, x_2) = (2x_1 + 3x_2, x_1 + 3x_2)$
$M : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$M(x_1, x_2) = (x_1 - x_2, -x_1 + 2x_2)$

B30 $L : P_1(\mathbb{R}) \to \mathbb{R}^2$ defined by
$L(a + bx) = (4a - 3b, -7a + 5b)$
$M : \mathbb{R}^2 \to P_1(\mathbb{R})$ defined by
$M(y_1, y_2) = (-5y_1 - 3y_2) + (-7y_1 - 4y_2)x$

B31 $L : \mathbb{R}^3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$L(a, b, c) = (a + b + 2c) + (a + 2c)x + (a + 2b + c)x^2$
$M : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by
$M(a + bx + cx^2) = (-4a + 3b + 2c, a - b, 2a - b - c)$

For Problems B32–B34, construct a linear mapping $L : \mathbb{V} \to \mathbb{W}$ that satisfies the given properties.

B32 $\mathbb{V} = P_2(\mathbb{R})$, $\mathbb{W} = \mathbb{R}^2$; $L(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $L(x) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $L\left(x^2\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

B33 $\mathbb{V} = M_{2\times 2}(\mathbb{R})$, $\mathbb{W} = \mathbb{R}^3$; nullity$(L) = 3$, rank$(L) = 1$, and $L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

B34 $\mathbb{V} = P_2(\mathbb{R})$, $\mathbb{W} = P_2(\mathbb{R})$; Range$(L) = P_2(\mathbb{R})$, $L(1 + x^2) = x$, $L(1 + x) = x + x^2$.

Conceptual Problems

C1 Prove if $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then $L(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$.

C2 Prove if $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then Null(L) is a subspace of $\mathbb{V}$.

C3 Prove if $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then Range(L) is a subspace of $\mathbb{W}$.

C4 Prove if $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ are linear mappings, then $M \circ L : \mathbb{V} \to \mathbb{U}$ is also a linear mapping.

C5 Prove if $L : \mathbb{V} \to \mathbb{W}$ is an invertible linear mapping, then L^{-1} is unique.

C6 (a) Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Prove that if $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is a linearly independent set in $\mathbb{W}$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set in $\mathbb{V}$.

(b) Give an example of a linear mapping $L : \mathbb{V} \to \mathbb{W}$, where $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent in $\mathbb{V}$ but $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is linearly dependent in $\mathbb{W}$.

C7 Let $\mathbb{V}$ and $\mathbb{W}$ be n-dimensional vector spaces and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Prove that Range$(L) = \mathbb{W}$ if and only if Null$(L) = \{\mathbf{0}\}$.

C8 Let $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$ be finite-dimensional vector spaces over $\mathbb{R}$ and let $L : \mathbb{V} \to \mathbb{U}$ and $M : \mathbb{U} \to \mathbb{W}$ be linear mappings.

(a) Prove that rank$(M \circ L) \leq$ rank(M).

(b) Prove that rank$(M \circ L) \leq$ rank(L).

(c) Construct an example such that the rank of the composition is strictly less than the maximum of the ranks.

C9 Let $\mathbb{U}$ and $\mathbb{V}$ be finite-dimensional vector spaces and let $L : \mathbb{V} \to \mathbb{U}$ be a linear mapping and $M : \mathbb{U} \to \mathbb{U}$ be a linear operator such that Null$(M) = \{\mathbf{0}_{\mathbb{U}}\}$. Prove that rank$(M \circ L) =$ rank(L).

C10 Let S denote the set of all infinite sequences of real numbers. A typical element of S is $\mathbf{x} = (x_1, x_2, \dots, x_n, \dots)$. Define addition $\mathbf{x} + \mathbf{y}$ and scalar multiplication $t\mathbf{x}$ in the obvious way. Then S is a vector space. Define the left shift $L : S \to S$ by $L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$ and the right shift $R : S \to S$ by $R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$. Then it is easy to verify that L and R are linear. Check that $(L \circ R)(\mathbf{x}) = \mathbf{x}$ but that $(R \circ L)(\mathbf{x}) \neq \mathbf{x}$. L has a right inverse, but it does not have a left inverse. It is important in this example that S is infinite-dimensional.

4.6 Matrix of a Linear Mapping

In Section 3.2, we defined the standard matrix of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ so that we could represent any such linear mapping as a matrix mapping. We now generalize this to finding a matrix representation of any linear mapping $L : \mathbb{V} \to \mathbb{V}$ with respect to any basis $\mathcal{B}$ of the vector space $\mathbb{V}$.

The Matrix of a Linear Mapping $L : \mathbb{R}^n \to \mathbb{R}^n$

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping and let $\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$ denote the standard basis for $\mathbb{R}^n$. Using the result of Exercise 4.4.2 on page 266, we can write the equations in Theorem 3.2.3 on page 176 regarding the standard matrix of L as

$$[L] = \begin{bmatrix} [L(\vec{e}_1)]_{\mathcal{S}} & \cdots & [L(\vec{e}_n)]_{\mathcal{S}} \end{bmatrix}$$

and

$$[L(\vec{x})]_{\mathcal{S}} = [L][\vec{x}]_{\mathcal{S}} \tag{4.4}$$

Our goal is to mimic these equations for any linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ with respect to any basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$. We follow the same method as in the proof of Theorem 3.2.3. For any $\vec{x} \in \mathbb{R}^n$, we can write $\vec{x} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$. Therefore,

$$L(\vec{x}) = L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) = b_1L(\vec{v}_1) + \cdots + b_nL(\vec{v}_n)$$

Taking $\mathcal{B}$-coordinates of both sides gives

$$\begin{aligned} [L(\vec{x})]_{\mathcal{B}} &= [b_1L(\vec{v}_1) + \cdots + b_nL(\vec{v}_n)]_{\mathcal{B}} \\ &= b_1[L(\vec{v}_1)]_{\mathcal{B}} + \cdots + b_n[L(\vec{v}_n)]_{\mathcal{B}} \\ &= \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \end{aligned}$$

Observe that $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [\vec{x}]_{\mathcal{B}}$, so this equation is the $\mathcal{B}$-coordinates version of equation (4.4) where the matrix $\begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}$ is taking the place of the standard matrix of L.

Definition
Matrix of a Linear Mapping

Suppose that $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is any basis for $\mathbb{R}^n$ and that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping. Define the **matrix of the linear mapping** L with respect to the basis $\mathcal{B}$ to be the matrix

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

It satisfies

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

Note that the columns of $[L]_{\mathcal{B}}$ are the $\mathcal{B}$-coordinate vectors of the images of the $\mathcal{B}$-basis vectors under L. The pattern is exactly the same as before, except that everything is done in terms of the basis $\mathcal{B}$. It is important to emphasize again that by "basis," we always mean *ordered basis*; the order of the vectors in the basis determines the order of the columns of the matrix $[L]_{\mathcal{B}}$.

Remark

It is important to always distinguish between the two very similar-looking notations: $[L]_{\mathcal{B}}$ is the notation for the matrix of the linear mapping with respect to the basis $\mathcal{B}$. $[L(\vec{x})]_{\mathcal{B}}$ is the notation for the $\mathcal{B}$-coordinates of $L(\vec{x})$ with respect to the basis $\mathcal{B}$.

EXAMPLE 4.6.1

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by

$$L(x_1, x_2, x_3) = (x_1 + 2x_2 - 2x_3, -x_2 + 2x_3, x_1 + 2x_2)$$

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ and let $\vec{x} \in \mathbb{R}^3$ such that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the matrix of L with respect to $\mathcal{B}$ and use it to determine $[L(\vec{x})]_{\mathcal{B}}$.

Solution: By definition, the columns of $[L]_{\mathcal{B}}$ are the $\mathcal{B}$-coordinates of the images of the vectors in $\mathcal{B}$ under L. So, we find these images and write them as a linear combination of the vectors in $\mathcal{B}$:

$$L(2, -1, -1) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$L(1, 1, 1) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = (0) \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$L(0, 0, -1) = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = (4/3) \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + (-2/3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(2, -1, -1)]_{\mathcal{B}} & [L(1, 1, 1)]_{\mathcal{B}} & [L(0, 0, -1)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -2/3 \\ -1 & -2 & -2 \end{bmatrix}$$

Thus, $[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -2/3 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$.

It is instructive to verify the answer in Example 4.6.1 by calculating $L(\vec{x})$ in two ways. First, if $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\vec{x} = 1 \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$, and, by definition of the mapping, we have $L(\vec{x}) = L(4, 1, -2) = (10, -5, 6)$.

Second, if $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$, then $L(\vec{x}) = 5 \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 11 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ 6 \end{bmatrix}$.

It is natural to wonder why we might want to represent a linear mapping with respect to a basis other than the standard basis. However, in some problems it is much more convenient to use a different basis than the standard basis. For example, a stretch by a factor of 3 in $\mathbb{R}^2$ in the direction of the vector $\begin{bmatrix}1\\2\end{bmatrix}$ is geometrically easy to understand. However, it would be awkward to determine the standard matrix of this stretch and then determine its effect on any other vector. It would be much better to have a basis that takes advantage of the vector $\begin{bmatrix}1\\2\end{bmatrix}$ and of the orthogonal vector $\begin{bmatrix}-2\\1\end{bmatrix}$, which remain unchanged under the stretch.

Consider the reflection $\operatorname{refl}_{\vec{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ in the plane $x_1 + 2x_2 - 3x_3 = 0$. It is easy to describe this by saying that it reverses the normal vector $\vec{n} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$ to $\begin{bmatrix}-1\\-2\\3\end{bmatrix}$ and leaves unchanged any vectors lying in the plane (such as $\begin{bmatrix}2\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\0\\1\end{bmatrix}$) as in Figure 4.6.1. Describing this reflection in terms of these vectors gives more geometrical information than describing it in terms of the standard basis vectors.

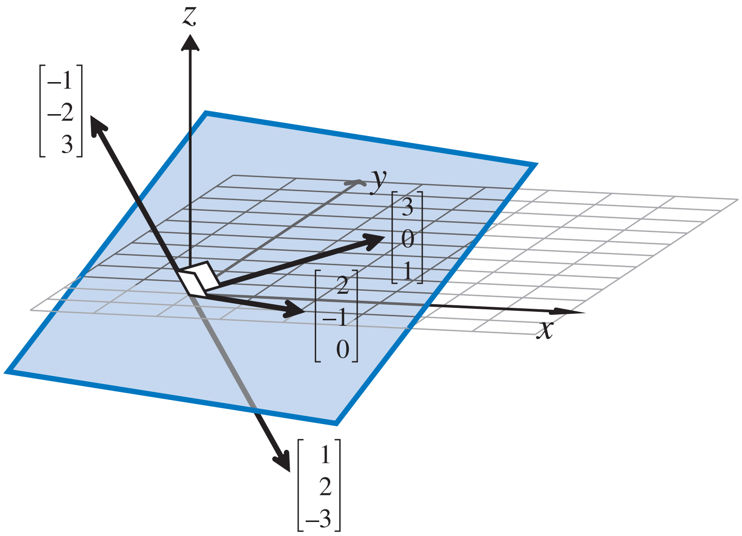


Figure 4.6.1 Reflection in the plane $x_1 + 2x_2 - 3x_3 = 0$.

EXERCISE 4.6.1

Let $\mathcal{B} = \left\{\begin{bmatrix}1\\2\\-3\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\right\}$. Find $[\operatorname{refl}_{\vec{n}}]_{\mathcal{B}}$ where $\operatorname{refl}_{\vec{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ is the reflection over the plane $x_1 + 2x_2 - 3x_3 = 0$.

Notice that in these examples, the geometry itself provides us with a preferred basis for the appropriate space.

EXAMPLE 4.6.2

Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. In Example 3.2.6, we found the standard matrix of the linear operator $\text{proj}_{\vec{v}} : \mathbb{R}^2 \to \mathbb{R}^2$ to be

$$[\text{proj}_{\vec{v}}]_{\mathcal{S}} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$$

Find the matrix of $\text{proj}_{\vec{v}}$ with respect to a basis that shows the geometry of the transformation more clearly.

Solution: For this linear transformation, it is natural to use a basis for $\mathbb{R}^2$ consisting of the vector $\vec{v}$, which is the direction vector for the projection, and a second vector orthogonal to $\vec{v}$, say $\vec{w} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. Taking $\mathcal{B} = \{\vec{v}, \vec{w}\}$ we find that

$$\text{proj}_{\vec{v}}(\vec{v}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1\vec{v} + 0\vec{w}$$

$$\text{proj}_{\vec{v}}(\vec{w}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v} + 0\vec{w}$$

Hence, $[\text{proj}_{\vec{v}}(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[\text{proj}_{\vec{v}}(\vec{w})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus,

$$[\text{proj}_{\vec{v}}]_{\mathcal{B}} = \begin{bmatrix} [\text{proj}_{\vec{v}}(\vec{v})]_{\mathcal{B}} & [\text{proj}_{\vec{v}}(\vec{w})]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We can use $[\text{proj}_{\vec{v}}]_{\mathcal{B}}$ to get a simple geometrical description of $\text{proj}_{\vec{v}}$. Consider $\text{proj}_{\vec{v}}(\vec{x})$ for any $\vec{x} = b_1\vec{v} + b_2\vec{w} \in \mathbb{R}^2$. We have

$$[\text{proj}_{\vec{v}}(\vec{x})]_{\mathcal{B}} = [\text{proj}_{\vec{v}}]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

Consequently, $\text{proj}_{\vec{v}}$ is described as the linear mapping that sends $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ to $\begin{bmatrix} b_1 \\ 0 \end{bmatrix}$ as shown in Figure 4.6.2.

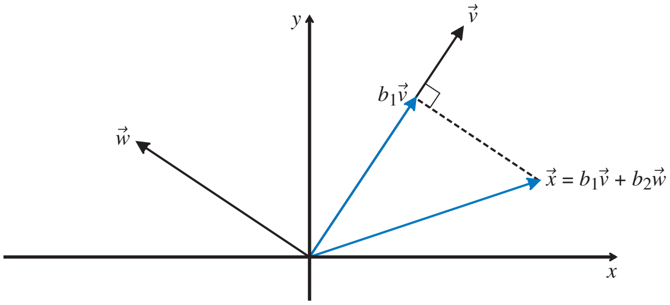


Figure 4.6.2 The geometry of $\text{proj}_{\vec{v}}$.

The Matrix of a General Linear Mapping

We now want to generalize our work above to find the matrix of any linear operator L on a vector space $\mathbb{V}$ with respect to any basis $\mathcal{B}$ for $\mathbb{V}$. To do this, we could repeat our argument from the beginning of this section. This is left as Problem C5.

We make the following definition.

Definition
Matrix of a General Linear Mapping

Suppose that $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is any basis for a vector space $\mathbb{V}$ and that $L : \mathbb{V} \to \mathbb{V}$ is a linear mapping. Define the **matrix of the linear mapping** L with respect to the basis $\mathcal{B}$ to be the matrix

$$[L]_\mathcal{B} = \begin{bmatrix} [L(\mathbf{v}_1)]_\mathcal{B} & \cdots & [L(\mathbf{v}_n)]_\mathcal{B} \end{bmatrix}$$

It satisfies

$$[L(\mathbf{x})]_\mathcal{B} = [L]_\mathcal{B}[\mathbf{x}]_\mathcal{B}$$

EXAMPLE 4.6.3

Let $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $L(a + bx + cx^2) = (a + b) + bx + (a + b + c)x^2$. Find the matrix of L with respect to the basis $\mathcal{B} = \{1, x, x^2\}$ and use it to calculate $[L(1 + 2x + 3x^2)]_\mathcal{B}$.

Solution: We have

$$\begin{aligned} L(1) &= 1 + x^2 \\ L(x) &= 1 + x + x^2 \\ L(x^2) &= x^2 \end{aligned}$$

Hence,

$$[L]_\mathcal{B} = \begin{bmatrix} [L(1)]_\mathcal{B} & [L(x)]_\mathcal{B} & [L(x^2)]_\mathcal{B} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Since $[1 + 2x + 3x^2]_\mathcal{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, we get that

$$[L(1 + 2x + 3x^2)]_\mathcal{B} = [L]_\mathcal{B}[1 + 2x + 3x^2]_\mathcal{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

We can check the the matrix $[L]_\mathcal{B}$ in Example 4.6.3 is correct by observing that $[L(a + bx + cx^2)]_\mathcal{B} = \begin{bmatrix} a + b \\ b \\ a + b + c \end{bmatrix}$ and

$$[L(a + bx + cx^2)]_\mathcal{B} = [L]_\mathcal{B}[a + bx + cx^2]_\mathcal{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b \\ a + b + c \end{bmatrix}$$

EXAMPLE 4.6.4

Let $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $L(a + bx + cx^2) = (a + b) + bx + (a + b + c)x^2$. Find the matrix of L with respect to the basis $\mathcal{B} = \{1 + x + x^2, 1 - x, 2\}$.

Solution: We have

$$\begin{aligned} L(1 + x + x^2) &= 2 + x + 3x^2 \\ L(1 - x) &= -x \\ L(2) &= 2 + 2x^2 \end{aligned}$$

In this case, we are not able to write the images as linear combinations of the vectors in $\mathcal{B}$ by inspection. So, we use the method of Section 4.4.

Consider

$$\begin{aligned} 2 + x + 3x^2 &= a_1(1 + x + x^2) + a_2(1 - x) + a_3(2) \\ 0 + (-1)x + 0x^2 &= b_1(1 + x + x^2) + b_2(1 - x) + b_3(2) \\ 2 + 2x^2 &= c_1(1 + x + x^2) + c_2(1 - x) + c_3(2) \end{aligned}$$

This gives us three systems of linear equations with the same coefficient matrix. Row reducing the triple augmented matrix gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 2 & 0 & 2 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 3 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & -3/2 & -1/2 & -1 \end{array}\right]$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(1 + x + x^2)]_{\mathcal{B}} & [L(1 - x)]_{\mathcal{B}} & [L(2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 2 \\ -3/2 & -1/2 & -1 \end{bmatrix}$$

EXERCISE 4.6.2

Let $L : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + b & a - b \\ c & a + b + d \end{bmatrix}$. Find the matrix of L with respect to the basis $\mathcal{B} = \left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

Observe that in each of the cases above, we have used the same basis for the domain and codomain of the linear mapping L. To make this as general as possible, we would like to define the matrix ${}_{C}[L]_{\mathcal{B}}$ of a linear mapping $L : \mathbb{V} \to \mathbb{W}$, where $\mathcal{B}$ is a basis for the vector space $\mathbb{V}$ and C is a basis for the vector space $\mathbb{W}$. This is left as Problem C6.

Change of Coordinates and Linear Mappings

In Example 4.6.2, we used special geometrical properties of the linear transformation and of the chosen basis $\mathcal{B}$ to determine the $\mathcal{B}$-coordinate vectors that make up the $\mathcal{B}$-matrix $[L]_{\mathcal{B}}$ of the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. In some applications of these ideas, the geometry does not provide such a simple way of determining $[L]_{\mathcal{B}}$. We need a general method for determining $[L]_{\mathcal{B}}$, given the standard matrix $[L]_{\mathcal{S}}$ of a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ and a new basis $\mathcal{B}$.

Theorem 4.6.1

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, let $\mathcal{S}$ denote the standard basis for $\mathbb{R}^n$, and let $\mathcal{B}$ be any other basis for $\mathbb{R}^n$. If P is the change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates, then

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P$$

Proof: By definition of the change of coordinates matrix and Theorem 4.4.2, we have

$$P[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{S}} \qquad \text{and} \qquad [\vec{x}]_{\mathcal{B}} = P^{-1}[\vec{x}]_{\mathcal{S}}$$

If we apply the change of coordinates equation to the vector $L(\vec{x})$ (which is in $\mathbb{R}^n$), we get

$$[L(\vec{x})]_{\mathcal{B}} = P^{-1}[L(\vec{x})]_{\mathcal{S}}$$

Substitute for $[L(\vec{x})]_{\mathcal{B}}$ and $[L(\vec{x})]_{\mathcal{S}}$ to get

$$[L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}[\vec{x}]_{\mathcal{S}}$$

But, $P[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{S}}$, so we have

$$[L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P[\vec{x}]_{\mathcal{B}}$$

Since this is true for every $[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^n$, we get, by the Matrices Equal Theorem, that

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P$$

■

We analyze Theorem 4.6.1 by comparing it to Example 4.6.2.

EXAMPLE 4.6.5

Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. In Example 4.6.2, we determined the matrix of $\text{proj}_{\vec{v}}$ with respect to a geometrically adapted basis $\mathcal{B}$. Let us verify that the change of basis method just described does transform the standard matrix $[\text{proj}_{\vec{v}}]_{\mathcal{S}}$ to the $\mathcal{B}$-matrix $[\text{proj}_{\vec{v}}]_{\mathcal{B}}$.

The matrix $[\text{proj}_{\vec{v}}]_{\mathcal{S}} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$. The basis $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$, so the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is $P = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$. The inverse is found to be $P^{-1} = \begin{bmatrix} 3/25 & 4/25 \\ -4/25 & 3/25 \end{bmatrix}$. Hence, the $\mathcal{B}$-matrix of $\text{proj}_{\vec{v}}$ is given by

$$P^{-1}[\text{proj}_{\vec{v}}]_{\mathcal{S}}P = \begin{bmatrix} 3/25 & 4/25 \\ -4/25 & 3/25 \end{bmatrix} \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, we get exactly the same $\mathcal{B}$-matrix $[\text{proj}_{\vec{v}}]_{\mathcal{B}}$ as we obtained by the earlier geometric argument.

To make sure we understand precisely what this means, let us calculate the $\mathcal{B}$-coordinates of the image of the vector $\vec{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ under $\text{proj}_{\vec{v}}$. We can do this in two ways.

Method 1. Use the fact that $[\text{proj}_{\vec{v}}\, \vec{x}]_{\mathcal{B}} = [\text{proj}_{\vec{v}}]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$. We need the $\mathcal{B}$-coordinates of $\vec{x}$:

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_{\mathcal{B}} = P^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 3/25 & 4/25 \\ -4/25 & 3/25 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 23/25 \\ -14/25 \end{bmatrix}$$

Hence,

$$[\text{proj}_{\vec{v}}(\vec{x})]_{\mathcal{B}} = [\text{proj}_{\vec{v}}]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 23/25 \\ -14/25 \end{bmatrix} = \begin{bmatrix} 23/25 \\ 0 \end{bmatrix}$$

Method 2. Use the fact that $[\text{proj}_{\vec{v}}\, \vec{x}]_{\mathcal{B}} = P^{-1}[\text{proj}_{\vec{v}}\, \vec{x}]_{\mathcal{S}}$:

$$[\text{proj}_{\vec{v}}(\vec{x})]_{\mathcal{S}} = [\text{proj}_{\vec{v}}]_{\mathcal{S}} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 69/25 \\ 92/25 \end{bmatrix}$$

Therefore,

$$[\text{proj}_{\vec{v}}\, \vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3/25 & 4/25 \\ -4/25 & 3/25 \end{bmatrix} \begin{bmatrix} 69/25 \\ 92/25 \end{bmatrix} = \begin{bmatrix} 23/25 \\ 0 \end{bmatrix}$$

The calculation is probably slightly easier if we use the first method, but that really is not the point. What is extremely important is that it is easy to get a geometrical understanding of what happens to vectors if you multiply by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (the $\mathcal{B}$-matrix); it is much more difficult to understand what happens if you multiply by $\begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$ (the standard matrix). Using a non-standard basis may make it much easier to understand the geometry of a linear transformation.

EXAMPLE 4.6.6

Let L be the linear mapping with standard matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Find the matrix of L with respect to the basis $\mathcal{B}$.

Solution: The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, and we have $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. It follows that the $\mathcal{B}$-matrix of L is

$$[L]_{\mathcal{B}} = P^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13/2 & 1/2 \\ 21/2 & 1/2 \end{bmatrix}$$

EXAMPLE 4.6.7

Let $L(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} -3 & 5 & -5 \\ -7 & 9 & -5 \\ -7 & 7 & -3 \end{bmatrix}$. Let $\mathcal{B}$ be the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Determine the matrix of L with respect to the basis $\mathcal{B}$ and use it to determine the geometry of L.

Solution: The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and we have $P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. Thus, the $\mathcal{B}$-matrix of the mapping L is

$$[L]_{\mathcal{B}} = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Observe that for any $\vec{x} \in \mathbb{R}^3$ such that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ we have

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2b_1 \\ -3b_2 \\ 4b_3 \end{bmatrix}$$

Thus, L is a stretch by a factor of 2 in the direction of the first basis vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, a reflection (because of the minus sign) and a stretch by a factor of 3 in the second basis vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and a stretch by a factor of 4 in the direction of the third basis vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. This gives a very clear geometrical picture of how the linear transformation maps vectors, which would not be obvious from looking at the standard matrix A.

CONNECTION

At this point it is natural to ask whether for any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ there exists a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix of L is in diagonal form, and how can we find such a basis if it exists? The answers to these questions are found in Chapter 6. However, in order to deal with these questions, one more computational tool is needed, the determinant, which is discussed in Chapter 5.

PROBLEMS 4.6

Practice Problems

A1 Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$ and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping such that $L(\vec{v}_1) = \vec{v}_2$ and $L(\vec{v}_2) = 2\vec{v}_1 - \vec{v}_2$. Determine $[L]_\mathcal{B}$ and use it to calculate $[L(4\vec{v}_1 + 3\vec{v}_2)]_\mathcal{B}$.

A2 Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $\mathbb{R}^3$ and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping such that $L(\vec{v}_1) = 2\vec{v}_1 - \vec{v}_3$, $L(\vec{v}_2) = 2\vec{v}_1 - \vec{v}_3$, and $L(\vec{v}_3) = 4\vec{v}_2 + 5\vec{v}_3$. Determine $[L]_\mathcal{B}$ and use it to calculate $[L(3\vec{v}_1 + 3\vec{v}_2 - \vec{v}_3)]_\mathcal{B}$.

For Problems A3 and A4, let $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$ be a basis of $\mathbb{R}^2$. Assume that L is a linear mapping. Compute $[L]_\mathcal{B}$.

A3 $L(1,1) = (-3,-3)$, $L(-1,2) = (-4,8)$

A4 $L(1,1) = (-1,2)$, $L(-1,2) = (2,2)$

For Problems A5–A9, compute $[L]_\mathcal{B}$ and use it to find $[L(\vec{x})]_\mathcal{B}$.

A5 $L(x_1,x_2) = (2x_1 + x_2, x_1 + 2x_2)$,
$\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, $\vec{x} = \begin{bmatrix}2\\0\end{bmatrix}$

A6 $L(x_1,x_2) = (2x_1 + 3x_2, 2x_1 + 3x_2)$,
$\mathcal{B} = \left\{\begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$, $\vec{x} = \begin{bmatrix}0\\2\end{bmatrix}$

A7 $L(x_1,x_2) = (x_1 + 3x_2, -8x_1 + 7x_2)$,
$\mathcal{B} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}\right\}$, $\vec{x} = \begin{bmatrix}0\\2\end{bmatrix}$

A8 $L(x_1,x_2,x_3) = (x_1 + x_2 + x_3, x_2, x_2 + x_3)$,
$\mathcal{B} = \left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$, $\vec{x} = \begin{bmatrix}0\\3\\2\end{bmatrix}$

A9 $L(x_1,x_2,x_3) = (3x_1 - 2x_3, x_1 + x_2 + x_3, 2x_1 + x_2)$,
$\mathcal{B} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$, $\vec{x} = \begin{bmatrix}3\\-2\\9\end{bmatrix}$

For Problems A10–A13, determine a geometrically natural basis $\mathcal{B}$ (as in Example 4.6.2) and determine the $\mathcal{B}$-matrix of the transformation.

A10 $\text{refl}_{(1,-2)} : \mathbb{R}^2 \to \mathbb{R}^2$

A11 $\text{perp}_{(1,-2)} : \mathbb{R}^2 \to \mathbb{R}^2$

A12 $\text{proj}_{(2,1,-1)} : \mathbb{R}^3 \to \mathbb{R}^3$

A13 $\text{refl}_{(-1,-1,1)} : \mathbb{R}^3 \to \mathbb{R}^3$

A14 Let $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$ and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping such that

$$L(1,0,1) = (1,2,4)$$
$$L(1,-1,0) = (0,1,2)$$
$$L(0,1,2) = (2,-2,0)$$

(a) Let $\vec{x} = \begin{bmatrix}1\\2\\4\end{bmatrix}$. Find $[\vec{x}]_\mathcal{B}$.
(b) Find $[L]_\mathcal{B}$.
(c) Use parts (a) and (b) to determine $L(\vec{x})$.

A15 Let $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping such that

$$L(1,0,-1) = (0,1,1)$$
$$L(1,2,0) = (-2,0,2)$$
$$L(0,1,1) = (5,3,-5)$$

(a) Let $\vec{x} = \begin{bmatrix}-1\\7\\6\end{bmatrix}$. Find $[\vec{x}]_\mathcal{B}$.
(b) Find $[L]_\mathcal{B}$.
(c) Use parts (a) and (b) to determine $L(\vec{x})$.

For Problems A16–A21, assume that the matrix A is the standard matrix of a linear mapping L. Determine the matrix of L with respect to the given basis $\mathcal{B}$ using Theorem 4.6.1. You may find it helpful to use a computer to find inverses and to multiply matrices.

A16 $A = \begin{bmatrix} 1 & 3 \\ -8 & 7 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

A17 $A = \begin{bmatrix} 1 & -6 \\ -4 & -1 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

A18 $A = \begin{bmatrix} 4 & -6 \\ 2 & 8 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\}$

A19 $A = \begin{bmatrix} 16 & -20 \\ 6 & -6 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$

A20 $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 2 \\ 1 & -1 & 5 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

A21 $A = \begin{bmatrix} 4 & 1 & -3 \\ 16 & 4 & -18 \\ 6 & 1 & -5 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

For Problems A22–A25, find the matrix of each of the linear mapping with respect to the given basis $\mathcal{B}$.

A22 $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$L(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 - x_3)$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

A23 $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$L(a + bx + cx^2) = a + (b + c)x^2$,
$\mathcal{B} = \{1 + x^2, -1 + x, 1 - x + x^2\}$

A24 $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$D(a + bx + cx^2) = b + 2cx$, $\mathcal{B} = \{1, x, x^2\}$

A25 $T : \mathbb{U} \to \mathbb{U}$, where $\mathbb{U}$ is the subspace of upper-triangular matrices in $M_{2\times 2}(\mathbb{R})$, defined by
$T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & b + c \\ 0 & a + b + c \end{bmatrix}$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

Homework Problems

B1 Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$ and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping such that $L(\vec{v}_1) = 2\vec{v}_1 + \vec{v}_2$, and $L(\vec{v}_2) = 3\vec{v}_1 + 4\vec{v}_2$. Determine $[L]_{\mathcal{B}}$ and use it to calculate $[L(-\vec{v}_1 + 3\vec{v}_2)]_{\mathcal{B}}$.

For Problems B2 and B3, let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $\mathbb{R}^3$, and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping. Determine $[L(\vec{x})]_{\mathcal{B}}$ for the given $[\vec{x}]_{\mathcal{B}}$.

B2 $L(\vec{v}_1) = 2\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3$, $L(\vec{v}_2) = 2\vec{v}_2 + \vec{v}_3$,
$L(\vec{v}_3) = 4\vec{v}_1 + 3\vec{v}_3$; $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

B3 $L(\vec{v}_1) = \vec{v}_1 + 2\vec{v}_2 - \vec{v}_3$, $L(\vec{v}_2) = 2\vec{v}_2 + \vec{v}_3$,
$L(\vec{v}_3) = \vec{v}_1 - 2\vec{v}_3$; $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

For Problems B4–B6, let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and assume that L is a linear mapping. Compute $[L]_{\mathcal{B}}$.

B4 $L(1, 2) = (0, 0)$, $L(-2, 1) = (-1, 3)$

B5 $L(1, 2) = (4, -2)$, $L(-2, 1) = (3, 6)$

B6 $L(1, 2) = (1, 2)$, $L(-2, 1) = (1, 2)$

For Problems B7–B11, compute $[L]_{\mathcal{B}}$ and use it to find $[L(\vec{x})]_{\mathcal{B}}$.

B7 $L(x_1, x_2) = (-2x_1 + 3x_2, 4x_1 - 3x_2)$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

B8 $L(x_1, x_2) = (3x_1 + 2x_2, 6x_1 - x_2)$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

B9 $L(x_1, x_2) = (2x_1 + 5x_2, 3x_1 - 2x_2)$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

B10 $L(x_1, x_2, x_3) = (x_1 + 6x_2 + 3x_3, -2x_2, 3x_1 + 6x_2 + x_3)$,
$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

B11 $L(x_1, x_2, x_3) = (2x_1 - 7x_2 - 3x_3, x_1 - 2x_2 - x_3, -2x_2 - x_3)$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

For Problems B12–B15, determine a geometrically natural basis $\mathcal{B}$ and determine the $\mathcal{B}$-matrix of the transformation.

B12 $\text{perp}_{(3,2)}$

B13 $\text{proj}_{(-1,-2)}$

B14 $\text{perp}_{(2,1,-2)}$

B15 $\text{refl}_{(1,2,3)}$

B16 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping such that

$$L(1,1,0) = (2,0,2)$$
$$L(0,1,1) = (1,1,2)$$
$$L(1,0,1) = (1,0,1)$$

(a) Let $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.

(b) Find $[L]_{\mathcal{B}}$.

(c) Use parts (a) and (b) to determine $L(\vec{x})$.

B17 Let $\mathcal{B} = \{2 + x, -1 + x^2, 1 + x\}$ and let $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear mapping such that

$$L(2 + x) = 4 + 2x$$
$$L(-1 + x^2) = 1 + x$$
$$L(1 + x) = 1 + x + x^2$$

(a) Let $\mathbf{p}(x) = 1 + 4x + 4x^2$. Find $[\vec{x}]_{\mathcal{B}}$.

(b) Find $[L]_{\mathcal{B}}$.

(c) Use parts (a) and (b) to determine $L(\mathbf{p})$.

For Problems B18–B21, assume that the matrix A is the standard matrix of a linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$. Determine the matrix of L with respect to the given basis $\mathcal{B}$ using Theorem 4.6.1. Use $[L]_{\mathcal{B}}$ to determine $[L(\vec{x})]_{\mathcal{B}}$ for the given $\vec{x}$. You may find it helpful to use a computer to find inverses and to multiply matrices.

B18 $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B19 $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

B20 $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$

B21 $A = \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

For Problems B22–B25, assume that the matrix A is the standard matrix of a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^3$. Determine the matrix of L with respect to the given basis $\mathcal{B}$ using Theorem 4.6.1. You may find it helpful to use a computer to find inverses and to multiply matrices.

B22 $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

B23 $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & -3 \\ 3 & -3 & 11 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}. \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

B24 $\begin{bmatrix} 3 & 6 & 1 \\ 5 & -4 & 5 \\ 3 & -6 & 5 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

B25 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

For Problems B26–B31, find the matrix of each of the linear mapping with respect to the given basis $\mathcal{B}$.

B26 $L : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ defined by
$L(a + bx) = (a + b) + (2a + 3b)x, \mathcal{B} = \{1 + x, 2 - x\}$

B27 $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$L(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 + 2x_2, x_1 + x_2 + x_3)$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

B28 $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$L(a + bx + cx^2) = (a + b + c) + (a + 2b)x + (a + c)x^2$,
$\mathcal{B} = \{1 + x, x^2, 1 + x^2\}$

B29 $L : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by
$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + d & b + c \\ 0 & d \end{bmatrix}$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

B30 $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$D(a + bx + cx^2) = b + 2cx$,
$\mathcal{B} = \{1 + 2x + 3x^2, -2x + x^2, 1 + x + 3x^2\}$

B31 $T : \mathbb{D} \to \mathbb{D}$, where $\mathbb{D}$ is the subspace of diagonal matrices in $M_{2\times 2}(\mathbb{R})$, defined by
$T\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} a + b & 0 \\ 0 & 2a + b \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \right\}$

Conceptual Problems

C1 Suppose that $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^n$ and that $\mathcal{S}$ is the standard basis of $\mathbb{R}^n$. Suppose that P is the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ and that Q is the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{S}$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Express the matrix $[L]_{\mathcal{C}}$ in terms of $[L]_{\mathcal{B}}$, P, and Q.

C2 Suppose that a 2×2 matrix A is the standard matrix of a linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$ and let P denote the change of coordinates matrix from $\mathcal{B}$ to the standard basis. What conditions will have to be satisfied by the vectors $\vec{v}_1$ and $\vec{v}_2$ in order for

$$P^{-1}AP = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = D$$

for some $d_1, d_2 \in \mathbb{R}$? (Hint: consider the equation $AP = PD$, or $A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} D$.)

C3 Let $\mathbb{V}$ be a finite dimensional vector space. Prove if $L : \mathbb{V} \to \mathbb{V}$ is an invertible linear mapping, then $[L]_{\mathcal{B}}$ is invertible for all bases $\mathcal{B}$ of $\mathbb{V}$.

C4 Let $\mathbb{V}$ be a finite dimensional vector space. Let $L : \mathbb{V} \to \mathbb{V}$ be a linear mapping and let $\mathcal{B}$ be any basis for $\mathbb{V}$.

(a) Prove if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for Null(L), then $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_k]_{\mathcal{B}}\}$ is a basis for Null($[L]_{\mathcal{B}}$).

(b) Prove that rank(L) = rank($[L]_{\mathcal{B}}$).

C5 Let $\mathbb{V}$ be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and let $L : \mathbb{V} \to \mathbb{V}$ be a linear mapping. Prove that the matrix $[L]_{\mathcal{B}}$ defined by

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

satisfies $[L(\mathbf{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

C6 Let $\mathbb{V}$ be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, let $\mathbb{W}$ be a vector space with basis $\mathcal{C}$, and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Prove that the matrix ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ defined by

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\mathbf{v}_1)]_{\mathcal{C}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

satisfies $[L(\mathbf{x})]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ and hence is the **matrix of** L with respect to basis $\mathcal{B}$ and $\mathcal{C}$.

For Problems C7–C10, using the definition given in Problem C6, determine the matrix of the linear mapping with respect to the given bases $\mathcal{B}$ and $\mathcal{C}$.

C7 $D : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $D(a + bx + cx^2) = b + 2cx$, $\mathcal{B} = \{1, x, x^2\}$, $\mathcal{C} = \{1, x\}$

C8 $L : \mathbb{R}^2 \to P_2(\mathbb{R})$ defined by $L(a_1, a_2) = (a_1 + a_2) + a_1 x^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, $\mathcal{C} = \{1 + x^2, 1 + x, -1 - x + x^2\}$

C9 $T : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a + b & 0 \\ 0 & a - b \end{bmatrix}$, $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

C10 $L : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by $L(a + bx + cx^2) = \begin{bmatrix} a + c \\ b - a \end{bmatrix}$, $\mathcal{B} = \{1 + x^2, 1 + x, -1 + x + x^2\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

4.7 Isomorphisms of Vector Spaces

Some of the ideas discussed in Chapters 3 and 4 lead to generalizations that are important in the further development of linear algebra (and also in abstract algebra). Some of these generalizations are outlined in this section. Most of the proofs are easy or simple variations on proofs given earlier, so they will be left as exercises.

One-to-One and Onto

Definition
One-to-one

A linear mapping $L : \mathbb{U} \to \mathbb{V}$ is said to be **one-to-one** if $L(\mathbf{u}_1) = L(\mathbf{u}_2)$ implies that $\mathbf{u}_1 = \mathbf{u}_2$.

That is, L is one-to-one if for each $\mathbf{w} \in \text{Range}(L)$ there is only one $\mathbf{v} \in \mathbb{V}$ such that $L(\mathbf{v}) = \mathbf{w}$. Figure 4.7.1 represents a one-to-one function, while Figure 4.7.2 represents a function that is not one-to-one (since there is an element in $\mathbb{W}$ that is mapped to by two vectors in $\mathbb{V}$).

Definition
Onto

A linear mapping $L : \mathbb{U} \to \mathbb{V}$ is said to be **onto** if for every $\mathbf{v} \in \mathbb{V}$ there exists some $\mathbf{u} \in \mathbb{U}$ such that $L(\mathbf{u}) = \mathbf{v}$.

By definition, L is onto if and only if $\text{Range}(L) = \mathbb{W}$. Thus, Figure 4.7.1 represents a function that is not onto (there is a vector in $\mathbb{W}$ that is not mapped to by a vector in $\mathbb{V}$), while Figure 4.7.2 represents an onto function.

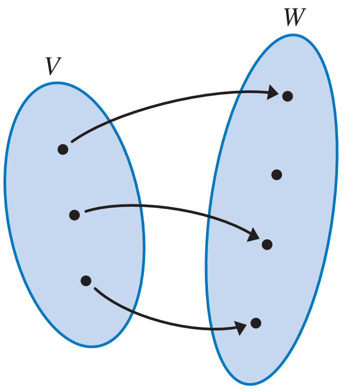


Figure 4.7.1 One-to-one, not onto.

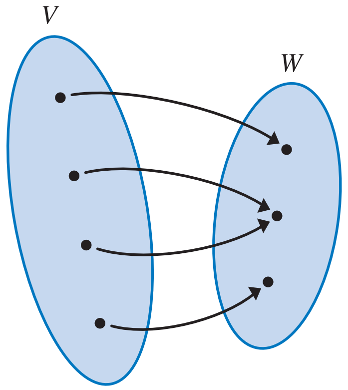


Figure 4.7.2 Onto, not one-to-one.

EXAMPLE 4.7.1

Prove that the linear mapping $L : \mathbb{R}^2 \to P_2(\mathbb{R})$ defined by $L(a_1, a_2) = a_1 + a_2x^2$ is one-to-one, but not onto.

Solution: If $L(a_1, a_2) = L(b_1, b_2)$, then $a_1 + a_2x^2 = b_1 + b_2x^2$ and hence $a_1 = b_1$ and $a_2 = b_2$. Thus, by definition, L is one-to-one.

L is not onto, since there is no $\vec{a} \in \mathbb{R}^2$ such that $L(\vec{a}) = x$.

EXAMPLE 4.7.2

Prove that the linear mapping $L : \mathbb{R}^3 \to P_1(\mathbb{R})$ defined by $L(y_1, y_2, y_3) = y_1 + (y_2 + y_3)x$ is onto, but not one-to-one.

Solution: Let $a + bx$ be any polynomial in $P_1(\mathbb{R})$. We need to find a vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$, such that $L(\vec{y}) = a + bx$. We observe that we have

$$L(a, b, 0) = a + (b + 0)x = a + bx$$

So, L is onto. Notice that we also have

$$L(a, 0, b) = a + (0 + b)x = a + bx$$

Hence, L is not one-to-one.

EXAMPLE 4.7.3

Prove that the linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x_1, x_2) = (x_1, x_1 + x_2)$ is one-to-one and onto.

Solution: Assume that $L(x_1, x_2) = L(y_1, y_2)$. Then we have $(x_1, x_1 + x_2) = (y_1, y_1 + y_2)$, and so $x_1 = y_1$ and $x_2 = y_2$. Thus, L is one-to-one.

Take any vector $(a, b) \in \mathbb{R}^2$. Observe that

$$L(a, b - a) = (a, a + (b - a)) = (a, b)$$

Thus, L is onto.

We have already observed the connection between L being onto and the range of L. We now establish a relationship between a mapping being one-to-one and its nullspace. Both of these connections will be exploited in some proofs below.

Theorem 4.7.1

A linear mapping $L : \mathbb{U} \to \mathbb{V}$ is one-to-one if and only if $\text{Null}(L) = \{\mathbf{0}\}$.

You are asked to prove Theorem 4.7.1 as Problem C1.

EXERCISE 4.7.1

Suppose that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set in $\mathbb{U}$ and $L : \mathbb{U} \to \mathbb{V}$ is one-to-one. Prove that $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_k)\}$ is linearly independent.

EXERCISE 4.7.2

Suppose that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a spanning set for $\mathbb{U}$ and $L : \mathbb{U} \to \mathbb{V}$ is onto. Prove that a spanning set for $\mathbb{V}$ is $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_k)\}$.

Theorem 4.7.2

The linear mapping $L : \mathbb{U} \to \mathbb{V}$ has an inverse linear mapping $L^{-1} : \mathbb{V} \to \mathbb{U}$ if and only if L is one-to-one and onto.

You are asked to prove Theorem 4.7.2 as Problem C4.

Isomorphisms

Our goal now is to identify the condition under which two vector spaces are "equal", and to create a mechanism for changing between two such vector spaces.

For two vector spaces $\mathbb{V}$ and $\mathbb{W}$ to be "equal" we need each vector $\mathbf{v} \in \mathbb{V}$ to correspond to a unique vector $\mathbf{w} \in \mathbb{W}$. However, to ensure that $\mathbb{V}$ and $\mathbb{W}$ have the same "structure", we also need to ensure that a linear combination of corresponding vectors gives corresponding results. That is, if $\mathbf{v}_i$ corresponds to $\mathbf{w}_i$ for $1 \leq i \leq n$, then $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ corresponds to $c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$.

To have a unique correspondence between vectors in $\mathbb{V}$ and $\mathbb{W}$, we need a mapping between $\mathbb{V}$ and $\mathbb{W}$ that is one-to-one and onto. To ensure that linear combinations of these vectors also correspond with one another, we need the mapping to be linear.

Definition
Isomorphism
Isomorphic

If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over $\mathbb{R}$, and if $L : \mathbb{V} \to \mathbb{W}$ is a linear, one-to-one, and onto mapping, then L is called an **isomorphism** (or a vector space isomorphism), and $\mathbb{V}$ and $\mathbb{W}$ are said to be **isomorphic**.

The word *isomorphism* comes from Greek words meaning "same form." The concept of an isomorphism is a very powerful and important one. It implies that the essential structure of the isomorphic vector spaces is the same. So, a vector space statement that is true in one space is immediately true in any isomorphic space. Of course, some vector spaces such as $M_{m\times n}(\mathbb{R})$ or $P_n(\mathbb{R})$ have some features that are not purely vector space properties (such as matrix multiplication and polynomial factorization), and these particular features cannot automatically be transferred.

EXAMPLE 4.7.4

Prove that $P_2(\mathbb{R})$ and $\mathbb{R}^3$ are isomorphic by constructing an explicit isomorphism L.

Solution: We define $L : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $L(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

We need to prove that L is linear, one-to-one, and onto.

Linear: Let $\mathbf{p}(x) = a_0 + a_1x + a_2x^2, \mathbf{q}(x) = b_0 + b_1x + b_2x^2 \in P_2(\mathbb{R})$, and let $s, t \in \mathbb{R}$. Then, L is linear since

$$\begin{aligned} L(s\mathbf{p} + t\mathbf{q}) &= L(sa_0 + tb_0 + (sa_1 + tb_1)x + (sa_2 + tb_2)x^2) \\ &= \begin{bmatrix} sa_0 + tb_0 \\ sa_1 + tb_1 \\ sa_2 + tb_2 \end{bmatrix} \\ &= s\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + t\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = sL(\mathbf{p}) + tL(\mathbf{q}) \end{aligned}$$

One-to-one: Let $a_0 + a_1x + a_2x^2 \in \text{Null}(L)$. Then, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = L(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.
Hence, $a_0 = a_1 = a_2 = 0$, so $\text{Null}(L) = \{0\}$ and thus L is one-to-one by Theorem 4.7.1.

Onto: For any $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^3$, we have $L(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. Hence, L is onto.

Thus, L is an isomorphism from $P_2(\mathbb{R})$ to $\mathbb{R}^3$. Hence $P_2(\mathbb{R})$ and $\mathbb{R}^3$ are isomorphic.

It is instructive to think carefully about the isomorphism in the example above. Observe that the images of the standard basis vectors for $P_2(\mathbb{R})$ are the standard basis vectors for $\mathbb{R}^3$. That is, the isomorphism is mapping basis vectors to basis vectors. We will keep this in mind when constructing an isomorphism in the next example.

EXAMPLE 4.7.5

Let $\mathbb{T} = \{\mathbf{p} \in P_2(\mathbb{R}) \mid \mathbf{p}(1) = 0\}$ and $\mathbb{D} = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a, b \in \mathbb{R}\right\}$. Prove that $\mathbb{T}$ and $\mathbb{S}$ are isomorphic.

Solution: By the Factor Theorem, every vector $\mathbf{p} \in \mathbb{T}$ has the form $(1-x)(a+bx)$. Consequently, a basis for $\mathbb{T}$ is $\mathcal{B} = \{1-x, x-x^2\}$. We see that a basis for $\mathbb{D}$ is $C = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. To define an isomorphism, we can map the vectors in $\mathcal{B}$ to the vectors in C. In particular, we define $L : \mathbb{T} \to \mathbb{S}$ by

$$L\Big(a(1-x) + b(x-x^2)\Big) = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear: Let $\mathbf{p}(x) = a_1(1-x) + b_1(x-x^2), \mathbf{q}(x) = a_2(1-x) + b_2(x-x^2) \in \mathbb{T}$. For any $s, t \in \mathbb{R}$ we get

$$\begin{aligned} L(s\mathbf{p} + t\mathbf{q}) &= L\Big((sa_1 + ta_2)(1-x) + (sb_1 + tb_2)(x-x^2)\Big) \\ &= (sa_1 + ta_2)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (sb_1 + tb_2)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= s\left(a_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b_1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) + t\left(a_2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b_2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= sL(\mathbf{p}) + tL(\mathbf{q}) \end{aligned}$$

Thus, L is linear.

One-to-one: Let $a(1-x) + b(x-x^2) \in \ker(L)$. Then,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L\Big(a(1-x) + b(x-x^2)\Big) = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Hence, $a = b = 0$, so $\mathrm{Null}(L) = \{\mathbf{0}\}$. Therefore, by Theorem 4.7.1, L is one-to-one.

Onto: Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathbb{S}$. We have

$$L\Big(a(1-x) + b(x-x^2)\Big) = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

So, L is onto.

Thus, L is an isomorphism, and hence $\mathbb{T}$ and $\mathbb{S}$ are isomorphic.

Observe in the examples above that the isomorphic vector spaces have the same dimension. This makes sense since if we are mapping basis vectors to basis vectors, then both vector spaces need to have the same number of basis vectors.

Theorem 4.7.3 Suppose that $\mathbb{U}$ and $\mathbb{V}$ are finite-dimensional vector spaces over $\mathbb{R}$. Then $\mathbb{U}$ and $\mathbb{V}$ are isomorphic if and only if they are of the same dimension.

You are asked to prove Theorem 4.7.3 as Problem C5.

EXAMPLE 4.7.6

1. The vector space $M_{m\times n}(\mathbb{R})$ is isomorphic to $\mathbb{R}^{mn}$.
2. The vector space $P_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n+1}$.
3. Every k-dimensional subspace of $\mathbb{R}^n$ is isomorphic to every k-dimensional subspace of $M_{m\times n}(\mathbb{R})$.

If we know that two vector spaces over $\mathbb{R}$ have the same dimension, then Theorem 4.7.3 says that they are isomorphic. However, even if we already know that two vector spaces are isomorphic, we may need to construct an explicit isomorphism between the two vector spaces. The following theorem shows that if we have two isomorphic vector spaces $\mathbb{U}$ and $\mathbb{V}$, then we only have to check if a linear mapping $L : \mathbb{U} \to \mathbb{V}$ is one-to-one or onto to prove that it is an isomorphism between these two spaces.

Theorem 4.7.4 If $\mathbb{U}$ and $\mathbb{V}$ are n-dimensional vector spaces over $\mathbb{R}$, then a linear mapping $L : \mathbb{U} \to \mathbb{V}$ is one-to-one if and only if it is onto.

You are asked to prove Theorem 4.7.4 as Problem C6.

Remark

If $\mathbb{V}$ and $\mathbb{W}$ are both n-dimensional, then it does **not** mean that every linear mapping $L : \mathbb{V} \to \mathbb{W}$ must be an isomorphism. For example, $L : \mathbb{R}^4 \to M_{2\times 2}(\mathbb{R})$ defined by $L(x_1, x_2, x_3, x_4) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is definitely not one-to-one nor onto!

We saw that we can define an isomorphism by mapping basis vectors of one vector space to the basis vectors of the other vector space. The following theorem shows that this property actually characterizes isomorphisms.

Theorem 4.7.5 Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{V}$ and let $\mathbb{V}$ and $\mathbb{W}$ be isomorphic vector spaces. A linear mapping $L : \mathbb{V} \to \mathbb{W}$ is an isomorphism if and only if $\{L(\vec{v}_1), \dots, L(\vec{v}_n)\}$ is a basis for $\mathbb{W}$.

You are asked to prove Theorem 4.7.5 as Problem C7.

PROBLEMS 4.7

Practice Problems

For Problems A1–A12, determine whether the linear mapping is one-to-one and/or onto.

A1 $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$L(x_1, x_2) = (2x_1 - x_2, 3x_1 - 5x_2)$

A2 $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by
$L(x_1, x_2) = (x_1 + 2x_2, x_1 - x_2, x_1 + x_2)$

A3 $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$

A4 $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$L(x_1, x_2, x_3) = \begin{bmatrix} x_1 + 3x_3 \\ x_1 + x_2 + x_3 \\ -x_1 - 2x_2 + x_3 \end{bmatrix}$$

A5 $L : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by
$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+c & b+c \\ a+d & b \end{bmatrix}$$

A6 $L : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^3$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+c \\ b-c \\ a+b \end{bmatrix}$

A7 $L : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ defined by
$L(a + bx) = (3a - b) + (-3a + b)x$

A8 $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$L(a + bx + cx^2) = (a + 4b) + (2b + c)x + (a - 2c)x^2$

A9 $L : P_1(\mathbb{R}) \to \mathbb{R}_2$ defined by $L(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \end{bmatrix}$

A10 $L : P_1(\mathbb{R}) \to \mathbb{R}_3$ defined by $L(a + bx) = \begin{bmatrix} a+b \\ a-b \\ a+b \end{bmatrix}$

A11 $L : P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by
$$L(a + bx + cx^2) = \begin{bmatrix} a-b & b-c \\ a-c & 2a+b-3c \end{bmatrix}$$

A12 $L : \mathbb{R}^3 \to P_2(\mathbb{R})$ defined by
$L(a, b, c) = (-b + 2c) + (a + 3b - c)x + (2a + 2b)x^2$

For Problems, A13–A21, determine whether the given vectors spaces are isomorphic. If so, then give, with proof, an explicit isomorphism.

A13 $P_2(\mathbb{R})$ and $\mathbb{R}^2$

A14 $P_3(\mathbb{R})$ and $\mathbb{R}^4$

A15 $M_{2\times2}(\mathbb{R})$ and $\mathbb{R}^4$

A16 $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$

A17 Any plane through the origin in $\mathbb{R}^3$ and $\mathbb{R}^2$

A18 The subspace $\mathbb{D}$ of diagonal matrices in $M_{2\times2}(\mathbb{R})$ and $P_1(\mathbb{R})$

A19 The subspace $\mathbb{T} = \{\vec{x} \in \mathbb{R}^4 \mid x_1 = x_4\}$ and $M_{2\times2}(\mathbb{R})$

A20 The subspace $\mathbb{S}$ of matrices in $M_{2\times2}(\mathbb{R})$ that satisfy $A^T = A$ and the subspace $\mathbb{V} = \{\vec{x} \in \mathbb{R}^3 \mid x_1 - x_2 = x_3\}$ of $\mathbb{R}^3$

A21 The subspace $\mathbb{U}$ of upper triangular matrices in $M_{2\times2}(\mathbb{R})$ and the subspace $\mathbb{P} = \{\mathbf{p} \in P_3(\mathbb{R}) \mid \mathbf{p}(1) = 0\}$ of $P_3(\mathbb{R})$

Homework Problems

For Problems B1–B7, determine whether the linear mapping is one-to-one and/or onto.

B1 $L : \mathbb{R}^2 \to P_1(\mathbb{R})$ defined by
$L(a, b) = (a + b) + (a + b)x$

B2 $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $L(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_2 \end{bmatrix}$

B3 $L : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$L(a + bx + cx^2 + dx^3) = (a + b) + (b + c)x + (c + d)x^2$

B4 $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by
$L(a + bx + cx^2) = (-2b - 3c) + (2a + c)x + (-3a + b)x^2$

B5 $L : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ defined by
$L(a + bx) = (3a - 2b) + (2a + b)x$

B6 $L : \mathbb{R}^2 \to M_{2\times2}(\mathbb{R})$ defined by
$$L(x_1, x_2) = \begin{bmatrix} x_1 & x_1 \\ x_1 & x_2 \end{bmatrix}$$

B7 Let $\mathbb{U}$ be the subspace of $M_{2\times2}(\mathbb{R})$ of upper triangular matrices. $L : \mathbb{U} \to \mathbb{R}^3$ defined by
$$L\left(\begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ 3x_2 + 2x_3 \\ -x_1 + x_2 \end{bmatrix}$$

For Problems, B8–B14, determine whether the given vectors spaces are isomorphic. If so, then give, with proof, an explicit isomorphism.

B8 $P_1(\mathbb{R})$ and $\mathbb{R}^2$

B9 $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$

B10 $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$

B11 $\mathbb{R}^2$ and the subspace $\mathcal{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ of $\mathbb{R}^3$

B12 The subspace $\mathbb{S} = \left\{\begin{bmatrix}x_1\\x_2\\0\\x_3\end{bmatrix} \mid x_1 - x_2 + x_3 = 0\right\}$ of $\mathbb{R}^4$ and the subspace $\mathbb{U} = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid a = b\}$ of $P_2(\mathbb{R})$

B13 The subspace $\mathbb{S}$ of matrices in $M_{2\times 2}(\mathbb{R})$ that satisfy $A^T = -A$ and the subspace $\mathbb{V} = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = x_3\}$ of $\mathbb{R}^3$

B14 Any n-dimensional vector space $\mathbb{V}$ and $\mathbb{R}^n$

Conceptual Problems

C1 Prove L is one-to-one if and only if $\text{Null}(L) = \{\mathbf{0}\}$.

C2 (a) Prove that if L and M are one-to-one, then $M \circ L$ is one-to-one.

(b) Give an example where M is not one-to-one but $M \circ L$ is one-to-one.

(c) Is it possible to give an example where L is not one-to-one but $M \circ L$ is one-to-one? Explain.

C3 Prove that if L and M are onto, then $M \circ L$ is onto.

C4 Prove that the linear mapping $L : \mathbb{U} \to \mathbb{V}$ has an inverse linear mapping $L^{-1} : \mathbb{V} \to \mathbb{U}$ if and only if L is one-to-one and onto.

C5 Suppose $\mathbb{U}$ and $\mathbb{V}$ are finite-dimensional vector spaces over $\mathbb{R}$ and that $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis for $\mathbb{U}$.

(a) Prove if $\mathbb{U}$ and $\mathbb{V}$ are isomorphic, then $\dim \mathbb{U} = \dim \mathbb{V}$ by using an isomorphism $L : \mathbb{U} \to \mathbb{V}$ to construct a basis for $\mathbb{V}$. Make sure to prove that your basis is linearly independent and spans $\mathbb{V}$).

(b) Prove if $\dim \mathbb{U} = \dim \mathbb{V}$, then $\mathbb{U}$ and $\mathbb{V}$ are isomorphic by defining an explicit isomorphism between them . Make sure to prove that this is an isomorphism.

C6 Prove if $\mathbb{U}$ and $\mathbb{V}$ are n-dimensional vector spaces over $\mathbb{R}$, then a linear mapping $L : \mathbb{U} \to \mathbb{V}$ is one-to-one if and only if it is onto.

C7 Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{V}$, let $\mathbb{V}$ and $\mathbb{W}$ be isomorphic vector spaces, and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Prove $\{L(\vec{v}_1), \dots, L(\vec{v}_n)\}$ is a basis for $\mathbb{W}$ if and only if L is an isomorphism.

For Problems C8–C10, recall the definition of the Cartesian product from Section 4.2 Problem C7.

C8 Prove that $\mathbb{U} \times \{\mathbf{0}_\mathbb{V}\}$ is a subspace of $\mathbb{U} \times \mathbb{V}$ that is isomorphic to $\mathbb{U}$.

C9 Prove that $\mathbb{R}^2 \times \mathbb{R}$ is isomorphic to $\mathbb{R}^3$.

C10 Prove that $\mathbb{R}^n \times \mathbb{R}^m$ is isomorphic to $\mathbb{R}^{n+m}$.

C11 Suppose that $L : \mathbb{U} \to \mathbb{V}$ is a vector space isomorphism and that $M : \mathbb{V} \to \mathbb{V}$ is a linear mapping. Prove that $L^{-1} \circ M \circ L$ is a linear mapping from $\mathbb{U}$ to $\mathbb{U}$. Describe the nullspace and range of $L^{-1} \circ M \circ L$ in terms of the nullspace and range of M.

For Problems C12–C15, prove or disprove the statement. Assume that $\mathbb{U}, \mathbb{V}, \mathbb{W}$ are all finite dimensional vector spaces.

C12 If $\dim \mathbb{V} = \dim \mathbb{U}$ and $L : \mathbb{U} \to \mathbb{V}$ is linear, then L is an isomorphism.

C13 If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping and $\dim \mathbb{V} < \dim \mathbb{W}$, then L cannot be onto.

C14 If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping and $\dim \mathbb{V} > \dim \mathbb{W}$, then L is onto.

C15 If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping and $\dim \mathbb{V} > \dim \mathbb{W}$, then L cannot be one-to-one.

CHAPTER REVIEW

Suggestions for Student Review

1 State the ten properties of a vector space over $\mathbb{R}$. Why is the empty set not a vector space? Describe two or three examples of vector spaces that are not subspaces of $\mathbb{R}^n$. (Section 4.2)

2 State the definition of a subspace of a vector space $\mathbb{V}$. (Section 4.2)

3 State the definitions of spanning, linear independence, and bases. State the two ways we have of finding a basis. (Section 4.3)

4 (a) Explain the concept of dimension. What theorem is required to ensure that the concept of dimension is well defined? (Section 4.3)

(b) Explain how knowing the dimension of a vector space is helpful when you have to find a basis for the vector space. (Section 4.3)

5 State the definition of $\mathcal{B}$-coordinates and the $\mathcal{B}$-coordinate vector of a vector $\mathbf{v}$ in a vector space $\mathbb{V}$. What theorem is required to ensure that the concept of coordinates is well defined? (Section 4.4)

6 Create and analyze an example as follows.

(a) Give a basis $\mathcal{B}$ for a three-dimensional subspace in $\mathbb{R}^5$. (Do not make it too easy by choosing any standard basis vectors, but do not make it too hard by choosing completely random components.) (Section 4.3)

(b) Determine the standard coordinates in $\mathbb{R}^5$ of the vector that has coordinate vector $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ with respect to your basis $\mathcal{B}$. (Section 4.4)

(c) Take the vector you found in (b) and carry out the standard procedure to determine its coordinates with respect to $\mathcal{B}$. Did you get the right answer, $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$? (Section 4.4)

(d) Pick any two vectors in $\mathbb{R}^5$ and determine whether they lie in your subspace. Determine the coordinates of any vector that is in the subspace. (Section 4.4)

7 Write a short explanation of how you use information about consistency of systems and uniqueness of solutions in testing for linear independence and in determining whether a vector belongs to a given subspace. (Sections 4.3 and 4.4)

8 Define the change of coordinates matrix. Create two different bases $\mathcal{B}$ and $\mathcal{C}$ for $P_1(\mathbb{R})$. Find the change of coordinates matrix P from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates and its inverse. Check your answer in as many ways as you can. (Section 4.4)

9 Give the definition of a linear mapping $L : \mathbb{V} \to \mathbb{W}$. Explain the procedure for determining if a vector $\mathbf{y}$ is in the range of L. Describe how to find a basis for the nullspace and a basis for the range of L. (Section 4.5)

10 State the definition of the zero mapping and the identity mapping. (Section 4.5)

11 State the Rank-Nullity Theorem and demonstrate it with an example. In what kinds of problems have we used the Rank-Nullity Theorem? (Section 4.5 and 4.7)

12 State the definition of the inverse of a linear mapping and explain how it is related to the content in Sections 4.6 and 4.7. (Section 4.5, 4.6, and 4.7)

13 State how to determine the standard matrix and the $\mathcal{B}$-matrix of a linear mapping $L : \mathbb{V} \to \mathbb{V}$. Explain how $[L(\vec{x})]_{\mathcal{B}}$ is determined in terms of $[L]_{\mathcal{B}}$. (Section 4.6)

14 State the relationship between the matrix of a linear mapping and the change of coordinates matrix. Demonstrate with an example. (Section 4.6)

15 State and explain the concepts of one-to-one and onto. (Section 4.7)

16 State the definition of an isomorphism of vector spaces and give some examples. Explain why a finite-dimensional vector space cannot be isomorphic to a proper subspace of itself. (Section 4.7)

Chapter Quiz

For Problems E1–E4, determine whether the set is a vector space. Explain briefly.

E1 The set of 4×3 matrices such that the sum of the entries in the first row is zero ($a_{11} + a_{12} + a_{13} = 0$) under standard addition and scalar multiplication of matrices.

E2 The set of polynomials $\mathbf{p}(x) \in P_3(\mathbb{R})$ such that $\mathbf{p}(1) = 0$ and $\mathbf{p}(2) = 0$ under standard addition and scalar multiplication of polynomials.

E3 The set of 2×2 matrices such that all entries are integers under standard addition and scalar multiplication of matrices.

E4 The set of all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ such that $x_1+x_2+x_3 = 0$ under standard addition and scalar multiplication of vectors in $\mathbb{R}^3$.

For Problems E5–E8, determine with proof whether the given set is a subspace of the given vector space. If the set is a subspace, find a basis for the subspace.

E5 $\mathbb{S} = \{\mathbf{p} \in P_2(\mathbb{R}) \mid \mathbf{p}(0) = 1\}$ of $P_2(\mathbb{R})$

E6 $\mathbb{S} = \left\{\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a + b = -2c\right\}$ of $M_{2\times 2}(\mathbb{R})$

E7 $\mathbb{S} = \{a + cx^2 \in P_2(\mathbb{R}) \mid a = c\}$ of $P_2(\mathbb{R})$

E8 $\mathbb{S} = \operatorname{Span}\{1 + x^2, 2 + x, x - 2x^2\}$ of $P_2(\mathbb{R})$

For Problems E9–E11, determine whether the given set is a basis for $M_{2\times 2}(\mathbb{R})$.

E9 $\left\{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}\right\}$

E10 $\left\{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 4 & -2 \end{bmatrix}\right\}$

E11 $\left\{\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\right\}$

E12 (a) Let $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}\right\}$. Find a basis $\mathcal{B}$ for $\mathbb{S}$ and hence determine the dimension of $\mathbb{S}$.

(b) Determine the $\mathcal{B}$-coordinates of $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \end{bmatrix}$.

E13 Let $\mathcal{B} = \{1 + x, 2 - x\}$ and $\mathcal{C} = \{1 - 3x, 1 + 2x\}$ both be basis for $P_1(\mathbb{R})$. Find the change of coordinates matrix Q from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates and the change of coordinates matrix P from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

E14 (a) Find a basis for the plane in $\mathbb{R}^3$ with equation $x_1 - x_3 = 0$.

(b) Extend the basis you found in (a) to a basis $\mathcal{B}$ for $\mathbb{R}^3$.

(c) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection in the plane from part (a). Determine $[L]_{\mathcal{B}}$.

(d) Using your result from part (c), determine the standard matrix $[L]_{\mathcal{S}}$ of the reflection.

E15 Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping with standard matrix $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$ and let $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$. Determine the matrix $[L]_{\mathcal{B}}$.

E16 Suppose that $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping with $\operatorname{Null}(L) = \{\mathbf{0}\}$. Suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set in $\mathbb{V}$. Prove that $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is a linearly independent set in $\mathbb{W}$.

For Problems E17–E22, decide whether each of the following statements is true or false. If it is true, explain briefly; if it is false, give an example to show that it is false.

E17 A subspace of $\mathbb{R}^n$ must have dimension less than n.

E18 A set of four polynomials in $P_2(\mathbb{R})$ cannot be a basis for $P_2(\mathbb{R})$.

E19 If $\mathcal{B}$ is a basis for a subspace of $\mathbb{R}^5$, then the $\mathcal{B}$-coordinate vector of some vector $\vec{x} \in \mathbb{R}^5$ has five components.

E20 For any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ and any basis $\mathcal{B}$ of $\mathbb{R}^n$, the rank of the matrix $[L]_{\mathcal{B}}$ is the same as the rank of the matrix $[L]_{\mathcal{S}}$.

E21 For any linear mapping $L : \mathbb{V} \to \mathbb{V}$ and any basis $\mathcal{B}$ of $\mathbb{V}$, the column space of $[L]_{\mathcal{B}}$ equals the range of L.

E22 If $L : \mathbb{V} \to \mathbb{W}$ is one-to-one, then $\dim \mathbb{V} = \dim \mathbb{W}$.

Further Problems

These problems are intended to be challenging.

F1 Let $\mathbb{S}$ be a subspace of an n-dimensional vector space $\mathbb{V}$. Prove that there exists a linear operator $L : \mathbb{V} \to \mathbb{V}$ such that $\text{Null}(L) = \mathbb{S}$.

F2 Use the ideas of this chapter to prove the uniqueness of the reduced row echelon form for a given matrix A. (Hint: begin by assuming that there are two reduced row echelon forms R and S. What can you say about the columns with leading 1s in the two matrices?)

F3 **Magic Squares—An Exploration of Their Vector Space Properties**

We say that any matrix $A \in M_{3\times 3}(\mathbb{R})$ is a 3×3 **magic square** if the three **row sums** (where each row sum is the sum of the entries in one row of A) of A, the three **column sums** of A, and the two **diagonal sums** of A ($a_{11} + a_{22} + a_{33}$ and $a_{13} + a_{22} + a_{31}$) all have the same value k. The common sum k is called the **weight** of the magic square A and is denoted by $wt(A) = k$.

For example, $A = \begin{bmatrix} 2 & 2 & -1 \\ -2 & 1 & 4 \\ 3 & 0 & 0 \end{bmatrix}$ is a magic square with $wt(A) = 3$.

The aim of this exploration is to find all 3×3 magic squares. The subset of $M_{3\times 3}(\mathbb{R})$ consisting of magic squares is denoted MS_3.

(a) Show that MS_3 is a subspace of $M_{3\times 3}(\mathbb{R})$.

(b) Observe that weight determines a map $wt : MS_3 \to \mathbb{R}$. Show that wt is linear.

(c) Compute the nullspace of wt. Suppose that

$$\underline{X}_1 = \begin{bmatrix} 1 & 0 & a \\ b & c & d \\ e & f & g \end{bmatrix}, \quad \underline{X}_2 = \begin{bmatrix} 0 & 1 & h \\ i & j & k \\ l & m & n \end{bmatrix}$$

and

$$\underline{0} = \begin{bmatrix} 0 & 0 & p \\ q & r & s \\ t & u & v \end{bmatrix}$$

are all in the nullspace, where $a, b, c, \dots, v$ denote unknown entries. Determine these unknown entries and prove that $\underline{X}_1$ and $\underline{X}_2$ form a basis for $\text{Null}(wt)$. (Hint: if $A \in \text{Null}(wt)$, consider $A - a_{11}\underline{X}_1 - a_{12}\underline{X}_2$.)

(d) Let $\underline{J} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Observe that $\underline{J}$ is a magic square with $wt(\underline{J}) = 3$. Show that all A in MS_3 that have weight k are of the form

$$(k/3)\underline{J} + p\underline{X}_1 + q\underline{X}_2, \quad \text{for some } p, q \in \mathbb{R}$$

(e) Show that $\mathcal{B} = \{\underline{J}, \underline{X}_1, \underline{X}_2\}$ is a basis for MS_3.

(f) Find the coordinates of $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ with respect to the basis $\mathcal{B}$.

Exercises F4–F8 require the following definitions.

If $\mathbb{S}$ and $\mathbb{T}$ are subspaces of the vector space $\mathbb{V}$, we define

$$\mathbb{S} + \mathbb{T} = \{\mathbf{s} + \mathbf{t} \mid \mathbf{s} \in \mathbb{S}, \mathbf{t} \in \mathbb{T}\}$$

If $\mathbb{S}$ and $\mathbb{T}$ are subspaces of $\mathbb{V}$ such that $\mathbb{S} + \mathbb{T} = \mathbb{V}$ and $\mathbb{S} \cap \mathbb{T} = \{\mathbf{0}\}$, then we say that $\mathbb{S}$ is a **complement** of $\mathbb{T}$ (and $\mathbb{T}$ is a complement of $\mathbb{S}$).

F4 Show that, in general, the complement of a subspace $\mathbb{T}$ is not unique.

F5 In the vector space of continuous real-valued functions of a real variable, show that the even functions and the odd functions form subspaces such that each is the complement of the other.

F6 (a) If $\mathbb{S}$ is a k-dimensional subspace of $\mathbb{R}^n$, show that any complement of $\mathbb{S}$ must be of dimension $n - k$.

(b) Suppose that $\mathbb{S}$ is a subspace of $\mathbb{R}^n$ that has a unique complement. Must it be true that $\mathbb{S}$ is either $\{\mathbf{0}\}$ or $\mathbb{R}^n$?

F7 Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors in a vector space $\mathbb{V}$. Suppose also that $\mathbb{S}$ is a subspace of $\mathbb{V}$. Let $\mathbb{T}$ be the subspace spanned by $\mathbf{v}$ and $\mathbb{S}$. Let $\mathbb{U}$ be the subspace spanned by $\mathbf{w}$ and $\mathbb{S}$. Prove that if $\mathbf{w}$ is in $\mathbb{T}$ but not in $\mathbb{S}$, then $\mathbf{v}$ is in $\mathbb{U}$.

F8 Show that if $\mathbb{S}$ and $\mathbb{T}$ are finite-dimensional subspaces of $\mathbb{V}$, then

$$\dim \mathbb{S} + \dim \mathbb{T} = \dim(\mathbb{S} + \mathbb{T}) + \dim(\mathbb{S} \cap \mathbb{T})$$

CHAPTER 5

Determinants

CHAPTER OUTLINE

In Chapter 3, we saw that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. That is, the value $ad - bc$ determines if A is invertible or not. What is surprising is that the area of the parallelogram formed by vectors $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ according to the parallelogram rule for addition is also $ad - bc$ (see Section 5.4). This quantity $ad - bc$, which we call the **determinant** *of the matrix, turns out to be extremely useful. For example, we will use it to find eigenvalues in Chapter 6, it can help us solve a variety of problems in geometry (Section 5.4), and it is even required in multivariable calculus.*

5.1 Determinants in Terms of Cofactors

Definition
Determinant of a 2×2 Matrix

The **determinant of a 2×2 matrix** $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is defined by

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

An Alternate Notation: The determinant is often denoted by vertical straight lines:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

One risk with this notation is that one may fail to distinguish between a matrix and the determinant of the matrix. This is a rather gross error.

EXAMPLE 5.1.1

Find the determinant of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and the determinant of $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$.

Solution: We have

$$\det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 1(4) - 3(2) = -2$$

$$\det \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = 2(4) - 2(4) = 0$$

EXERCISE 5.1.1

Calculate the following determinants.

(a) $\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}$

The 3×3 *Case*

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. We can show through elimination (with some effort) that A is invertible if and only if

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0$$

We would like to reorganize this expression so that we can remember it more easily, and so that we can determine how to generalize it to the $n \times n$ case. Notice that a_{11} is a common factor in the first pair of terms in D, a_{12} is a common factor in the second pair, and a_{13} is a common factor in the third pair. Thus, D can be rewritten as

$$\begin{aligned} D &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned} \tag{5.1}$$

Observe that the determinant being multiplied by a_{11} in equation (5.1) is the determinant of the 2×2 matrix formed by removing the first row and first column of A. Similarly, a_{12} is being multiplied by (-1) times the determinant of the matrix formed by removing the first row and second column of A, and a_{13} is being multiplied by the determinant of the matrix formed by removing the first row and third column of A. Hence, we make the following definitions.

Definition
Cofactors of a 3×3 Matrix

Let $A \in M_{3\times 3}(\mathbb{R})$ and let $A(i, j)$ denote the 2×2 submatrix obtained from A by deleting the i-th row and j-th column. Define the (i, j)-**cofactor** of A to be

$$C_{ij} = (-1)^{(i+j)} \det A(i, j)$$

EXAMPLE 5.1.2

Find all nine of the cofactors of $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ -2 & -3 & 4 \end{bmatrix}$.

Solution: The $(1,1)$-cofactor C_{11} of A is defined by $C_{11} = (-1)^{1+1} \det A(1,1)$, where $A(1,1)$ is the matrix obtained from A by deleting the first row and first column. That is

$$A(1,1) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ -2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 4 \end{bmatrix}$$

Hence, $C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 3 \\ -3 & 4 \end{vmatrix} = (1)[(-1)(4) - 3(-3)] = 5.$

The $(1,2)$-cofactor C_{12} is defined by $C_{12} = (-1)^{1+2} \det A(1,2)$, where $A(1,2)$ is the matrix obtained from A by deleting the first row and second column. We have

$$A(1,2) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ -2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$$

Thus, $C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 3 \\ -2 & 4 \end{vmatrix} = (-1)[0(4) - 3(-2)] = -6.$

Similarly, the $(1,3)$-cofactor C_{13} is defined by $C_{13} = (-1)^{1+3} \det A(1,3)$, where $A(1,3)$ is the matrix obtained from A by deleting the first row and third column. We have

$$A(1,3) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ -2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix}$$

Thus, $C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} = (1)[0(-3) - (-1)(-2)] = -2.$

Continuing in this way, we can find the other cofactors are:

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ -3 & 4 \end{vmatrix} = (-1)[0(4) - 2(-3)] = -6$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = (1)[1(4) - 2(-2)] = 8$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ -2 & -3 \end{vmatrix} = (-1)[1(-3) - 0(-2)] = 3$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = (1)[0(3) - 2(-1)] = 2$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = (-1)[1(3) - 2(0)] = -3$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = (1)[1(-1) - 0(0)] = -1$$

CONNECTION

Comparing the cross product of the second and third column of the matrix in Example 5.1.2 with the cofactors from the first column of the matrix shows us that

$$\begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$$

That is, the cross product is defined in terms of 2×2 determinants. We will explore this relationship a little further in Section 5.3.

Definition
Determinant of a 3×3 Matrix

The **determinant of a 3×3 matrix** A is defined by

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

EXAMPLE 5.1.3

Let $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 3 & 5 \\ 1 & 0 & 6 \end{bmatrix}$. Calculate the cofactors of the first row of A and use them to find the determinant of A.

Solution: By definition, the $(1, 1)$-cofactor C_{11} is $(-1)^{1+1}$ times the determinant of the matrix obtained from A by deleting the first row and first column. Thus,

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix} = 3(6) - 5(0) = 18$$

The $(1, 2)$-cofactor C_{12} is $(-1)^{1+2}$ times the determinant of the matrix obtained from A by deleting the first row and second column. So,

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = -[2(6) - 5(1)] = -7$$

Finally, the $(1, 3)$-cofactor C_{13} is

$$C_{13} = (-1)^{1+3} \det \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = 2(0) - 3(1) = -3$$

Hence,

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 4(18) + (-1)(-7) + 1(-3) = 76$$

EXERCISE 5.1.2

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 4 & 0 & -3 \end{bmatrix}$. Calculate the cofactors of the first row of A and use them to find the determinant of A.

Generally, when expanding a determinant, we write the steps more compactly, as in the next example.

EXAMPLE 5.1.4

Calculate $\det\begin{bmatrix} 1 & 2 & 3 \\ -2 & 2 & 1 \\ 5 & 0 & -1 \end{bmatrix}$.

Solution: By definition, we have

$$\begin{aligned}
\det\begin{bmatrix} 1 & 2 & 3 \\ -2 & 2 & 1 \\ 5 & 0 & -1 \end{bmatrix} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= 1(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} -2 & 1 \\ 5 & -1 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} -2 & 2 \\ 5 & 0 \end{vmatrix} \\
&= 1[2(-1) - 1(0)] - 2[(-2)(-1) - 1(5)] + 3[(-2)(0) - 2(5)] \\
&= -2 + 6 - 30 = -26
\end{aligned}$$

We now define the determinant of an $n \times n$ matrix by following the pattern of the definition for the 3×3 case.

Definition

Determinant of a $n \times n$ Matrix
Cofactors of an $n \times n$ Matrix

Let $A \in M_{n\times n}(\mathbb{R})$ with $n > 2$. Let $A(i, j)$ denote the $(n-1) \times (n-1)$ submatrix obtained from A by deleting the i-th row and j-th column.
The **determinant** of $A \in M_{n\times n}(\mathbb{R})$ is defined by

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

where the (i, j)-**cofactor** of A is defined to be

$$C_{ij} = (-1)^{(i+j)} \det A(i, j)$$

Remarks

1. This definition of the determinant is called the **Cofactor (Laplace) expansion of the determinant along the first row**. As we shall see in Theorem 5.1.1 below, a determinant can be expanded along any row or column.

2. The signs attached to cofactors can cause trouble if you are not careful. One helpful way to remember which sign to attach to which cofactor is to take a blank matrix and put a + in the top left corner and then alternate − and + both across and down: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$. This is shown for a 3×3 matrix, but it works for a square matrix of any size.

3. This is a recursive definition. The result for the $n \times n$ case is defined in terms of the $(n-1) \times (n-1)$ case, which in turn must be calculated in terms of the $(n-2) \times (n-2)$ case, and so on, until we get back to the 2×2 case, for which the result is given explicitly. Note that this also works with the 1×1 case since we define $\det\begin{bmatrix} a \end{bmatrix} = a$.

EXAMPLE 5.1.5

We calculate the following determinant by using the definition of the determinant. Note that * and ** represent cofactors whose values are irrelevant because they are multiplied by 0.

$$\begin{aligned}
\begin{vmatrix} 0 & 2 & 3 & 0 \\ 1 & 5 & 6 & 7 \\ -2 & 3 & 0 & 4 \\ -5 & 1 & 2 & 3 \end{vmatrix} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} \\
&= 0(*) + 2(-1)^{1+2}\begin{vmatrix} 1 & 6 & 7 \\ -2 & 0 & 4 \\ -5 & 2 & 3 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 1 & 5 & 7 \\ -2 & 3 & 4 \\ -5 & 1 & 3 \end{vmatrix} + 0(**) \\
&= -2\left(1(-1)^{1+1}\begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix} + 6(-1)^{1+2}\begin{vmatrix} -2 & 4 \\ -5 & 3 \end{vmatrix} + 7(-1)^{1+3}\begin{vmatrix} -2 & 0 \\ -5 & 2 \end{vmatrix}\right) \\
&\quad +3\left(1(-1)^{1+1}\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + 5(-1)^{1+2}\begin{vmatrix} -2 & 4 \\ -5 & 3 \end{vmatrix} + 7(-1)^{1+3}\begin{vmatrix} -2 & 3 \\ -5 & 1 \end{vmatrix}\right) \\
&= -2((0-8) - 6(-6+20) + 7(-4-0)) \\
&\quad +3((9-4) - 5(-6+20) + 7(-2+15)) \\
&= -2(-8-84-28) + 3(5-70+91) \\
&= -2(-120) + 3(26) = 318
\end{aligned}$$

It is apparent that evaluating the determinant of a 4×4 matrix is a fairly lengthy calculation and will get worse for larger matrices. In applications it is not uncommon to have a matrix with thousands (or even millions) of columns. Thus, in this section and the next section, we will examine some theorems which help us evaluate determinants more efficiently.

Theorem 5.1.1

The determinant of $A \in M_{n\times n}(\mathbb{R})$ may be obtained by a **cofactor expansion** along any row or any column. In particular, the expansion of the determinant along the i-th row of A is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The expansion of the determinant along the j-th column of A is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

We omit a proof here since there is no conceptually helpful proof, and it would be a bit grim to verify the result in the general case.

Theorem 5.1.1 is a very practical result. It allows us to *choose* from A the row or column along which we are going to expand. If one row or column has many zeros, it is sensible to expand along it since we do not have to evaluate the cofactors of the zero entries. This was demonstrated in Example 5.1.5, where we had to compute only two cofactors in the first step.

EXAMPLE 5.1.6

Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 0 & -1 \\ 0 & 6 & 0 & 0 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{bmatrix}$, and

$$C = \begin{bmatrix} 4 & 2 & 1 & -1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Solution: For A, we expand along the third column to get

$$\begin{aligned} \det A &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \\ &= (-1)(-1)^{1+3}\begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix} + 0 + 0 \\ &= -1(15 - (-1)) \\ &= -16 \end{aligned}$$

For B, we expand along the second row to get

$$\begin{aligned} \det B &= b_{21}C_{21} + b_{22}C_{22} + b_{23}C_{23} + b_{24}C_{24} \\ &= 0 + 6(-1)^{2+2}\begin{vmatrix} 3 & 0 & -1 \\ 4 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix} + 0 + 0 \end{aligned}$$

We now expand the 3×3 determinant along the second column to get

$$\begin{aligned} \det B &= 0 + 6\left(2(-1)^{2+2}\begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix}\right) + 0 \\ &= 6(2)(3 - (-3)) \\ &= 72 \end{aligned}$$

For C we continuously expand along the bottom row to get

$$\begin{aligned} \det C &= c_{41}C_{41} + c_{42}C_{42} + c_{43}C_{43} + c_{44}C_{44} \\ &= 0 + 0 + 0 + 4(-1)^{4+4}\begin{vmatrix} 4 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{vmatrix} \\ &= 4\left(0 + 0 + (-1)(-1)^{3+3}\begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix}\right) \\ &= 4(-1)(4(2) - 0) = 4(-1)(4)(2) \\ &= -32 \end{aligned}$$

EXERCISE 5.1.3

Calculate the determinant of $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 3 & 5 & -1 & 0 \\ -2 & 2 & -4 & 0 \end{bmatrix}$ by

(a) Expanding along the first column
(b) Expanding along the second row
(c) Expanding along the fourth column

Exercise 5.1.3 demonstrates the usefulness of doing a cofactor expansion along the row or column with the most zeros. Of course, if one row or column contains only zeros, then it is really easy.

Theorem 5.1.2 If one row (or column) of $A \in M_{n\times n}(\mathbb{R})$ contains only zeros, then $\det A = 0$.

Proof: If the i-th row of A contains only zeros, then expanding the determinant along the i-th row of A gives

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = 0 + 0 + \cdots + 0 = 0$$ ■

As we saw with the matrix C in Example 5.1.6, another useful special case is when the matrix is upper or lower triangular.

Theorem 5.1.3 If $A \in M_{n\times n}(\mathbb{R})$ is an upper or lower triangular matrix, then the determinant of A is the product of the diagonal entries of A. That is,

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

The proof is left as Problem C1.

Finally, recall that taking the transpose of a matrix A turns rows into columns and vice versa. That is, the columns of A^T are identical to the rows of A. Thus, if we expand the determinant of A^T along its first column, we will get the same cofactors and coefficients we would get by expanding the determinant of A along its first row. We get the following theorem.

Theorem 5.1.4 If $A \in M_{n\times n}(\mathbb{R})$, then $\det A = \det A^T$.

With the tools we have so far, evaluation of determinants is still a very tedious business. Properties of the determinant with respect to elementary row operations make the evaluation much easier. These properties are discussed in the next section.

PROBLEMS 5.1

Practice Problems

For Problems A1–A9, evaluate the determinant.

A1 $\begin{vmatrix} 2 & -4 \\ 7 & 5 \end{vmatrix}$ **A2** $\begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix}$ **A3** $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$

A4 $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$ **A5** $\begin{vmatrix} 7 & 5 \\ -3 & 7 \end{vmatrix}$ **A6** $\begin{vmatrix} -2 & -1 \\ -5 & -11 \end{vmatrix}$

A7 $\begin{vmatrix} 1 & 3 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix}$ **A8** $\begin{vmatrix} 3 & 4 & 0 & 7 \\ 3 & 4 & 0 & 2 \\ 1 & 5 & 0 & 5 \\ 1 & 2 & 0 & 0 \end{vmatrix}$ **A9** $\begin{vmatrix} 5 & 0 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix}$

For Problems A10–A15, evaluate the determinant by expanding along the first row.

A10 $\begin{vmatrix} 3 & 4 & 0 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{vmatrix}$ **A11** $\begin{vmatrix} 3 & 2 & 1 \\ -1 & 4 & 5 \\ 3 & 2 & 1 \end{vmatrix}$

A12 $\begin{vmatrix} 0 & 5 & 0 \\ 1 & 8 & -9 \\ 0 & \sqrt{2} & 1 \end{vmatrix}$ **A13** $\begin{vmatrix} 5 & 0 & 0 \\ 8 & 1 & -9 \\ \sqrt{2} & 0 & 1 \end{vmatrix}$

A14 $\begin{vmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 1 \\ -4 & 0 & 2 & -2 \\ 3 & -5 & 2 & 1 \end{vmatrix}$ **A15** $\begin{vmatrix} 1 & 0 & 4 & 0 \\ 2 & -3 & 4 & 1 \\ -1 & 3 & 2 & 4 \\ 1 & 1 & -2 & 4 \end{vmatrix}$

For Problems A16 and A17, show that $\det A$ is equal to $\det A^T$ by expanding along the second column of A and the second row of A^T.

A16 $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 5 \end{bmatrix}$ **A17** $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 5 \\ 3 & 0 & 1 & 4 \\ 4 & 5 & 1 & -2 \end{bmatrix}$

For Problems A18–A27, evaluate the determinant by expanding along the row or column of your choice.

A18 $\begin{vmatrix} 3 & 5 & 0 \\ -2 & 6 & 0 \\ 4 & 1 & 0 \end{vmatrix}$ **A19** $\begin{vmatrix} -5 & 2 & -4 \\ 2 & -4 & 6 \\ -6 & 2 & -3 \end{vmatrix}$

A20 $\begin{vmatrix} 1 & -3 & 4 \\ 9 & 5 & 0 \\ 0 & -2 & 0 \end{vmatrix}$ **A21** $\begin{vmatrix} 1 & -3 & 4 \\ 0 & -2 & 0 \\ 9 & 5 & 0 \end{vmatrix}$

A22 $\begin{vmatrix} 2 & 1 & 5 \\ 4 & 3 & -1 \\ 0 & 1 & -2 \end{vmatrix}$ **A23** $\begin{vmatrix} -3 & 4 & 0 & 1 \\ 4 & -1 & 0 & -6 \\ 1 & -1 & 0 & -3 \\ 4 & -2 & 3 & 6 \end{vmatrix}$

A24 $\begin{vmatrix} 1 & 5 & -7 & 8 \\ 2 & -1 & 3 & 0 \\ -4 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$ **A25** $\begin{vmatrix} 8 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 2 & 1 & 0 \end{vmatrix}$

A26 $\begin{vmatrix} 0 & 6 & 1 & 2 \\ 0 & 5 & -1 & 1 \\ 3 & -5 & -3 & -5 \\ 5 & 6 & -3 & -6 \end{vmatrix}$ **A27** $\begin{vmatrix} 1 & 3 & 4 & -5 & 7 \\ 0 & -3 & 1 & 2 & 3 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 4 & 3 \end{vmatrix}$

For Problems A28–A31, calculate the determinant of the elementary matrix.

A28 $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **A29** $E_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A30 $E_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **A31** $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Homework Problems

For Problems B1–B9, evaluate the determinant.

B1 $\begin{vmatrix} 5 & 3 \\ -1 & 7 \end{vmatrix}$ **B2** $\begin{vmatrix} 2 & 1 \\ -1 & 1/2 \end{vmatrix}$ **B3** $\begin{vmatrix} 3 & -4 \\ -6 & 8 \end{vmatrix}$

B4 $\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}$ **B5** $\begin{vmatrix} 5 & 7 \\ -2 & 4 \end{vmatrix}$ **B6** $\begin{vmatrix} 2 & 0 & 0 \\ \sqrt{2} & 5 & 0 \\ -1 & 2 & 3 \end{vmatrix}$

B7 $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{vmatrix}$ **B8** $\begin{vmatrix} 3 & 5 & 2 & 1 \\ 3 & 3 & 9 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$ **B9** $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 1 & 5 & 0 \\ 3 & 1 & 2 & 1 \end{vmatrix}$

For Problems B10–B13, evaluate the determinant by expanding along the first row.

B10 $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 6 & 1 \\ 4 & 1 & 7 \end{vmatrix}$ **B11** $\begin{vmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

B12 $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 3 & 1 \\ 2 & 2 & 1 \end{vmatrix}$ **B13** $\begin{vmatrix} 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & x \end{vmatrix}$

For Problems B14 and B15, show that $\det A$ is equal to $\det A^T$ by expanding along the third row of A and the third column of A^T.

B14 $A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 6 & 4 \\ -2 & 7 & 0 \end{bmatrix}$

B15 $A = \begin{bmatrix} 1 & 8 & -1 & 1 \\ 4 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 8 & -1 & -1 \end{bmatrix}$

For Problems B16–B23, evaluate the determinant by expanding along the row or column of your choice.

B16 $\begin{vmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ 5 & 0 & 4 \end{vmatrix}$

B17 $\begin{vmatrix} 0 & -1 & 2 \\ 3 & 0 & 4 \\ 6 & 9 & 0 \end{vmatrix}$

B18 $\begin{vmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

B19 $\begin{vmatrix} 1 & 0 & -5 \\ 1/2 & 1 & 0 \\ 0 & 8 & 1/2 \end{vmatrix}$

B20 $\begin{vmatrix} 0 & 3 & \pi \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$

B21 $\begin{vmatrix} 4 & 1 & -2 & 5 \\ 0 & 0 & -2 & 0 \\ 3 & 6 & 9 & 0 \\ 2 & 1 & 7 & 1 \end{vmatrix}$

B22 $\begin{vmatrix} 6 & 2 & 0 & 3 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{vmatrix}$

B23 $\begin{vmatrix} 4 & 8 & 6 & 5 & 3 \\ 2 & 4 & 3 & 5 & 0 \\ 1 & 4 & 3 & 8 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{vmatrix}$

For Problems B24–B27, calculate the determinant of the elementary matrix.

B24 $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

B25 $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

B26 $E_3 = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B27 $E_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For Problems B28–B31, find all the cofactors of the matrix.

B28 $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

B29 $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

B30 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

B31 $\begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 1 \\ 1 & 1 & -7 \end{bmatrix}$

For Problems B32–B35, let $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Compute all the expressions and compare.

B32 $\det(A + B)$, $\det A + \det B$

B33 $(\det A)(\det B)$, $\det(AB)$, $\det(BA)$

B34 $\det(3A)$, $3 \det A$

B35 $\det A$, $\det A^{-1}$, $\det A^T$

Conceptual Problems

C1 Prove Theorem 5.1.3.

C2 Prove that if $A \in M_{n\times n}(\mathbb{R})$ is a diagonal matrix, then

$$\det A = a_{11}a_{22} \cdots a_{nn}$$

For Problems C3–C5, find $A, B \in M_{2\times 2}(\mathbb{R})$ such that the statement holds, and find another pair $A, B \in M_{2\times 2}(\mathbb{R})$ such that the statement does not hold.

C3 $\det(A + B) = \det A + \det B$

C4 $\det(cA) = c \det A$

C5 $\det A^{-1} = \det A$

C6 Use induction to prove that if $A \in M_{n\times n}(\mathbb{R})$ with $n \geq 2$ that has two identical rows, then $\det A = 0$.

C7 Prove if $E_1 \in M_{n\times n}(\mathbb{R})$ with $n \geq 2$ is an elementary matrix corresponding to cR_i, $c \neq 0$, then $\det E_1 = c$.

C8 Prove if $E_2 \in M_{n\times n}(\mathbb{R})$ with $n \geq 2$ is an elementary matrix corresponding to $R_i + cR_j$, then $\det E_2 = 1$.

C9 Prove if $E_3 \in M_{n\times n}(\mathbb{R})$ with $n \geq 2$ is an elementary matrix corresponding to $R_i \updownarrow R_j$, then $\det E_3 = -1$.

C10 (a) Consider the points (a_1, a_2) and (b_1, b_2) in $\mathbb{R}^2$. Show that the equation $\det \begin{bmatrix} x_1 & x_2 & 1 \\ a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \end{bmatrix} = 0$ is the equation of the line containing the two points.

(b) Write an equation for the plane in $\mathbb{R}^3$ that contains the non-collinear points (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3).

5.2 Properties of the Determinant

Calculating the determinant of a large matrix with few zeros can be very lengthy. Theorem 5.1.3 suggests that an effective strategy for evaluating a determinant of a matrix is to row reduce the matrix to upper triangular form. We now look at how applying elementary row operations changes the determinant. This will lead us to some important properties of the determinant.

Elementary Row Operations and the Determinant

To see what happens to the determinant of a matrix A when we multiply a row of A by a constant, we first consider a 3×3 example. Following the example, we state and prove the general result.

EXAMPLE 5.2.1

Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 4 & -1 \\ 5 & 2 & 6 \end{bmatrix}$ and let B be the matrix obtained from A by multiplying the third row of A by 3. Show that $\det B = 3 \det A$.

Solution: We have $B = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 4 & -1 \\ 15 & 6 & 18 \end{bmatrix}$. Expand the determinant of B along its third row. The cofactors for this row are

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}, \quad C_{32} = (-1)^{3+2} \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix}, \quad C_{33} = (-1)^{3+3} \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}$$

Observe that these are also the cofactors for the third row of A. Hence,

$$\begin{aligned} \det B &= 15C_{31} + 6C_{32} + 18C_{33} \\ &= 3(5C_{31} + 2C_{32} + 6C_{33}) \\ &= 3 \det A \end{aligned}$$

Theorem 5.2.1 Let $A \in M_{n\times n}(\mathbb{R})$. If B is the matrix obtained from A by multiplying the i-th row of A by the real number r, then $\det B = r \det A$.

Proof: As in the example, we expand the determinant of B along the i-th row. Notice that the cofactors of the elements in this row are exactly the cofactors of the i-th row of A since all the other rows of B are identical to the corresponding rows in A. Therefore,

$$\det B = ra_{i1}C_{i1} + \cdots + ra_{in}C_{in} = r\left(a_{i1}C_{i1} + \cdots + a_{in}C_{in}\right) = r \det A$$

■

Remark

It is important to be careful when using this theorem. It is not uncommon to incorrectly use the reciprocal $(1/r)$ of the factor. One way to counter this error is to think of factoring out the value of r from a row of the matrix. Keep this in mind when reading the following example.

EXAMPLE 5.2.2

Given that $\det\begin{bmatrix}1 & 1 & -2\\ 1 & 2 & 1\\ 0 & 3 & 1\end{bmatrix} = -8$, find $\det\begin{bmatrix}-2 & -2 & 4\\ 1 & 2 & 1\\ 0 & 3 & 1\end{bmatrix}$.

Solution: By Theorem 5.2.1,

$$\det\begin{bmatrix}-2 & -2 & 4\\ 1 & 2 & 1\\ 0 & 3 & 1\end{bmatrix} = (-2)\det\begin{bmatrix}1 & 1 & -2\\ 1 & 2 & 1\\ 0 & 3 & 1\end{bmatrix} = (-2)(-8) = 16$$

EXERCISE 5.2.1

Let $A \in M_{3\times 3}(\mathbb{R})$ and let $r \in \mathbb{R}$. Use Theorem 5.2.1 to show that $\det(rA) = r^3 \det A$.

Next, we consider the effect of swapping two rows. For a general 2×2 matrix we get

$$\det\begin{bmatrix}c & d\\ a & b\end{bmatrix} = cb - da = -(ad - bc) = -\det\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

Hence, it seems that swapping two rows of a matrix multiplies the determinant by -1.

EXAMPLE 5.2.3

Let $A = \begin{bmatrix}0 & 1 & 3\\ 1 & 4 & -1\\ 5 & 2 & 6\end{bmatrix}$ and let B be the matrix obtained from A by swapping the first and third row. Show that $\det B = -\det A$.

Solution: Expand the determinant of $B = \begin{bmatrix}5 & 2 & 6\\ 1 & 4 & -1\\ 0 & 1 & 3\end{bmatrix}$ along the row that was not swapped. The cofactors for this row are

$$C_{21} = (-1)^{2+1}\begin{vmatrix}2 & 6\\ 1 & 3\end{vmatrix}, \quad C_{22} = (-1)^{2+2}\begin{vmatrix}5 & 6\\ 0 & 3\end{vmatrix}, \quad C_{23} = (-1)^{2+3}\begin{vmatrix}5 & 2\\ 0 & 1\end{vmatrix}$$

Observe that these are just the cofactors of the second row of A with their rows swapped. Since these cofactors are determinants of 2×2 matrices, we know from our work above that these will just be negatives of the cofactors of A. That is, if we let C^*_{2j} denote the cofactors of A, then we have

$$C_{2j} = -C^*_{2j}, \qquad \text{for } j = 1, 2, 3$$

Hence,

$$\begin{aligned}
\det B &= 1C_{21} + 4C_{22} + (-1)C_{23} && \text{by the cofactor expansion along the second row}\\
&= 1(-C^*_{21}) + 4(-C^*_{22}) + (-1)(-C^*_{23}) && \text{since } C_{2j} = -C^*_{2j}\\
&= -\left(1C^*_{21} + 4C^*_{22} + (-1)C^*_{23}\right)\\
&= -\det A
\end{aligned}$$

as this is the cofactor expansion of A along its second row.

Indeed, this result holds in general.

Theorem 5.2.2 Let $A \in M_{n\times n}(\mathbb{R})$. If B is the matrix obtained from A by swapping two rows, then $\det B = -\det A$.

Example 5.2.3 shows how the proof of the theorem will work. In particular, we see that in the 3×3 case we needed to refer back to the 2×2 case. This indicates that a proof by induction would be appropriate. We leave this as Problem C1.

EXAMPLE 5.2.4

Given that $\det \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} = -8$, find $\det \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$.

Solution: Since we can obtain $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$ from the original matrix by swapping the first and second rows, Theorem 5.2.2 tells us that

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} = (-1)(-8) = 8$$

Theorem 5.2.3 If two rows of $A \in M_{n\times n}(\mathbb{R})$ are equal, then $\det A = 0$.

Proof: Let B be the matrix obtained from A by interchanging the two equal rows. Obviously $B = A$, so $\det B = \det A$. But, by Theorem 5.2.2, $\det B = -\det A$, so $\det A = -\det A$. This implies that $\det A = 0$. ■

Finally, we show that the third type of elementary row operation is particularly useful as it does not change the determinant. We again begin by considering the 2×2 case.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let B be the matrix obtained from A by adding r times the first row to the second row. We get

$$\begin{aligned} \det B &= \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix} \\ &= a(d + rb) - b(c + ra) \\ &= ad + arb - bc - arb \\ &= ad - bc \\ &= \det A \end{aligned}$$

Hence, in the 2×2 case, adding a multiple of one row to another does not change the determinant.

EXAMPLE 5.2.5

Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 4 & -1 \\ 5 & 2 & 6 \end{bmatrix}$ and let B be the matrix obtained from A by adding r times the first row to the second row.

Solution: Expand the determinant of $B = \begin{bmatrix} 0 & 1 & 3 \\ 1 + 0r & 4 + r & -1 + 3r \\ 5 & 2 & 6 \end{bmatrix}$ along the third row . The cofactors for this row are

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 4 + r & -1 + 3r \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 0 & 3 \\ 1 + 0r & -1 + 3r \end{vmatrix}$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 0 & 1 \\ 1 + 0r & 4 + r \end{vmatrix}$$

Observe that these are the 2×2 cofactors C^*_{3j} of A with the operation $R_2 + rR_1$ applied to them. Hence, our work above shows us that

$$C_{3j} = C^*_{3j}, \qquad \text{for } j = 1, 2, 3$$

Hence, $$\det B = 5C_{31} + 2C_{32} + 6C_{33} = 5C^*_{31} + 2C^*_{32} + 6C^*_{33} = \det A$$

Theorem 5.2.4

Let $A \in M_{n\times n}(\mathbb{R})$. If B is the matrix obtained from A by adding r times the i-th row of A to the k-th row, then $\det B = \det A$.

Once again the example outlines how one can do the proof. The proof is left as Problem C2.

EXAMPLE 5.2.6

Given that $\det \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} = -8$, find $\det \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 7 & 3 \end{bmatrix}$.

Solution: Since we can obtain the new matrix from the original matrix by performing $R_3 + 2R_2$, Theorem 5.2.4 tells us that

$$\det \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 7 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} = -8$$

Theorems 5.2.1, 5.2.2, and 5.2.4 confirm that an effective strategy for evaluating the determinant of a matrix is to row reduce the matrix to upper triangular form keeping track of how the elementary row operations change the determinant. For $n > 3$, it can be shown that in general, this strategy will require fewer arithmetic operations than using only cofactor expansions. The following example illustrates this strategy.

EXAMPLE 5.2.7

Let $A = \begin{bmatrix} 1 & 3 & 1 & 5 \\ 1 & 3 & -3 & -3 \\ 0 & 3 & 1 & 0 \\ 1 & 6 & 2 & 11 \end{bmatrix}$. Find $\det A$.

Solution: By Theorem 5.2.4, performing the row operations $R_2 - R_1$ and $R_4 - R_1$ do not change the determinant, so

$$\det A = \begin{vmatrix} 1 & 3 & 1 & 5 \\ 0 & 0 & -4 & -8 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 6 \end{vmatrix}$$

By Theorem 5.2.2, performing $R_2 \updownarrow R_3$ gives

$$\det A = (-1) \begin{vmatrix} 1 & 3 & 1 & 5 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -4 & -8 \\ 0 & 3 & 1 & 6 \end{vmatrix}$$

By Theorem 5.2.1, performing $(-1/4)R_3$ gives

$$\det A = (-1)(-4) \begin{vmatrix} 1 & 3 & 1 & 5 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & 1 & 6 \end{vmatrix}$$

Finally, by Theorem 5.2.4, performing the row operation $R_4 - R_2$ gives

$$\det A = 4 \begin{vmatrix} 1 & 3 & 1 & 5 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix} = 4\,(1)(3)(1)(6) = 72$$

EXERCISE 5.2.2

Let $A = \begin{bmatrix} 2 & 4 & -2 & 6 \\ -6 & -6 & -2 & 5 \\ 1 & 1 & 3 & -1 \\ 4 & 6 & -2 & 5 \end{bmatrix}$. Find $\det A$.

In some cases, it may be appropriate to use some combination of row operations and cofactor expansions. We demonstrate this in the following example.

EXAMPLE 5.2.8

Find the determinant of $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 1 & 8 & 7 & 9 \\ 1 & 5 & 6 & 10 \\ 0 & 1 & 4 & -2 \end{bmatrix}$.

Solution: By Theorem 5.2.4, performing $R_2 - R_1$ and $R_3 - R_1$ gives

$$\det A = \begin{vmatrix} 1 & 5 & 6 & 7 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 4 & -2 \end{vmatrix}$$

Expanding along the first column gives

$$\det A = (-1)^{1+1} \begin{vmatrix} 3 & 1 & 2 \\ 0 & 0 & 3 \\ 1 & 4 & -2 \end{vmatrix}$$

Expanding along the second row gives

$$\det A = 1(3)(-1)^{2+3} \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = (-3)(12 - 1) = -33$$

In the next example we show how the fact that $\det A = \det A^T$ can be used to give us even more options for simplifying a determinant.

EXAMPLE 5.2.9

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$. Find $\det A$.

Solution: Observe that the sum of the first and second columns equals the third column. If we could just add the first column to the second column, we would have two identical columns which should give us a determinant of 0 (as we know that a matrix with two identical rows has determinant 0).

We can simulate a column operation by taking the transpose and using a row operation. In particular, taking the transpose and then using $R_2 + R_1$ gives

$$\det A = \det A^T = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 0$$

Rather that having to take the transpose of A, we just allow the use of column operations when simplifying a determinant. We get

1. Adding a multiple of one column to another does not change the determinant.
2. Swapping two columns multiplies the determinant by -1.
3. Multiplying a column by a non-zero scalar c multiplies the determinant by c.

EXAMPLE 5.2.10

Evaluate the determinant of $A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ -3 & 2 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 2 & -3 & 2 & -3 \end{bmatrix}$.

Solution: Using the row operation $R_3 + R_1$ gives

$$\det A = \begin{vmatrix} 1 & 3 & -1 & 1 \\ -3 & 2 & 1 & 2 \\ 3 & 2 & 0 & 2 \\ 2 & -3 & 2 & -3 \end{vmatrix}$$

We now use the column operation $C_2 - C_4$ to get

$$\det A = \begin{vmatrix} 1 & 2 & -1 & 1 \\ -3 & 0 & 1 & 2 \\ 3 & 0 & 0 & 2 \\ 2 & 0 & 2 & -3 \end{vmatrix}$$

Using the cofactor expansion along the second column gives

$$\det A = 2(-1)^{1+2} \begin{vmatrix} -3 & 1 & 2 \\ 3 & 0 & 2 \\ 2 & 2 & -3 \end{vmatrix}$$

Applying $R_3 - 2R_1$ and then using the cofactor expansion along the second column we get

$$\det A = -2 \begin{vmatrix} -3 & 1 & 2 \\ 3 & 0 & 2 \\ 8 & 0 & -7 \end{vmatrix} = (-2)(1)(-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 8 & -7 \end{vmatrix} = -74$$

EXERCISE 5.2.3

Find the determinant of $A = \begin{bmatrix} -6 & -2 & 4 & -5 \\ 3 & 2 & -4 & 3 \\ -6 & 4 & 0 & 0 \\ -3 & 2 & -3 & -4 \end{bmatrix}$.

CONNECTION

We will see in Section 5.4 that there are very simple geometric reasons why row and column operations affect the determinant the way that they do.

At this time, we will also stress that determinants play a very important role in Chapter 6 (and hence in Chapters 8 and 9, which both use the content in Chapter 6). Being able to simplify a determinant effectively will be of great assistance in those chapters.

Properties of Determinants

It follows from Theorem 5.2.1, Theorem 5.2.2, and Theorem 5.2.4 that there is a connection between the determinant of a square matrix, its rank, and whether it is invertible. Thus, we extend the Invertible Matrix Theorem from Section 3.5.

Theorem 5.2.5 Invertible Matrix Theorem continued

If $A \in M_{n\times n}(\mathbb{R})$, then the following are equivalent:

(2) $\operatorname{rank}(A) = n$
(10) $\det A \neq 0$

Proof: Theorem 5.2.1, Theorem 5.2.2, and Theorem 5.2.4 indicate that applying an elementary row operation can only multiply the determinant of a matrix by $c \neq 0$, -1, or 1. Thus, $\det A \neq 0$ if and only if its reduced row echelon form has a non-zero determinant. But, since A is $n \times n$, the reduced row echelon form has no zero rows if and only if there is a leading one in every row. That is, $\det A \neq 0$ if and only if $\operatorname{rank}(A) = n$. ■

We shall see how to use the determinant in calculating the inverse in the next section. It is worth noting that Theorem 5.2.5 implies that "almost all" square matrices are invertible; a square matrix fails to be invertible only if it satisfies the special condition $\det A = 0$.

Determinant of a Product

Often it is necessary to calculate the determinant of the product of two square matrices A and B. When you remember that each entry of AB is the dot product of a row from A and a column from B, and that the rule for calculating determinants is quite complicated, you might expect a very complicated rule. So, Theorem 5.2.7 should be a welcome surprise. The next result will help us prove Theorem 5.2.7.

Theorem 5.2.6 If $E \in M_{n\times n}(\mathbb{R})$ is an elementary matrix and $B \in M_{n\times n}(\mathbb{R})$, then

$$\det(EB) = (\det E)(\det B)$$

Proof: If E is an elementary matrix, then, since E is obtained by performing a single row operation on the identity matrix, we get by Theorem 5.2.1, Theorem 5.2.2, or Theorem 5.2.4 that there exists $c \in \mathbb{R}$ such that $\det E = c$ depending on the elementary row operation used. Moreover, since EB is the matrix obtained by performing that row operation on B, we get by Theorem 5.2.1, Theorem 5.2.2, or Theorem 5.2.4 that

$$\det(EB) = c \det B = \det E \det B$$

■

Theorem 5.2.7 If $A, B \in M_{n\times n}(\mathbb{R})$, then $\det(AB) = (\det A)(\det B)$.

Proof: If $\det A = 0$, then A is not invertible by the Invertible Matrix Theorem. Assume that $\det(AB) \neq 0$. Then, AB is invertible by the Invertible Matrix Theorem. Hence, there exists $C \in M_{n\times n}(\mathbb{R})$ such that

$$I = (AB)C = A(BC)$$

But, then Theorem 3.5.2 implies that A is invertible, which is a contradiction. Thus, we must have

$$\det(AB) = 0 = 0 \det B = \det A \det B$$

If $\det A \neq 0$, then A is invertible by the Invertible Matrix Theorem. Thus, by Theorem 3.6.3, there exists a sequence of elementary matrices $E_1, \dots, E_k$ such that

$$A = E_1^{-1} \cdots E_k^{-1}$$

Hence, by repeated use of Theorem 5.2.6, we get

$$\det(AB) = \det(E_1^{-1} \cdots E_k^{-1} B) = (\det E_1^{-1}) \cdots (\det E_k^{-1}) \det B = (\det A)(\det B)$$

■

EXAMPLE 5.2.11 Verify Theorem 5.2.7 for $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 4 \\ 5 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 4 \\ 7 & 1 & 0 \\ 1 & -2 & 3 \end{bmatrix}$.

Solution: Using $R_2 - 4R_1$ and a cofactor expansion along the third column gives

$$\det A = \begin{vmatrix} 3 & 0 & 1 \\ -10 & -1 & 0 \\ 5 & 2 & 0 \end{vmatrix} = 1(-1)^{1+3} \begin{vmatrix} -10 & -1 \\ 5 & 2 \end{vmatrix} = -15$$

Using $R_2 + 7R_1$ and $R_3 + R_1$ gives

$$\det B = \begin{vmatrix} -1 & 2 & 4 \\ 0 & 15 & 28 \\ 0 & 0 & 7 \end{vmatrix} = -105$$

So, $(\det A)(\det B) = (-15)(-105) = 1575$. Similarly, using $-\frac{1}{5}R_2$, $R_1 + 2R_2$, $R_3 - 9R_2$, and a cofactor expansion along the first column gives

$$\begin{aligned} \det(AB) &= \det \begin{bmatrix} -2 & 4 & 15 \\ -5 & -5 & 20 \\ 9 & 12 & 20 \end{bmatrix} = (-5) \begin{vmatrix} -2 & 4 & 15 \\ 1 & 1 & -4 \\ 9 & 12 & 20 \end{vmatrix} \\ &= (-5) \begin{vmatrix} 0 & 6 & 7 \\ 1 & 1 & -4 \\ 0 & 3 & 56 \end{vmatrix} = (-5)(-1)^{2+1} \begin{vmatrix} 6 & 7 \\ 3 & 56 \end{vmatrix} \\ &= (5)(315) = 1575 \end{aligned}$$

PROBLEMS 5.2

Practice Problems

For Problems A1–A6, use row operations and triangular form to compute the determinants of the matrix. Show your work clearly. Decide whether the matrix is invertible.

A1 $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$

A2 $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

A3 $\begin{bmatrix} 5 & 2 & -1 & 1 \\ 1 & 2 & -1 & 1 \\ 3 & 2 & 1 & 4 \\ -2 & 0 & 3 & 5 \end{bmatrix}$

A4 $\begin{bmatrix} 1 & 1 & 3 & 1 \\ -2 & -2 & -4 & -1 \\ 2 & 2 & 8 & 3 \\ 1 & 1 & 7 & 3 \end{bmatrix}$

A5 $\begin{bmatrix} 5 & 10 & 5 & -5 \\ 1 & 3 & 5 & 7 \\ 1 & 2 & 6 & 3 \\ -1 & 7 & 1 & 1 \end{bmatrix}$

A6 $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 5 & 0 & 5 \\ 2 & 4 & 4 & 6 \\ 1 & -1 & -4 & -5 \end{bmatrix}$

For Problems A7–A20, evaluate the determinant.

A7 $\begin{vmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{vmatrix}$

A8 $\begin{vmatrix} 2 & 3 & 5 \\ -1 & 1 & 0 \\ 7 & -6 & 1 \end{vmatrix}$

A9 $\begin{vmatrix} 3 & 3 & 3 \\ 1 & 2 & -2 \\ -1 & 5 & -7 \end{vmatrix}$

A10 $\begin{vmatrix} 2 & 4 & 5 \\ 1 & 1 & 3 \\ 3 & 5 & 5 \end{vmatrix}$

A11 $\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -2 \\ 1 & 2 & 3 \end{vmatrix}$

A12 $\begin{vmatrix} 2 & 4 & 2 \\ 4 & 2 & 1 \\ -2 & 2 & 2 \end{vmatrix}$

A13 $\begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 5 & 9 \\ 1 & 3 & 4 & 3 \end{vmatrix}$

A14 $\begin{vmatrix} 2 & 0 & -2 & -6 \\ 2 & -6 & -4 & -1 \\ -3 & -4 & 5 & 3 \\ -2 & -1 & -3 & 2 \end{vmatrix}$

A15 $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & -8 \\ 2 & -5 & 2 \end{vmatrix}$

A16 $\begin{vmatrix} 2 & 3 & 1 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

A17 $\begin{vmatrix} 6 & 8 & -8 \\ 7 & 5 & 8 \\ -2 & -4 & 8 \end{vmatrix}$

A18 $\begin{vmatrix} 1 & 10 & 7 & -9 \\ 7 & -7 & 7 & 7 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{vmatrix}$

A19 $\begin{vmatrix} -1 & 2 & 6 & 4 \\ 0 & 3 & 5 & 6 \\ 1 & -1 & 4 & -2 \\ 1 & 2 & 1 & 2 \end{vmatrix}$

A20 $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$

For Problems A21–A28, determine all values of p such that the matrix is invertible.

A21 $\begin{bmatrix} 1 & p \\ 2 & 4 \end{bmatrix}$

A22 $\begin{bmatrix} p & -2 \\ 1 & p \end{bmatrix}$

A23 $\begin{bmatrix} 1 & 0 & 0 \\ p & p & 0 \\ -1 & 3 & 1 \end{bmatrix}$

A24 $\begin{bmatrix} 3 & 2 & 2 \\ -8 & p & p \\ 5 & -3 & -1 \end{bmatrix}$

A25 $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ 4 & p & -2 \end{bmatrix}$

A26 $\begin{bmatrix} 2 & 3 & 1 & p \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 7 & 6 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

A27 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & p \end{bmatrix}$

A28 $\begin{bmatrix} 2 & p & 1 & 1 \\ 2 & 2 & p & 2 \\ 0 & -2 & -p & 0 \\ 3 & 2 & p & 3 \end{bmatrix}$

For Problems A29 and A30, find $\det A$, $\det B$, and $\det(AB)$. Verify that Theorem 5.2.7 holds.

A29 $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$

A30 $A = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 5 \\ 4 & 1 & 1 \end{bmatrix}$

For Problems A31–A36, determine all values of λ such that the determinant is 0. [Hint: for the 3×3 determinants, use row and column operations to simplify the determinant so that you do not have to factor a cubic polynomial.]

A31 $\begin{vmatrix} 3-\lambda & 2 \\ 4 & 5-\lambda \end{vmatrix}$

A32 $\begin{vmatrix} 4-\lambda & -3 \\ 3 & -4-\lambda \end{vmatrix}$

A33 $\begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$

A34 $\begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 3-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{vmatrix}$

A35 $\begin{vmatrix} -3-\lambda & 6 & -2 \\ -1 & 2-\lambda & -1 \\ 1 & -3 & -\lambda \end{vmatrix}$

A36 $\begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix}$

A37 Suppose that A is an $n \times n$ matrix and $r \in \mathbb{R}$. Determine $\det(rA)$.

A38 If A is invertible, prove that $\det A^{-1} = \dfrac{1}{\det A}$.

A39 If $A^3 = I$, prove that A is invertible.

Homework Problems

For Problems B1–B8, use row operations and triangular form to compute the determinants of the matrix. Show your work clearly. Decide whether the matrix is invertible.

B1 $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 8 \\ 1 & 5 & 2 \end{bmatrix}$

B2 $\begin{bmatrix} 3 & -2 & 1 \\ -6 & 4 & 1 \\ 9 & -6 & -7 \end{bmatrix}$

B3 $\begin{bmatrix} 3 & -1 & 0 \\ -3 & 7 & 2 \\ 6 & 1 & 1 \end{bmatrix}$

B4 $\begin{bmatrix} -2 & -4 & -2 \\ 1 & 6 & 3 \\ 2 & 8 & 2 \end{bmatrix}$

B5 $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 3 & 7 \\ 3 & 1 & 2 & 8 \\ 1 & -3 & 3 & 6 \end{bmatrix}$

B6 $\begin{bmatrix} 2 & 2 & 1 & 2 \\ -2 & -5 & 0 & -3 \\ 0 & -6 & 3 & -1 \\ 1 & 4 & 1 & 4 \end{bmatrix}$

B7 $\begin{bmatrix} 1 & -2 & -1 & -2 \\ -1 & 3 & 2 & 3 \\ 2 & -2 & 1 & 1 \\ 1 & 1 & 5 & 10 \end{bmatrix}$

B8 $\begin{bmatrix} 1 & 5 & -4 & 2 \\ 1 & -3 & 8 & 6 \\ 1 & -1 & 5 & 7 \\ 1 & 1 & 2 & 1 \end{bmatrix}$

For Problems B9–B20, evaluate the determinant.

B9 $\begin{vmatrix} 2 & -3 & 4 \\ 1 & 1 & 4 \\ 5 & 2 & 4 \end{vmatrix}$

B10 $\begin{vmatrix} 3 & 3 & 6 \\ 3 & 2 & 1 \\ -2 & 4 & 2 \end{vmatrix}$

B11 $\begin{vmatrix} 2 & 3 & -6 \\ 5 & 5 & -5 \\ -4 & -3 & 0 \end{vmatrix}$

B12 $\begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & -6 \\ 3 & -2 & 5 \end{vmatrix}$

B13 $\begin{vmatrix} 0 & 2 & 3 & 3 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 3 & 4 & 5 & 5 \end{vmatrix}$

B14 $\begin{vmatrix} 1 & 2 & 4 & -2 \\ 2 & 5 & 1 & -2 \\ 1 & 3 & -2 & 1 \\ -1 & 3 & 1 & 7 \end{vmatrix}$

B15 $\begin{vmatrix} 3 & 2 & 2 & 1 \\ 4 & 2 & 3 & 2 \\ 1 & 4 & -1 & 1 \\ 5 & 2 & 5 & 2 \end{vmatrix}$

B16 $\begin{vmatrix} 0 & 3 & 1 & 3 \\ 1 & 4 & 2 & 5 \\ 2 & 3 & -4 & -6 \\ 2 & 6 & -3 & -2 \end{vmatrix}$

B17 $\begin{vmatrix} 3 & 2 & 3 & 3 \\ 6 & 7 & 5 & 7 \\ -3 & 9 & 7 & 9 \\ -6 & 10 & 10 & 12 \end{vmatrix}$

B18 $\begin{vmatrix} 11 & 3 & 9 & 5 \\ 3 & -2 & 3 & 1 \\ 8 & -4 & 8 & 4 \\ 5 & 0 & 5 & 5 \end{vmatrix}$

B19 $\begin{vmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & 4 & -8 \end{vmatrix}$

B20 $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix}$

For Problems B21–B28, determine all values of p such that the matrix is invertible.

B21 $\begin{bmatrix} p & 2 & 1 \\ p & 3 & 4 \\ 1 & 1 & -2 \end{bmatrix}$

B22 $\begin{bmatrix} 4 & 2 & 3 \\ p & 5 & 4 \\ 3 & 2 & 3 \end{bmatrix}$

B23 $\begin{bmatrix} 3 & 4 & p \\ 2 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

B24 $\begin{bmatrix} 4 & 8 & 3 \\ 1 & 1 & 1 \\ 2p & 4p & 2p \end{bmatrix}$

B25 $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 2 & 1 & -1 & p \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$

B26 $\begin{bmatrix} 1 & 8 & 6 & 6 \\ p & p & 1 & 2 \\ 2 & 12 & 2 & 2 \\ 3 & 18 & 3 & 3 \end{bmatrix}$

B27 $\begin{bmatrix} p & 2 & 3 & 0 \\ 5 & -4 & -3 & 5 \\ 6 & -6 & p & 6 \\ -9 & 7 & 6 & -9 \end{bmatrix}$

B28 $\begin{bmatrix} 0 & 2 & 3 & 2 \\ 3 & 5 & 4 & 2 \\ 4 & p & 5 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

For Problems B29–B32, find $\det A$, $\det B$, and $\det(AB)$. Verify that Theorem 5.2.7 holds.

B29 $A = \begin{bmatrix} 3 & 1 \\ -4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$

B30 $A = \begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix}, B = \begin{bmatrix} 5 & -3 \\ -7 & 2 \end{bmatrix}$

B31 $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 2 & 4 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

B32 $A = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 & -1 \\ 6 & -11 & -2 \\ -5 & 9 & 2 \end{bmatrix}$

For Problems B33–B36, determine all values of λ such that the determinant is 0. [Hint: for the 3×3 determinants, use row and column operations to simplify the determinant so that you do not have to factor a cubic polynomial.]

B33 $\begin{vmatrix} 6-\lambda & 1 \\ 5 & 2-\lambda \end{vmatrix}$

B34 $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$

B35 $\begin{vmatrix} 3-\lambda & -1 & 3 \\ -1 & 9-\lambda & -3 \\ 3 & -3 & 11-\lambda \end{vmatrix}$

B36 $\begin{vmatrix} 1-\lambda & 2 & -2 \\ 2 & 1-\lambda & 2 \\ -2 & 2 & -3-\lambda \end{vmatrix}$

Conceptual Problems

C1 Let $A \in M_{n\times n}(\mathbb{R})$. Prove that if B is the matrix obtained from A by swapping two rows, then

$$\det B = -\det A$$

C2 Let $A \in M_{n\times n}(\mathbb{R})$. Prove that if B is the matrix obtained from A by adding r times the j-th row of A to the k-th row, then

$$\det B = \det A$$

C3 A square matrix A is a **skew-symmetric** if $A^T = -A$. If $A \in M_{n\times n}(\mathbb{R})$ is skew-symmetric matrix, with n odd, prove that $\det A = 0$.

C4 A matrix A is called **orthogonal** if $A^T = A^{-1}$.
(a) If A is orthogonal, prove that $\det A = \pm 1$.
(b) Give an example of an orthogonal matrix for which $\det A = -1$.

C5 Two matrices $A, B \in M_{n\times n}(\mathbb{R})$ are said to be **similar** if there exists an invertible matrix P such that $P^{-1}AP = B$. Prove that if A and B are similar, then

$$\det A = \det B$$

C6 Assume that $A, B \in M_{n\times n}(\mathbb{R})$ are invertible. Prove that $\det A = \det B$ if and only if $A = UB$, where U is a matrix with $\det U = 1$.

C7 Use determinants to prove that if $A, B \in M_{n\times n}(R)$ such that $AB = -BA$, A is invertible and n is odd, then B is not invertible.

For Problems C8–C13, let $A, B \in M_{n\times n}(\mathbb{R})$. Determine whether the statement is true or false. Justify your answer.

C8 If the columns of A are linearly independent, then $\det A \neq 0$.

C9 $\det(A + B) = \det A + \det B$.

C10 $\det(A + B^T) = \det(A^T + B)$.

C11 The system of equations $A\vec{x} = \vec{b}$ is consistent only if $\det A \neq 0$.

C12 If $\det(AB) = 0$, then $\det A = 0$ or $\det B = 0$.

C13 If $A^2 = I$, then $\det A = 1$.

C14 (a) Prove that $\det \begin{bmatrix} a+p & b+q & c+r \\ d & e & f \\ g & h & k \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} + \det \begin{bmatrix} p & q & r \\ d & e & f \\ g & h & k \end{bmatrix}$.

(b) Use part (a) to express $\det \begin{bmatrix} a+p & b+q & c+r \\ d+x & e+y & f+z \\ g & h & k \end{bmatrix}$ as the sum of determinants of matrices whose entries are not sums.

C15 Prove that

$$\det \begin{bmatrix} a+b & p+q & u+v \\ b+c & q+r & v+w \\ c+a & r+p & w+u \end{bmatrix} = 2\det \begin{bmatrix} a & p & u \\ b & q & v \\ c & r & w \end{bmatrix}.$$

C16 Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1+2a \\ 1 & (1+a)^2 & (1+2a)^2 \end{bmatrix} = 2a^3$.

5.3 Inverse by Cofactors, Cramer's Rule

The Invertible Matrix Theorem shows us that there is a close connection between whether a square matrix A is invertible, the number of solutions of the system of linear equations $A\vec{x} = \vec{b}$, and the determinant of A. We now examine this relationship a little further by looking at how to use determinants to find the inverse of a matrix and to solve systems of linear equations.

Inverse by Cofactors

Observe that we can write the inverse of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in the form

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

That is, the entries of A^{-1} are scalar multiples of the cofactors of A. In particular,

$$(A^{-1})_{ij} = \frac{1}{\det A}C_{ji} \tag{5.2}$$

Take careful notice of the change of order in the subscripts in the line above.
It is instructive to verify this result by multiplying out A and A^{-1}.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\left(\frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}\right) \\ &= \frac{1}{\det A}\begin{bmatrix} a_{11}C_{11} + a_{12}C_{12} & a_{11}C_{21} + a_{12}C_{22} \\ a_{21}C_{11} + a_{22}C_{12} & a_{21}C_{21} + a_{22}C_{22} \end{bmatrix} \\ &= \frac{1}{\det A}\begin{bmatrix} \det A & a_{11}(-a_{12}) + a_{12}a_{11} \\ a_{21}a_{22} + a_{22}(-a_{21}) & \det A \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Our goal now is to prove that equation (5.2) holds in the $n \times n$ case. We begin with a useful theorem.

Theorem 5.3.1 False Expansion Theorem

If $A \in M_{n\times n}(\mathbb{R})$ and $i \neq k$, then

$$a_{i1}C_{k1} + \cdots + a_{in}C_{kn} = 0$$

Proof: Let B be the matrix obtained from A by replacing (not swapping) the k-th row of A by the i-th row of A. Then the i-th row of B is identical to the k-th row of B, hence $\det B = 0$ by Theorem 5.2.3. Since the cofactors C^*_{kj} of B are equal to the cofactors C_{kj} of A, and the coefficients b_{kj} of the k-th row of B are equal to the coefficients a_{ij} of the i-th row of A, we get

$$0 = \det(B) = b_{k1}C^*_{k1} + \cdots + b_{kn}C^*_{kn} = a_{i1}C_{k1} + \cdots + a_{in}C_{kn}$$

as required. ■

The False Expansion Theorem says that if we try to do a cofactor expansion of the determinant, but use the coefficients from one row and the cofactors from another row, then we will always get 0. Of course, this also applies to a false expansion along any column of the matrix. That is, if $j \neq k$, then

$$a_{1j}C_{1k} + \cdots + a_{nj}C_{nk} = 0 \tag{5.3}$$

EXAMPLE 5.3.1

Find a non-zero vector $\vec{n} \in \mathbb{R}^4$ that is orthogonal to $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$.

Solution: Rather than setting up a system of linear equations, we will solve this by using the False Expansion Theorem. Consider the matrix

$$A = \begin{bmatrix} \vec{0} & \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Let $\vec{n} = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \end{bmatrix}$. Using equation (5.3) with $j = 2$ and $k = 1$ we get

$$0 = 1C_{11} + 1C_{21} + 1C_{31} + 1C_{41} = \vec{v}_1 \cdot \vec{n}$$

Using equation (5.3) with $j = 3$ and $k = 1$ we get

$$0 = 2C_{11} + 1C_{21} + 0C_{31} + 1C_{41} = \vec{v}_2 \cdot \vec{n}$$

Using equation (5.3) with $j = 4$ and $k = 1$ we get

$$0 = -1C_{11} + 3C_{21} + 1C_{31} + 2C_{41} = \vec{v}_3 \cdot \vec{n}$$

That is, $\vec{n}$ is orthogonal to $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Calculating the cofactors, we find that

$$\vec{n} = \begin{bmatrix} 1 \\ 4 \\ 1 \\ -6 \end{bmatrix}$$

This can be generalized to find a vector $\vec{x} \in \mathbb{R}^n$ that is orthogonal to $n - 1$ vectors $\vec{v}_1, \dots, \vec{v}_{n-1} \in \mathbb{R}^n$. See Problem C1. The case where $n = 3$ is the cross product.

Theorem 5.3.2 If $A \in M_{n\times n}(\mathbb{R})$ is invertible, then $(A^{-1})_{ij} = \dfrac{1}{\det A} C_{ji}$.

Proof: Let B be the $n \times n$ matrix such that

$$b_{ij} = C_{ji} \qquad \text{(the } (j,i)\text{-cofactor of } A)$$

Observe that the cofactors of the i-th row of A form the i-th column of B. Thus, the dot product of the i-th row of A and the i-th column of B is

$$a_{i1}C_{i1} + \cdots + a_{in}C_{in} = \det A$$

For $i \neq j$, the dot product of the i-th row of A and the j-th column of B is

$$a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = 0$$

by the False Expansion Theorem. Hence, if $\vec{a}_i^T$ represents the i-th row of A and $\vec{b}_j$ represents the j-th column of B, then

$$\begin{aligned} AB &= \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \\ &= \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \cdots & \vec{a}_1 \cdot \vec{b}_n \\ \vdots & \ddots & \vdots \\ \vec{a}_n \cdot \vec{b}_1 & \cdots & \vec{a}_n \cdot \vec{b}_n \end{bmatrix} \\ &= \begin{bmatrix} \det A & 0 & \cdots & 0 \\ 0 & \det A & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \det A \end{bmatrix} = (\det A)I \end{aligned}$$

Thus, by Theorem 3.5.2, $A^{-1} = \dfrac{1}{\det A}B$ as required. ■

Because of this result, we make the following definition.

Definition
Adjugate

Let $A \in M_{n\times n}(\mathbb{R})$. The **adjugate** of A is the matrix $\operatorname{adj}(A)$ defined by

$$\big(\operatorname{adj}(A)\big)_{ij} = C_{ji}$$

From the proof of Theorem 5.3.2, we have that

$$A \operatorname{adj}(A) = (\det A)I$$

Or, if A is invertible, then

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$$

EXAMPLE 5.3.2

Determine the adjugate of $A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 6 & -2 & 5 \end{bmatrix}$ and verify that $A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$.

Solution: The nine cofactors of A are

$$C_{11} = (1)\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} = 17 \qquad C_{12} = (-1)\begin{vmatrix} 0 & 1 \\ 6 & 5 \end{vmatrix} = 6 \qquad C_{13} = (1)\begin{vmatrix} 0 & 3 \\ 6 & -2 \end{vmatrix} = -18$$

$$C_{21} = (-1)\begin{vmatrix} 4 & -1 \\ -2 & 5 \end{vmatrix} = -18 \qquad C_{22} = (1)\begin{vmatrix} 2 & -1 \\ 6 & 5 \end{vmatrix} = 16 \qquad C_{23} = (-1)\begin{vmatrix} 2 & 4 \\ 6 & -2 \end{vmatrix} = 28$$

$$C_{31} = (1)\begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix} = 7 \qquad C_{32} = (-1)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = -2 \qquad C_{33} = (1)\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = 6$$

Hence,

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & 28 & 6 \end{bmatrix}$$

Multiplying we find

$$A\,\mathrm{adj}(A) = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 6 & -2 & 5 \end{bmatrix}\begin{bmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & 28 & 6 \end{bmatrix} = \begin{bmatrix} 76 & 0 & 0 \\ 0 & 76 & 0 \\ 0 & 0 & 76 \end{bmatrix}$$

Hence, $\det A = 76$ and

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A) = \frac{1}{76}\begin{bmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & 28 & 6 \end{bmatrix}$$

EXERCISE 5.3.1

Use $\mathrm{adj}(A)$ to find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}$.

For 3×3 matrices, finding $\mathrm{adj}(A)$ requires the evaluation of nine 2×2 determinants. This is manageable, but it is more work than would be required by row reducing $\left[\begin{array}{c|c} A & I \end{array}\right]$ to reduced row echelon form. Finding the inverse of a 4×4 matrix by finding $\mathrm{adj}(A)$ would require the evaluation of sixteen 3×3 determinants; this method becomes extremely unattractive. However, it can be useful in some theoretic applications as it gives a formula for the entries of the inverse.

Cramer's Rule

Consider the system of n linear equations in n variables, $A\vec{x} = \vec{b}$. If $\det A \neq 0$ so that A is invertible, then the solution may be written in the form

$$\vec{x} = A^{-1}\vec{b} = \left(\frac{1}{\det A}\operatorname{adj}(A)\right)\vec{b}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1i} & C_{2i} & \cdots & C_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

By our first definition of matrix-vector multiplication, the value of the i-th component of $\vec{x}$ is

$$x_i = \frac{1}{\det A}(b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni})$$

Now, let N_i be the matrix obtained from A by replacing the i-th column of A by $\vec{b}$. Then the cofactors of the i-th column of N_i will equal the cofactors of the i-th column of A, and hence we get

$$\det N_i = b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}$$

Therefore, the i-th component of $\vec{x}$ in the solution of $A\vec{x} = \vec{b}$ is

$$x_i = \frac{\det N_i}{\det A}$$

This is called Cramer's Rule (or Method). We now demonstrate Cramer's Rule with a couple of examples.

EXAMPLE 5.3.3 Use Cramer's Rule to solve the system of equations.

$$\begin{aligned} x_1 + x_2 - x_3 &= b_1 \\ 2x_1 + 4x_2 + 5x_3 &= b_2 \\ x_1 + x_2 + 2x_3 &= b_3 \end{aligned}$$

Solution: The coefficient matrix is $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$, so

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{vmatrix} = (1)(2)(3) = 6$$

EXAMPLE 5.3.3 (continued)

Hence,

$$x_1 = \frac{\det N_1}{\det A} = \frac{1}{6}\begin{vmatrix} b_1 & 1 & -1 \\ b_2 & 4 & 5 \\ b_3 & 1 & 2 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} b_1 & 1 & -1 \\ b_2+5b_1 & 9 & 0 \\ b_3+2b_1 & 3 & 0 \end{vmatrix} = \frac{3b_1 - 3b_2 + 9b_3}{6}$$

$$x_2 = \frac{\det N_2}{\det A} = \frac{1}{6}\begin{vmatrix} 1 & b_1 & -1 \\ 2 & b_2 & 5 \\ 1 & b_3 & 2 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & b_1 & -1 \\ 7 & b_2+5b_1 & 0 \\ 3 & b_3+2b_1 & 0 \end{vmatrix} = \frac{b_1 + 3b_2 - 7b_3}{6}$$

$$x_3 = \frac{\det N_3}{\det A} = \frac{1}{6}\begin{vmatrix} 1 & 1 & b_1 \\ 2 & 4 & b_2 \\ 1 & 1 & b_3 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & 1 & b_1 \\ 0 & 2 & b_2-2b_1 \\ 0 & 0 & b_3-b_1 \end{vmatrix} = \frac{-2b_1 + 2b_3}{6}$$

To solve a system of n equations in n variables by using Cramer's Rule would require the evaluation of the determinant of $n+1$ matrices where each matrix is $n \times n$. Thus, solving a system by using Cramer's Rule requires far more calculation than elimination. However, Cramer's Rule is sometimes used to write a formula for the solution of a problem. This is used, for example, in electrical engineering, control engineering, and economics. It is particularly useful when the system contains variables.

EXAMPLE 5.3.4

Let $A = \begin{bmatrix} a & 1 & 2 \\ 0 & b & -1 \\ c & 1 & d \end{bmatrix}$. Assuming that $\det A \neq 0$, solve $A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Solution: We have

$$\det A = a(bd+1) + c(-1-2b) = abd + a - c - 2cb$$

Thus,

$$x_1 = \frac{1}{\det A}\begin{vmatrix} 1 & 1 & 2 \\ -1 & b & -1 \\ 1 & 1 & d \end{vmatrix} = \frac{1}{\det A}\begin{vmatrix} 1 & 1 & 2 \\ 0 & b+1 & 1 \\ 0 & 0 & d-2 \end{vmatrix} = \frac{(b+1)(d-2)}{abd + a - c - 2cb}$$

$$x_2 = \frac{1}{\det A}\begin{vmatrix} a & 1 & 2 \\ 0 & -1 & -1 \\ c & 1 & d \end{vmatrix} = \frac{1}{\det A}\begin{vmatrix} a & 0 & 1 \\ 0 & -1 & -1 \\ c & 0 & d-1 \end{vmatrix} = \frac{-ad + a + c}{abd + a - c - 2cb}$$

$$x_3 = \frac{1}{\det A}\begin{vmatrix} a & 1 & 1 \\ 0 & b & -1 \\ c & 1 & 1 \end{vmatrix} = \frac{1}{\det A}\begin{vmatrix} a & 0 & 1 \\ 0 & b+1 & -1 \\ c & 0 & 1 \end{vmatrix} = \frac{(b+1)(a-c)}{abd + a - c - 2cb}$$

That is, the solution is $\vec{x} = \dfrac{1}{abd + a - c - 2cb}\begin{bmatrix} (b+1)(d-2) \\ -ad + a + c \\ (b+1)(a-c) \end{bmatrix}$.

PROBLEMS 5.3

Practice Problems

For Problems A1–A4, find a non-zero vector $\vec{x} \in \mathbb{R}^4$ that is orthogonal to the given vectors.

A1 $\begin{bmatrix} 2\\2\\1\\-1 \end{bmatrix}, \begin{bmatrix} 1\\2\\2\\2 \end{bmatrix}, \begin{bmatrix} 2\\1\\-1\\-1 \end{bmatrix}$

A2 $\begin{bmatrix} 2\\2\\2\\1 \end{bmatrix}, \begin{bmatrix} 2\\1\\1\\2 \end{bmatrix}, \begin{bmatrix} 2\\2\\-1\\-1 \end{bmatrix}$

A3 $\begin{bmatrix} 1\\0\\-1\\1 \end{bmatrix}, \begin{bmatrix} 1\\2\\0\\0 \end{bmatrix}, \begin{bmatrix} 2\\4\\1\\1 \end{bmatrix}$

A4 $\begin{bmatrix} 3\\1\\2\\-3 \end{bmatrix}, \begin{bmatrix} 4\\1\\2\\1 \end{bmatrix}, \begin{bmatrix} -4\\-1\\-3\\2 \end{bmatrix}$

For Problems A5–A10:

(a) Determine adj(A).

(b) Calculate $A\,\text{adj}(A)$. Determine $\det A$, and find A^{-1}.

A5 $A = \begin{bmatrix} 1 & 3 & 0\\ 2 & 1 & -1\\ 1 & 1 & 0 \end{bmatrix}$

A6 $A = \begin{bmatrix} 1 & 2 & 3\\ 0 & 1 & -5\\ 0 & 0 & 7 \end{bmatrix}$

A7 $A = \begin{bmatrix} a & 0 & d\\ 0 & c & 0\\ b & 0 & e \end{bmatrix}$

A8 $A = \begin{bmatrix} a & 2 & 3\\ 0 & 1 & -5\\ 0 & 0 & b \end{bmatrix}$

A9 $A = \begin{bmatrix} 2 & -3 & t\\ -2 & 0 & 1\\ 3 & 1 & 1 \end{bmatrix}$

A10 $A = \begin{bmatrix} 3 & 2 & t\\ 1 & t & 1\\ 1 & 1 & 0 \end{bmatrix}$

For Problems A11–A17, determine the inverse of the matrix by finding adj(A). Verify your answer by using multiplication.

A11 $\begin{bmatrix} 1 & 3\\ 4 & 10 \end{bmatrix}$

A12 $\begin{bmatrix} 3 & -5\\ 2 & -1 \end{bmatrix}$

A13 $\begin{bmatrix} 4 & 1 & 7\\ 2 & -3 & 1\\ -2 & 6 & 0 \end{bmatrix}$

A14 $\begin{bmatrix} 4 & 0 & -4\\ 0 & -1 & 1\\ -2 & 2 & -1 \end{bmatrix}$

A15 $\begin{bmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 4 & 1 & 2 \end{bmatrix}$

A16 $\begin{bmatrix} 4 & 0 & -2\\ 0 & -1 & 2\\ -4 & 1 & -1 \end{bmatrix}$

A17 $\begin{bmatrix} 1 & 5 & 3\\ 3 & 1 & 1\\ -6 & -2 & 2 \end{bmatrix}$

For Problems A18–A23, use Cramer's Rule to solve the system.

A18 $\begin{aligned} 2x_1 - 3x_2 &= 6\\ 3x_1 + 5x_2 &= 7 \end{aligned}$

A19 $\begin{aligned} 3x_1 + 3x_2 &= 2\\ 2x_1 - 3x_2 &= 5 \end{aligned}$

A20 $\begin{aligned} 7x_1 + x_2 - 4x_3 &= 3\\ -6x_1 - 4x_2 + x_3 &= 0\\ 4x_1 - x_2 - 2x_3 &= 6 \end{aligned}$

A21 $\begin{aligned} 2x_1 + 3x_2 - 5x_3 &= 2\\ 3x_1 - x_2 + 2x_3 &= 1\\ 5x_1 + 4x_2 - 6x_3 &= 3 \end{aligned}$

A22 $\begin{aligned} 5x_1 + 3x_2 + 5x_3 &= 2\\ 2x_1 + 4x_2 + 5x_3 &= 1\\ 7x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$

A23 $\begin{aligned} 2x_1 + 9x_2 + 3x_3 &= 1\\ 2x_1 - 2x_2 + 3x_3 &= 1\\ 3x_2 + 3x_3 &= -5 \end{aligned}$

Homework Problems

For Problems B1–B6, find a non-zero vector $\vec{x} \in \mathbb{R}^4$ that is orthogonal to the given vectors.

B1 $\begin{bmatrix} 3\\1\\-1\\0 \end{bmatrix}, \begin{bmatrix} 3\\3\\1\\1 \end{bmatrix}, \begin{bmatrix} 2\\1\\0\\2 \end{bmatrix}$

B2 $\begin{bmatrix} 4\\1\\2\\1 \end{bmatrix}, \begin{bmatrix} 3\\2\\2\\1 \end{bmatrix}, \begin{bmatrix} 1\\1\\-1\\-1 \end{bmatrix}$

B3 $\begin{bmatrix} 2\\3\\4\\3 \end{bmatrix}, \begin{bmatrix} -1\\2\\-2\\1 \end{bmatrix}, \begin{bmatrix} 1\\0\\3\\1 \end{bmatrix}$

B4 $\begin{bmatrix} 0\\1\\3\\5 \end{bmatrix}, \begin{bmatrix} -1\\2\\1\\3 \end{bmatrix}, \begin{bmatrix} 2\\1\\3\\-1 \end{bmatrix}$

B5 $\begin{bmatrix} 2\\1\\-4\\3 \end{bmatrix}, \begin{bmatrix} 2\\1\\-4\\4 \end{bmatrix}, \begin{bmatrix} 2\\1\\-5\\5 \end{bmatrix}$

B6 $\begin{bmatrix} 4\\-3\\6\\2 \end{bmatrix}, \begin{bmatrix} 4\\-3\\2\\3 \end{bmatrix}, \begin{bmatrix} -4\\2\\-7\\3 \end{bmatrix}$

For Problems B7–B14:

(a) Determine adj(A).

(b) Calculate $A\,\text{adj}(A)$ and determine $\det A$ and A^{-1}.

B7 $A = \begin{bmatrix} a & b\\ 0 & c \end{bmatrix}$

B8 $A = \begin{bmatrix} 5 & 0 & 0\\ 3 & 10 & 0\\ -9 & -3 & 1 \end{bmatrix}$

B9 $A = \begin{bmatrix} 1 & 1 & 3\\ 0 & 2 & 1\\ 1 & 1 & -1 \end{bmatrix}$

B10 $A = \begin{bmatrix} 2 & 5 & t\\ 1 & 1 & 3\\ -1 & 1 & 3 \end{bmatrix}$

B11 $A = \begin{bmatrix} 1 & 4 & 3\\ 3 & t & 0\\ -2 & 1 & 1 \end{bmatrix}$

B12 $A = \begin{bmatrix} t & -2 & 3\\ 1 & 1 & 2\\ 1 & 1 & 3 \end{bmatrix}$

B13 $A = \begin{bmatrix} 2 & t & 3\\ 0 & -1 & 1\\ 1 & 1 & t \end{bmatrix}$

B14 $A = \begin{bmatrix} t & 1 & 1\\ 1 & t & 1\\ 1 & 1 & t \end{bmatrix}$

For Problems B15–B22, determine the inverse of the matrix by finding adj(A). Verify your answer by using multiplication.

B15 $\begin{bmatrix} 7 & -5 \\ 3 & 9 \end{bmatrix}$

B16 $\begin{bmatrix} 6 & 5 \\ 8 & 5 \end{bmatrix}$

B17 $\begin{bmatrix} 6 & 3 & 1 \\ 4 & 5 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

B18 $\begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 4 \\ -1 & 4 & 3 \end{bmatrix}$

B19 $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -4 \\ 0 & 0 & 11 \end{bmatrix}$

B20 $\begin{bmatrix} 3 & -2 & 7 \\ 3 & -3 & 9 \\ 1 & 1 & 7 \end{bmatrix}$

B21 $\begin{bmatrix} 0 & 3 & 7 \\ 2 & 0 & -8 \\ 3 & 1 & 0 \end{bmatrix}$

B22 $\begin{bmatrix} 3 & 2 & 4 & 5 \\ 1 & 1 & -2 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

For Problems B23–B30, use Cramer's Rule to solve the system.

B23 $\begin{aligned} x_1 + 2x_2 &= -5 \\ 4x_1 + 9x_2 &= -24 \end{aligned}$

B24 $\begin{aligned} 2x_1 - 3x_2 &= 4 \\ -7x_1 + 3x_2 &= 9 \end{aligned}$

B25 $\begin{aligned} 3x_1 + 5x_2 &= 7 \\ 11x_1 - 6x_2 &= 13 \end{aligned}$

B26 $\begin{aligned} 8x_1 - 3x_2 &= 15 \\ 6x_1 + 8x_2 &= 14 \end{aligned}$

B27 $\begin{aligned} x_1 + 3x_2 - x_3 &= 4 \\ -x_1 + x_2 + 2x_3 &= 1 \\ x_1 + 3x_2 + 4x_3 &= 4 \end{aligned}$

B28 $\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 3x_2 + 6x_3 &= 1 \\ -2x_1 + x_2 + 5x_3 &= 1 \end{aligned}$

B29 $\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 1 \\ 2x_1 - 6x_2 + 8x_3 &= 2 \\ 4x_1 + 4x_2 + 4x_3 &= 9 \end{aligned}$

B30 $\begin{aligned} 5x_1 + x_2 + 4x_3 &= 1 \\ 3x_1 + 3x_3 &= 3 \\ x_1 - 2x_2 + 2x_3 &= -2 \end{aligned}$

Conceptual Problems

C1 Let $\vec{v}_1, \dots, \vec{v}_{n-1} \in \mathbb{R}^n$, let $A = \begin{bmatrix} \vec{0} & \vec{v}_1 & \cdots & \vec{v}_{n-1} \end{bmatrix}$, and let $\vec{n} = \begin{bmatrix} C_{11} \\ C_{21} \\ \vdots \\ C_{n1} \end{bmatrix}$ where C_{ij} is the (i, j)-cofactor of A.

(a) Prove that $\vec{n}$ is orthogonal to $\vec{v}_i$ for $1 \leq i \leq n-1$.

(b) Assume that $\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$ is linearly independent. Find an equation for the hyperplane in $\mathbb{R}^n$ spanned by $\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$.

C2 Let $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 3 \end{bmatrix}$. Calculate $(A^{-1})_{23}$ and $(A^{-1})_{42}$. (If you calculate more than these two entries of A^{-1}, you have missed the point.)

C3 Let L be an invertible 3×3 upper triangular matrix. Prove that L^{-1} is also upper triangular.

C4 Suppose that $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ is an invertible $n \times n$ matrix.

(a) Verify by Cramer's Rule that the system of equations $A\vec{x} = \vec{a}_j$ has the unique solution $\vec{x} = \vec{e}_j$ (the j-th standard basis vector).

(b) Explain the result of part (a) in terms of linear transformations and/or matrix multiplication.

C5 Prove if $A \in M_{n\times n}(\mathbb{R})$ is not invertible, then adj A is not invertible.

For Problems C6–C10, let $A, B \in M_{n\times n}(\mathbb{R})$ with $n \geq 3$ both be invertible.

C6 Prove that $\text{adj}(A^{-1}) = \dfrac{1}{\det A} A = (\text{adj}(A))^{-1}$.

C7 Prove that $\det(\text{adj}(A)) = (\det A)^{n-1}$.

C8 Prove that $\text{adj}(\text{adj}(A)) = \det(A)^{n-2} A$.

C9 Prove that $\text{adj}(AB) = \text{adj}(B)\,\text{adj}(A)$.

C10 Prove that $\text{adj}(A^T) = (\text{adj}(A))^T$.

5.4 Area, Volume, and the Determinant

As mentioned at the beginning of this chapter, the value of a determinant has a very nice geometric interpretation. As a result, we can use determinants to solve a variety of geometric problems.

Area and the Determinant

In Chapter 1, we saw that we could construct a parallelogram from two vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ by making the vectors $\vec{u}$ and $\vec{v}$ as adjacent sides and having $\vec{u} + \vec{v}$ as the vertex of the parallelogram, opposite the origin. This is called the **parallelogram induced by $\vec{u}$ and $\vec{v}$**.

Theorem 5.4.1

The area of the parallelogram induced by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ is

$$\text{Area} = \left|\det \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\right|$$

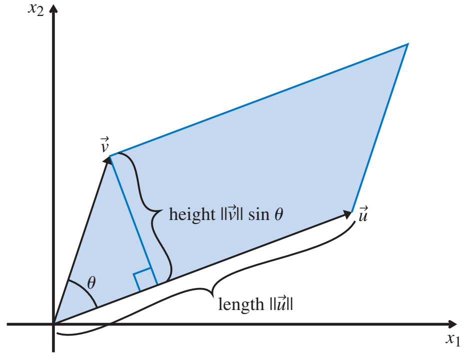
Proof: We have that

$$\text{Area} = \|\vec{u}\|\,\|\vec{v}\|\,|\sin\theta|$$

Recall from Section 1.3 that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

Using this gives



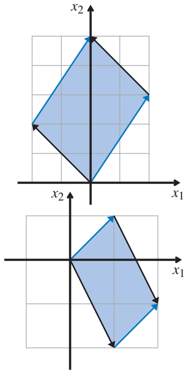
$$\begin{aligned}
(\text{Area})^2 &= \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta \\
&= \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta) \\
&= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta \\
&= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2 \\
&= (u_1^2+u_2^2)(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2 \\
&= u_1^2v_2^2 + u_2^2v_1^2 - 2(u_1v_2u_2v_1) \\
&= (u_1v_2-u_2v_1)^2 \\
&= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}^2
\end{aligned}$$

Taking the square root of both sides gives $\text{Area} = \left|\det\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}\right|$. ■

EXAMPLE 5.4.1

Draw the parallelogram induced by the following vectors and determine its area.

(a) $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ (b) $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



Solution: For (a), we have

$$\text{Area}(\vec{u}, \vec{v}) = \left| \det \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} \right| = |2(2) - (-2)(3)| = 10$$

For (b), we have

$$\text{Area}(\vec{u}, \vec{v}) = \left| \det \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \right| = |1(-2) - 1(1)| = 3$$

Theorem 5.4.2

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\text{Area} = \frac{1}{2} \left| \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \right|$$

Proof: If we perform a translation by $-x_1$ in the x-direction and by $-y_1$ in the y-direction so that (x_1, y_1) is relocated to the origin, then the area of the resulting triangle will be half the area of the parallelogram induced by the vectors $\vec{u} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \end{bmatrix}$ as in the diagram below. That is,

$$\text{Area} = \frac{1}{2} \left| \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \right|$$

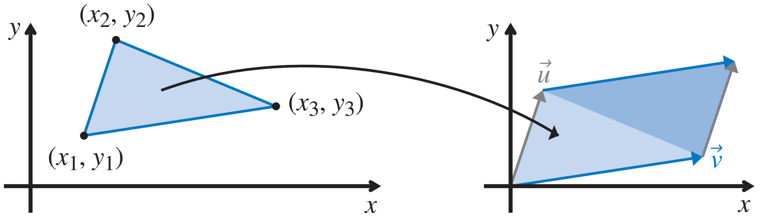
Rather than leaving it in this form, we expand the determinant to get

$$\text{Area} = \frac{1}{2} |(x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)|$$

which can be obtained from

$$\text{Area} = \frac{1}{2} \left| \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \right|$$

by performing a cofactor expansion along the third column.



■

EXAMPLE 5.4.2

Find the area of the triangle with vertices $(-1,-1)$, $(2,3)$, and $(4,2)$.

Solution: We have

$$\text{Area} = \frac{1}{2}\left|\det\begin{bmatrix} -1 & -1 & 1 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}\right| = \frac{11}{2}$$

Notice that the area of the triangle will be 0 if and only if the three points are colinear. Hence, we get the following theorem.

Theorem 5.4.3

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

If A is the standard matrix of a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$, then the images of $\vec{u}$ and $\vec{v}$ under L are $L(\vec{u}) = A\vec{u}$ and $L(\vec{v}) = A\vec{v}$. Moreover, the area of the **image parallelogram** is

$$\begin{aligned}
\text{Area}\,(A\vec{u}, A\vec{v}) &= \left|\det\begin{bmatrix} A\vec{u} & A\vec{v} \end{bmatrix}\right| && \text{by Theorem 5.4.1} \\
&= \left|\det\left(A\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\right)\right| && \text{by definition of matrix multiplication} \\
&= |\det A|\left|\det\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\right| && \text{since } \det(AB) = \det A \det B \\
&= |\det A|\,\text{Area}(\vec{u}, \vec{v}) && \text{by Theorem 5.4.1} \qquad (5.4)
\end{aligned}$$

In words: the absolute value of the determinant of the standard matrix A of a linear transformation is the factor by which area is changed under the linear transformation L. The result is illustrated in Figure 5.4.1.

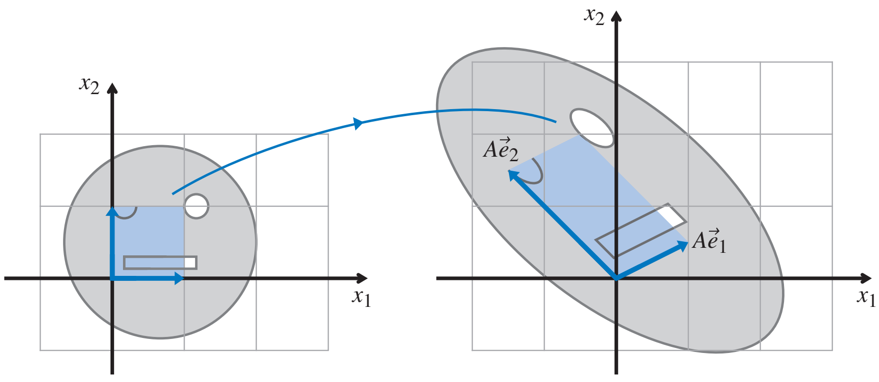


Figure 5.4.1 Under a linear transformation with matrix A, the area of a figure is changed by factor $|\det A|$.

EXAMPLE 5.4.3

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and L be the linear mapping $L(\vec{x}) = A\vec{x}$. Determine the image of $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ under L and compute the area determined by the image vectors in two ways.

Solution: The image of each vector under L is

$$L(\vec{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \qquad L(\vec{v}) = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Hence, the area determined by the image vectors is

$$\text{Area}\,(L(\vec{u}), L(\vec{v})) = \left|\det\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}\right| = |8 - 4| = 4$$

Or, using equation (5.4) gives

$$\text{Area}\,(L(\vec{u}), L(\vec{v})) = |\det A|\,\text{Area}\,(\vec{u}, \vec{v}) = \left|\det\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\right|\left|\det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right| = 2(2) = 4$$

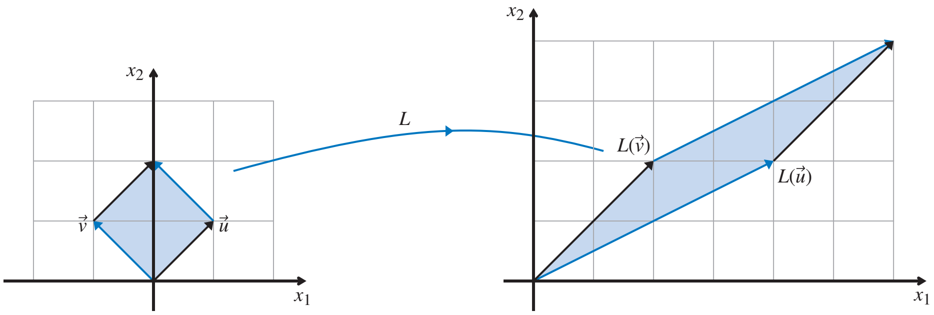


Figure 5.4.2 Illustration of Example 5.4.3.

EXERCISE 5.4.1

Let $A = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ be the standard matrix of the stretch $S : \mathbb{R}^2 \to \mathbb{R}^2$ in the x_1 direction by a factor of $t > 0$. Determine the image of the standard basis vectors $\vec{e}_1$ and $\vec{e}_2$ under S and compute the area determined by $A\vec{e}_1$ and $A\vec{e}_2$ by computing $\left|\det\begin{bmatrix} A\vec{e}_1 & A\vec{e}_2 \end{bmatrix}\right|$ and by using equation (5.4). Illustrate with a sketch.

The Determinant and Volume

Recall from Chapter 1 that if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, then the volume of the parallelepiped induced by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$\text{Volume}(\vec{u}, \vec{v}, \vec{w}) = |\vec{w} \cdot (\vec{u} \times \vec{v})|$$

Now observe that if $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, then

$$|\vec{w} \cdot (\vec{u} \times \vec{v})| = |w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)| = \left|\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right|$$

Hence, the volume of the parallelepiped induced by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$\text{Volume}(\vec{u}, \vec{v}, \vec{w}) = \left|\det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\right|$$

EXAMPLE 5.4.4

Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$. Calculate the volume of the parallelepiped induced by $\vec{u}$, $\vec{v}$, and $\vec{w}$ and the volume of the parallelepiped induced by $A\vec{u}$, $A\vec{v}$, and $A\vec{w}$.

Solution: The volume determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$\text{Volume}(\vec{u}, \vec{v}, \vec{w}) = \left|\det \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 5 \\ 0 & 2 & 1 \end{bmatrix}\right| = |-7| = 7$$

The volume determined by $A\vec{u}$, $A\vec{v}$, and $A\vec{w}$ is

$$\begin{aligned} \text{Volume}(A\vec{u}, A\vec{v}, A\vec{w}) &= \left|\det \begin{bmatrix} A\vec{u} & A\vec{v} & A\vec{w} \end{bmatrix}\right| \\ &= \left|\det \begin{bmatrix} 3 & -1 & 0 \\ -2 & 6 & 19 \\ 0 & 9 & 8 \end{bmatrix}\right| = |-385| = 385 \end{aligned}$$

Moreover, $\det A = 55$, so

$$\text{Volume}\,(A\vec{u}, A\vec{v}, A\vec{w}) = |\det A|\text{Volume}\,(\vec{u}, \vec{v}, \vec{w})$$

which coincides with the result for the 2×2 case.

In general, if $\vec{v}_1, \dots, \vec{v}_n$ are n vectors in $\mathbb{R}^n$, then we say that they induce an **n-dimensional parallelotope** (the n-dimensional version of a parallelogram or parallelepiped). The **n-volume** of the parallelotope is

$$n\text{-Volume}(\vec{v}_1, \dots, \vec{v}_n) = \left|\det \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\right|$$

and if A is the standard matrix of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$, then

$$n\text{-Volume}\,(A\vec{v}_1, \dots, A\vec{v}_n) = |\det A|n\text{-Volume}\,(\vec{v}_1, \dots, \vec{v}_n)$$

PROBLEMS 5.4

Practice Problems

For Problems A1–A3:

(a) Calculate the area of the parallelogram induced by $\vec{u}$ and $\vec{v}$.

(b) Determine the area of the parallelogram induced by $A\vec{u}$ and $A\vec{v}$ by computing $\left|\det\begin{bmatrix} A\vec{u} & A\vec{v}\end{bmatrix}\right|$ and by using equation (5.4).

A1 $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

A2 $\vec{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

A3 $\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, A = \begin{bmatrix} -4 & 6 \\ 3 & 2 \end{bmatrix}$

A4 Given that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix of the reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ over the line $x_2 = x_1$, find the image of $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ under R, and compute the area determined by $A\vec{u}$ and $A\vec{v}$.

For Problems A5 and A6:

(a) Calculate the area of the parallelogram induced by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

(b) Determine the volume of the parallelepiped induced by $A\vec{u}$, $A\vec{v}$, and $A\vec{w}$.

A5 $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$

A6 $\vec{u} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}, A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \\ 1 & 4 & 5 \end{bmatrix}$

For Problems A7–A11, determine whether the given points are collinear. If they are not collinear, find the area of the triangle which has those points as its vertices.

A7 $(-1,3), (2,5), (3,7)$

A8 $(-4,-1), (2,3), (5,5)$

A9 $(0,2), (3,3), (4,5)$

A10 $(1,4), (6,7), (9,0)$

A11 $(-3,-26), (1,2), (3,16)$

A12 (a) Calculate the 4-volume of the 4-dimensional parallelotope determined by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 5 \end{bmatrix}.$$

(b) Calculate the 4-volume of the image of this parallelotope under the linear mapping with standard matrix $A = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 5 & 4 & 3 & 0 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

A13 Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be vectors in $\mathbb{R}^n$. Prove that the n-volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_n$ is the same as the volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_{n-1}, \vec{v}_n + t\vec{v}_1$.

Homework Problems

For Problems B1–B3:

(a) Calculate the area of the parallelogram induced by $\vec{u}$ and $\vec{v}$.

(b) Determine the area of the parallelogram induced by $A\vec{u}$ and $A\vec{v}$ by computing $\left|\det\begin{bmatrix} A\vec{u} & A\vec{v}\end{bmatrix}\right|$ and by using equation (5.4).

B1 $\vec{u} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \vec{v} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}, A = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

B2 $\vec{u} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 2 & 3 \\ 5 & 3 \end{bmatrix}$

B3 $\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, A = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$

For Problems B4–B10, determine the area of the parallelogram induced by the images of the standard basis vectors $\vec{e}_1$ and $\vec{e}_2$ of $\mathbb{R}^2$ under the given transformation.

B4 A rotation through angle $\theta = \pi/4$.

B5 A stretch in the x_2-direction by a factor of 3.

B6 A dilation by a factor of 2.

B7 A horizontal shear by an amount of 5.

B8 A vertical shear by an amount of -2.

B9 A stretch in the x_1-direction by a factor of 4.

B10 A horizontal shear by an amount of 3 followed by a stretch in the x_2-direction by a factor of 2.

For Problems B11 and B12:

(a) Calculate the volume of the parallelepiped induced by $\vec{u}$, $\vec{v}$, and $\vec{w}$.
(b) Determine the volume of the parallelepiped induced by $A\vec{u}$, $A\vec{v}$, and $A\vec{w}$.

B11 $\vec{u} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

B12 $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, A = \begin{bmatrix} 2 & 4 & 4 \\ 1 & 3 & 3 \\ 1 & -1 & 4 \end{bmatrix}$

For Problems B13–B19, determine whether the given points are collinear. If they are not collinear, find the area of the triangle which has those points as its vertices.

B13 $(-3, 1), (-2, 3), (1, -1)$
B14 $(0, -3), (1, 2), (3, 5)$
B15 $(-2, 2), (2, -1), (6, -4)$
B16 $(-3, -7), (1, -1), (4, 4)$
B17 $(-1, 2), (2, -1), (4, 3)$
B18 $(-5, 7), (1, -1), (4, -5)$
B19 $(-4, -2), (-1, 0), (3, 1)$

For Problems B20 and B21:

(a) Calculate the 4-volume of the 4-dimensional parallelotope determined by $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.
(b) Calculate the 4-volume of the image of this parallelotope under the linear mapping with standard matrix $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 2 & 2 & 4 & 2 \\ 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 3 \end{bmatrix}$.

B20 $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix}$

B21 $\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$

Conceptual Problems

C1 Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be vectors in $\mathbb{R}^n$. Prove that the n-volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_n$ is half the volume of the parallelotope induced by $2\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

C2 Suppose that $L, M : \mathbb{R}^3 \to \mathbb{R}^3$ are linear mappings with standard matrices A and B, respectively. Prove that the factor by which a volume is multiplied under the composite map $M \circ L$ is $|\det BA|$.

CHAPTER REVIEW

Suggestions for Student Review

1 Define *cofactor* and explain cofactor expansion. Be especially careful about signs. (Section 5.1)

2 State as many facts as you can that simplify the evaluation of determinants. For each fact, explain why it is true. (Sections 5.1, 5.2)

3 List the properties of determinants, and list everything you can about a matrix A that is equivalent to $\det A \neq 0$. (Sections 5.2, 5.3)

4 Explain and justify the cofactor method for finding a matrix inverse. Write down a 3×3 matrix A and calculate $A(\text{cof } A)^T$. (Section 5.3)

5 State Cramer's Rule. Create a system of 3 linear equations in 3 variables that contains at least two unknown values a and b, and use Cramer's Rule to solve the system. (Section 5.3)

6 How are determinants connected to areas and volumes? (Section 5.4)

Chapter Quiz

For Problems E1–E6, find the determinant of the matrix. Decide whether the matrix is invertible.

E1 $\begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}$

E2 $\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & \sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}$

E3 $\begin{bmatrix} 3 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 4 & -1 \end{bmatrix}$

E4 $\begin{bmatrix} -2 & 4 & 0 & 0 \\ 1 & -2 & 2 & 9 \\ -3 & 6 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

E5 $\begin{bmatrix} 3 & 2 & 7 & -8 \\ -6 & -1 & -9 & 20 \\ 3 & 8 & 21 & -17 \\ 3 & 5 & 12 & 1 \end{bmatrix}$

E6 $\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 & 6 \end{bmatrix}$

E7 Determine all values of k such that the matrix $\begin{bmatrix} k & 2 & 1 \\ 0 & 3 & k \\ 2 & -4 & 1 \end{bmatrix}$ is invertible.

For Problems E8–E12, suppose that $A \in M_{5\times 5}(\mathbb{R})$ and $\det A = 7$.

E8 If B is obtained from A by multiplying the fourth row of A by 3, what is $\det B$?

E9 If C is obtained from A by moving the first row to the bottom and moving all other rows up, what is $\det C$? Justify.

E10 What is $\det(2A)$?

E11 What is $\det(A^{-1})$?

E12 What is $\det(A^T A)$?

E13 Let $A = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{bmatrix}$. Determine all values of λ such that the determinant of A is 0.

E14 Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix}$.

(a) Determine $\operatorname{adj}(A)$.

(b) Calculate $A\operatorname{adj}(A)$ and determine $\det A$.

(c) Determine $(A^{-1})_{31}$.

E15 Determine x_2 by using Cramer's Rule if

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= -1 \\ -2x_1 \qquad + 2x_3 &= 1 \end{aligned}$$

E16 (a) What is the volume of the parallelepiped induced by $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$?

(b) If $A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & -4 \end{bmatrix}$, what is the volume of the parallelepiped induced by $A\vec{u}$, $A\vec{v}$, and $A\vec{w}$?

E17 Find a non-zero vector $\vec{x} \in \mathbb{R}^4$ that is orthogonal to

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

For Problems E18–E20, determine if the given points are collinear. If they are not collinear, find the area of the triangle which has those points as its vertices.

E18 $(-2,-1), (1,8), (2,11)$

E19 $(-1,5), (2,7), (3,3)$

E20 $(-3,4), (0,3), (2,2)$

Further Problems

These exercises are intended to be challenging.

F1 Suppose that $A \in M_{n\times n}(\mathbb{R})$ with all row sums equal to zero. (That is, $\sum_{j=1}^{n} a_{ij} = 0$ for $1 \leq i \leq n$.) Prove that $\det A = 0$.

F2 Suppose that A and A^{-1} both have all integer entries. Prove that $\det A = \pm 1$.

F3 Consider a triangle in the plane with side lengths a, b, and c. Let the angles opposite the sides with lengths a, b, and c be denoted by A, B, and C, respectively. By using trigonometry, show that

$$c = b\cos A + a\cos B$$

Write similar equations for the other two sides. Use Cramer's Rule to show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

F4 Suppose that $A \in M_{3\times 3}(\mathbb{R})$ and $B \in M_{2\times 2}(\mathbb{R})$.

(a) Show that $\det \left[\begin{array}{c|c} A & O_{3,2} \\ \hline O_{2,3} & B \end{array}\right] = \det A \det B$.

(b) What is $\det \left[\begin{array}{c|c} O_{2,3} & B \\ \hline A & O_{3,2} \end{array}\right]$?

F5 (a) Let $V_3(a,b,c) = \det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$. Without expanding, argue that $(a-b)$, $(b-c)$, and $(c-a)$ are all factors of $V_3(a,b,c)$. By considering the cofactor of c^2, argue that

$$V_3(a,b,c) = (c-a)(c-b)(b-a)$$

(b) Let $V_4(a,b,c,d) = \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}$. By using arguments similar to those in part (a) (and without expanding the determinant), argue that

$$V_4(a,b,c,d) = (d-a)(d-b)(d-c)V_3(a,b,c)$$

The determinant $V_n(x_1,\dots,x_n)$ is called the **Vandermonde determinant**. We will use the related Vandermonde matrix when we do the Method of Least Squares in Chapter 7.

F6 Suppose that $A \in M_{4\times 4}(\mathbb{R})$ is partitioned into 2×2 blocks:

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]$$

(a) If $A_3 = O_{2,2}$ (the 2×2 zero matrix), show that $\det A = \det A_1 \det A_4$.

(b) Give an example to show that, in general,

$$\det A \neq \det A_1 \det A_4 - \det A_2 \det A_3$$

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CHAPTER 6

Eigenvectors and Diagonalization

CHAPTER OUTLINE

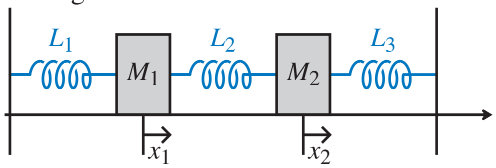
An eigenvector *is a special vector of a matrix that is mapped by the matrix to a scalar multiple of itself. Eigenvectors play an important role in many applications in the natural and physical sciences. In this chapter we will synthesize many of the concepts from the previous five chapters to develop this powerful tool.*

6.1 Eigenvalues and Eigenvectors

EXAMPLE 6.1.1 Consider the spring-mass system depicted below. Assume that the masses are identical and are attached to identical springs with spring constant k. As we did in Chapter 2, we denote the equilibrium displacements by x_1 and x_2 with a positive displacement being to the right. We would like to determine how to describe the motion of the masses that would result from the masses being displaced and then released.

According to Hooke's Law, the net force acting on the first mass is

$$-kx_1 + k(x_2 - x_1)$$



and the net force acting on the second mass is

$$-k(x_2 - x_1) - kx_2$$

By Newton's second law, the net force acting on a mass is also equal to

$$F = ma = m\frac{d^2x}{dt^2}$$

Thus, we have

$$-kx_1 + k(x_2 - x_1) = m\frac{d^2x_1}{dt^2}$$
$$-k(x_2 - x_1) - kx_2 = m\frac{d^2x_2}{dt^2}$$

EXAMPLE 6.1.1 (continued)

Experience tells us that the displacements x_1 and x_2 should undergo exponential decay. Thus, we take $x_1 = ae^{pt}$ and $x_2 = be^{pt}$. Substituting these into the equations above and rearranging, we can get

$$\begin{aligned} -2a + b &= \frac{mp^2}{k}a \\ a - 2b &= \frac{mp^2}{k}b \end{aligned}$$

Defining $\lambda = \dfrac{mp^2}{k}$ and rewriting the system in matrix form gives

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

Solving this for λ gives us the value of p, which tells us the frequencies of oscillation.

Many problems in science and engineering require us to solve a system of linear equations of the form

$$A\vec{x} = \lambda\vec{x}$$

This is often referred to as the **eigenvalue problem**.

Definition
Eigenvector
Eigenvalue

Suppose that $A \in M_{n\times n}(\mathbb{R})$. If there exists a non-zero vector $\vec{v} \in \mathbb{R}^n$ such that

$$A\vec{v} = \lambda\vec{v}$$

then the scalar λ is called an **eigenvalue** of A and $\vec{v}$ is called an **eigenvector** of A corresponding to λ.

Remarks

1. The pairing of eigenvalues and eigenvectors is not one-to-one. In particular, we will see that each eigenvector of A will correspond to a distinct eigenvalue, while each eigenvalue will have infinitely many eigenvectors.
2. We have restricted our definition of eigenvectors (and hence eigenvalues) to be real. In Chapter 9 we will consider the case where we allow eigenvalues and eigenvectors to be complex.
3. The restriction that an eigenvector $\vec{v}$ be non-zero is natural and important. It is natural because $A\vec{0} = \vec{0}$ for any matrix A, so it is uninteresting to consider $\vec{0}$ as an eigenvector. It is important because most of the applications of eigenvectors make sense only for non-zero vectors.

Finding Eigenvalues and Eigenvectors

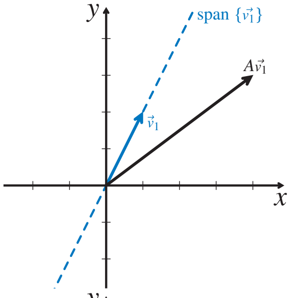
We now look at how to find eigenvalues and eigenvectors of a matrix. We begin by observing that it is easy to verify if a given vector is an eigenvector of a matrix.

EXAMPLE 6.1.2

Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Determine whether $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A. If so, give the corresponding eigenvalue.

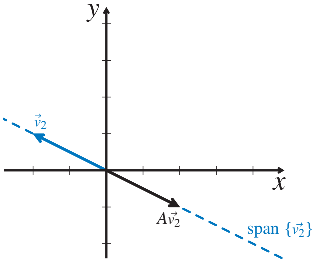
Solution: For $\vec{v}_1$, we find that

$$A\vec{v}_1 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Since, $A\vec{v}_1$ is not a scalar multiple of $\vec{v}_1$, $\vec{v}_1$ is not an eigenvector of A.
For $\vec{v}_2$, we find that

$$A\vec{v}_2 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Thus, $A\vec{v}_2$ is scalar multiple of $\vec{v}_2$ (it is -1 times $\vec{v}_2$), so $\vec{v}_2$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

If eigenvalues are going to be of any use, we need a systematic method for finding them. We first look at how to do this in a simple example before exploring the general procedure.

EXAMPLE 6.1.3

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find all eigenvalues of A.

Solution: For $\lambda \in \mathbb{R}$ to be an eigenvalue of A there must exist $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \vec{0}$ such that

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

We can rewrite this as the system of linear equations

$$\begin{aligned} 2v_1 + 2v_2 &= \lambda v_1 \\ 1v_1 + 3v_2 &= \lambda v_2 \end{aligned}$$

We put this into standard form of a system of linear equations by collecting the variables v_1 and v_2 on the left side. We get

$$\begin{aligned} (2-\lambda)v_1 + 2v_2 &= 0 \\ 1v_1 + (3-\lambda)v_2 &= 0 \end{aligned}$$

EXAMPLE 6.1.3 (continued)

For λ to be an eigenvalue, we need this system to have non-zero solutions so that $\vec{v}$ is a non-zero vector. By the Invertible Matrix Theorem, we know that this will only occur when the coefficient matrix $\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$ is not invertible. Hence, we can just determine when the determinant of this matrix equals 0. That is, we need to solve

$$0 = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2(1) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

Hence, $A\vec{v} = \lambda\vec{v}$ has non-zero solutions if and only if $\lambda = 4$ or $\lambda = 1$. Therefore, the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 1$.

Our work in Example 6.1.3 also shows us how to find the eigenvectors corresponding to an eigenvalue. For example, to find all infinitely many eigenvectors of the matrix A in Example 6.1.3 corresponding to $\lambda_1 = 4$, we just need to find the solution space of the homogeneous system with coefficient matrix

$$\begin{bmatrix} 2-\lambda_1 & 2 \\ 1 & 3-\lambda_1 \end{bmatrix}$$

In general, for any $A \in M_{n\times n}(\mathbb{R})$ a non-zero vector $\vec{v} \in \mathbb{R}^n$ is an eigenvector of A if and only if $A\vec{v} = \lambda\vec{v}$. So, as in Example 6.1.3, this condition can be rewritten as

$$(A - \lambda I)\vec{v} = \vec{0}$$

Thus, the eigenvector $\vec{v}$ is any non-trivial solution (since it cannot be the zero vector) of the homogeneous system of linear equations with coefficient matrix $(A-\lambda I)$. By the Invertible Matrix Theorem, the system $(A - \lambda I)\vec{v} = \vec{0}$ has non-trivial solutions if and only if the coefficient matrix $(A-\lambda I)$ has determinant equal to 0. Hence, for λ to be an eigenvalue, we must have $\det(A - \lambda I) = 0$. This is the key result in the procedure for finding the eigenvalues and eigenvectors, so it is worth summarizing as a theorem.

Theorem 6.1.1

Suppose that $A \in M_{n\times n}(\mathbb{R})$. A real number λ is an eigenvalue of A if and only if λ satisfies the equation

$$\det(A - \lambda I) = 0$$

If λ is an eigenvalue of A, then all non-zero solutions of the homogeneous system

$$(A - \lambda I)\vec{v} = \vec{0}$$

are all the eigenvectors of A that correspond to λ.

Observe that the set of all eigenvectors corresponding to an eigenvalue λ is just the nullspace of $A - \lambda I$, excluding the zero vector. In particular, the set containing all eigenvectors corresponding to λ and the zero vector is a subspace of $\mathbb{R}^n$. We make the following definition.

Definition
Eigenspace

Let λ be an eigenvalue of a matrix A. The set containing the zero vector and all eigenvectors of A corresponding to λ is called the **eigenspace** of λ and is denoted E_λ. In particular, we have

$$E_\lambda = \mathrm{Null}(A - \lambda I)$$

Remark

From our work preceding the theorem, we see that the eigenspace of any eigenvalue λ must contain at least one non-zero vector. Hence, the dimension of the eigenspace must be at least 1.

EXAMPLE 6.1.4

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix}$.

Solution: We have

$$A - \lambda I = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17-\lambda & -15 \\ 20 & -18-\lambda \end{bmatrix}$$

(You should set up your calculations like this: you will need $A - \lambda I$ later when you find the eigenvectors.) Then

$$\det(A - \lambda I) = \begin{vmatrix} 17-\lambda & -15 \\ 20 & -18-\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$$

Since $\det(A - \lambda I) = 0$ when $\lambda = -3, 2$, the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 2$.

To find all the eigenvectors corresponding to $\lambda_1 = -3$, we solve $(A - \lambda_1 I)\vec{v} = \vec{0}$. Writing $A - \lambda_1 I$ and row reducing gives

$$A - (-3)I = \begin{bmatrix} 20 & -15 \\ 20 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$$

The general solution of $(A - \lambda_1 I)\vec{v} = \vec{0}$ is $\vec{v} = t\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. Thus, all eigenvectors of A corresponding to $\lambda_1 = -3$ are $\vec{v} = t\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ for any *non-zero* value of t, and the eigenspace for $\lambda_1 = -3$ is $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}\right\}$.

We repeat the process for the eigenvalue $\lambda_2 = 2$:

$$A - 2I = \begin{bmatrix} 15 & -15 \\ 20 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The general solution of $(A - \lambda_2 I)\vec{v} = \vec{0}$ is $\vec{v} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$, so the eigenspace for $\lambda_2 = 2$ is $E_{\lambda_2} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. In particular, all eigenvectors of A corresponding to $\lambda_2 = 2$ are all non-zero multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Observe in Example 6.1.4 that $\det(A - \lambda I)$ gave us a degree 2 polynomial. This motivates the following definition.

Definition
Characteristic Polynomial

Let $A \in M_{n\times n}(\mathbb{R})$. We call $C(\lambda) = \det(A - \lambda I)$ the **characteristic polynomial** of A.

For $A \in M_{n\times n}(\mathbb{R})$, the characteristic polynomial $C(\lambda)$ is of degree n, and the roots of $C(\lambda)$ are the eigenvalues of A. Note that the term of highest degree λ^n has coefficient $(-1)^n$; some other books prefer to work with the polynomial $\det(\lambda I - A)$ so that the coefficient of λ^n is always 1. In our notation, the constant term in the characteristic polynomial is $\det A$. See Section 6.2 Problem C7.

It is relevant here to recall some facts about the roots of an n-th degree polynomial with real coefficients:

(1) λ_1 is a root of $C(\lambda)$ if and only if $(\lambda - \lambda_1)$ is a factor of $C(\lambda)$.

(2) The total number of roots (real and complex, counting repetitions) is n.

(3) Complex roots of the equation occur in "conjugate pairs," so that the total number of complex roots must be even.

(4) If n is odd, there must be at least one real root.

(5) If the entries of A are integers, then, since the leading coefficient of the characteristic polynomial is ± 1, any rational root must be an integer.

EXAMPLE 6.1.5

Find the eigenvalues and a basis for each eigenspace of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution: The characteristic polynomial is

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)$$

So, $\lambda = 1$ is a double root (that is, $(\lambda - 1)$ appears as a factor of $C(\lambda)$ twice). Thus, $\lambda_1 = 1$ is the only distinct eigenvalue of A.
For $\lambda_1 = 1$, we have

$$A - \lambda_1 I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has the general solution $\vec{v} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$. Thus, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

EXERCISE 6.1.1

Find the eigenvalues and a basis for each eigenspace of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

EXAMPLE 6.1.6

Find the eigenvalues and a basis for each eigenspace of $A = \begin{bmatrix} -3 & 5 & -5 \\ -7 & 9 & -5 \\ -7 & 7 & -3 \end{bmatrix}$.

Solution: We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 5 & -5 \\ -7 & 9-\lambda & -5 \\ -7 & 7 & -3-\lambda \end{vmatrix}$$

Expanding this determinant along some row or column will involve a fair number of calculations. Also, we will end up with a degree 3 polynomial, which may not be easy to factor. So, we use properties of determinants to make it easier. Using $R_2 - R_1$ and then $C_1 + C_2$ gives

$$C(\lambda) = \begin{vmatrix} -3-\lambda & 5 & -5 \\ -4+\lambda & 4-\lambda & 0 \\ -7 & 7 & -3-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 5 & -5 \\ 0 & 4-\lambda & 0 \\ 0 & 7 & -3-\lambda \end{vmatrix}$$

Now, expanding this determinant along the first column gives

$$\begin{aligned} C(\lambda) &= (2-\lambda)[(4-\lambda)(-3-\lambda) - 0(7)] \\ &= (2-\lambda)(4-\lambda)(-3-\lambda) \end{aligned}$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 4$, and $\lambda_3 = -3$.
For $\lambda_1 = 2$,

$$A - \lambda_1 I = \begin{bmatrix} -5 & 5 & -5 \\ -7 & 7 & -5 \\ -7 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of $(A - \lambda_1 I)\vec{v} = \vec{0}$ is $\vec{v} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$. So, $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for E_{λ_1}.

For $\lambda_2 = 4$,

$$A - \lambda_2 I = \begin{bmatrix} -7 & 5 & -5 \\ -7 & 5 & -5 \\ -7 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of $(A - \lambda_2 I)\vec{v} = \vec{0}$ is $\vec{v} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. So, $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for E_{λ_2}.

For $\lambda_3 = -3$,

$$A - \lambda_3 I = \begin{bmatrix} 0 & 5 & -5 \\ -7 & 12 & -5 \\ -7 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of $(A - \lambda_3 I)\vec{v} = \vec{0}$ is $\vec{v} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. So, $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for E_{λ_3}.

EXAMPLE 6.1.7

Find the eigenvalues and a basis for each eigenspace of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution: Using $R_3 - R_2$ and then $C_2 + C_3$ gives

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 3\lambda) = -\lambda^2(\lambda - 3)$$

Therefore, the eigenvalues of A are $\lambda_1 = 0$ (which occurs twice) and $\lambda_2 = 3$.
For $\lambda_1 = 0$,

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of $\lambda_1 = 0$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 3$,

$$A - \lambda_2 I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of $\lambda_2 = 3$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

These examples motivate the following definitions.

Definition
Algebraic Multiplicity
Geometric Multiplicity

Let $A \in M_{n\times n}(\mathbb{R})$ with eigenvalue λ. The **algebraic multiplicity** of λ is the number of times λ is repeated as a root of the characteristic polynomial. The **geometric multiplicity** of λ is the dimension of the eigenspace of λ.

EXAMPLE 6.1.8

In Example 6.1.5, the eigenvalue $\lambda_1 = 1$ has algebraic multiplicity 2 since the characteristic polynomial is $(\lambda - 1)(\lambda - 1)$, and it has geometric multiplicity 1 since a basis for its eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

In Example 6.1.6, all three eigenvalues have algebraic multiplicity 1 and they all have geometric multiplicity 1.

In Example 6.1.7, the eigenvalue $\lambda_1 = 0$ has algebraic and geometric multiplicity 2, and the eigenvalue $\lambda_2 = 3$ has algebraic and geometric multiplicity 1.

EXERCISE 6.1.2

Let $A = \begin{bmatrix} 5 & -3 & 2 \\ 0 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix}$. Show that $\lambda_1 = 5$ and $\lambda_2 = -2$ are both eigenvalues of A and determine the algebraic and geometric multiplicity of both of these eigenvalues.

These definitions lead to some theorems that will be very important in the next section.

Theorem 6.1.2

If λ is an eigenvalue of a matrix $A \in M_{n\times n}(\mathbb{R})$, then

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}$$

The proof of Theorem 6.1.2 is beyond the scope of this course.

If the geometric multiplicity of an eigenvalue is less than its algebraic multiplicity, then we say that the eigenvalue is **deficient**. However, if $A \in M_{n\times n}(\mathbb{R})$ with distinct eigenvalues $\lambda_1, \dots, \lambda_k$, which all have the property that their geometric multiplicity equals their algebraic multiplicity, then the sum of the geometric multiplicities of all eigenvalues equals the sum of the algebraic multiplicities, which equals n (since an n-th degree polynomial has exactly n roots). Hence, if we collect the basis vectors from the eigenspaces of all k eigenvalues, we will end up with n vectors in $\mathbb{R}^n$. The next theorem states that eigenvectors from eigenspaces of different eigenvalues are necessarily linearly independent, and hence this collection of n eigenvectors will form a basis for $\mathbb{R}^n$.

Theorem 6.1.3

If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of a matrix $A \in M_{n\times n}(\mathbb{R})$, with corresponding eigenvectors $\vec{v}_1, \dots, \vec{v}_k$, respectively, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

The proof of Theorem 6.1.3 is left as Problem F4 at the end of the chapter.

Remark

In this book, most eigenvalues turn out to be integers. This is very unrealistic; in real world applications, eigenvalues are often not rational numbers. Effective computer methods for finding eigenvalues depend on the theory of eigenvectors and eigenvalues.

Theorem 6.1.4

Invertible Matrix Theorem continued

If $A \in M_{n\times n}(\mathbb{R})$, then the following are equivalent:

(1) A is invertible.
(11) $\lambda = 0$ is not an eigenvalue of A.

The proof of Theorem 6.1.4 is left as Problem C6.

Eigenvalues and Eigenvectors of Linear Mappings

The geometric meaning of eigenvectors and eigenvalues becomes much clearer when we think of them as belonging to linear transformations.

Definition
Eigenvector
Eigenvalue

Suppose that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping. If there exists a non-zero vector $\vec{v} \in \mathbb{R}^n$ such that $L(\vec{v}) = \lambda\vec{v}$, then λ is called an **eigenvalue** of L and $\vec{v}$ is called an **eigenvector** of L corresponding λ.

EXAMPLE 6.1.9

Eigenvectors and Eigenvalues of Projections and Reflections in $\mathbb{R}^3$

1. Because $\text{proj}_{\vec{n}}(\vec{n}) = 1\vec{n}$, $\vec{n}$ is an eigenvector of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 1. If $\vec{v} \neq \vec{0}$ is orthogonal to $\vec{n}$, then $\text{proj}_{\vec{n}}(\vec{v}) = \vec{0} = 0\vec{v}$, so $\vec{v}$ is an eigenvector of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 0. Observe that this means there is a whole plane of eigenvectors corresponding to the eigenvalue 0 as the set of vectors orthogonal to $\vec{n}$ is a plane in $\mathbb{R}^3$. For an arbitrary vector $\vec{u} \neq \vec{0}$, $\text{proj}_{\vec{n}}(\vec{u})$ is a multiple of $\vec{n}$, so that $\vec{u}$ is definitely *not* an eigenvector of $\text{proj}_{\vec{n}}$ unless it is a multiple of $\vec{n}$ or orthogonal to $\vec{n}$.

2. On the other hand, $\text{perp}_{\vec{n}}(\vec{n}) = \vec{0} = 0\vec{n}$, so $\vec{n}$ is an eigenvector of $\text{perp}_{\vec{n}}$ with eigenvalue 0. For $\vec{v} \neq \vec{0}$ orthogonal to $\vec{n}$, $\text{perp}_{\vec{n}}(\vec{v}) = 1\vec{v}$, so such a $\vec{v}$ is an eigenvector of $\text{perp}_{\vec{n}}$ with eigenvalue 1.

3. We have $\text{refl}_{\vec{n}}(\vec{n}) = -1\vec{n}$, so $\vec{n}$ is an eigenvector of $\text{refl}_{\vec{n}}$ with eigenvalue -1. For $\vec{v} \neq \vec{0}$ orthogonal to $\vec{n}$, $\text{refl}_{\vec{n}}(\vec{v}) = 1\vec{v}$. Hence, such a $\vec{v}$ is an eigenvector of $\text{refl}_{\vec{n}}$ with eigenvalue 1.

EXAMPLE 6.1.10

Eigenvectors and Eigenvalues of Rotations in $\mathbb{R}^2$

Consider the rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where θ is not an integer multiple of π. Thinking geometrically, it is clear that there is no non-zero vector $\vec{v}$ in $\mathbb{R}^2$ such that $R_\theta(\vec{v}) = \lambda\vec{v}$ for some real number λ. This linear transformation has no real eigenvalues or real eigenvectors. In Chapter 9 we will see that it does have complex eigenvalues and complex eigenvectors.

EXERCISE 6.1.3

Let $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ denote the rotation in $\mathbb{R}^3$ with matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where θ is not an integer multiple of π. Determine any real eigenvectors of R_θ and the corresponding eigenvalues.

The Power Method of Determining Eigenvalues

Practical applications of eigenvalues often involve larger matrices with non-integer entries. Such problems often require efficient computer methods for determining eigenvalues. A thorough discussion of such methods is beyond the scope of this book, but we can indicate how powers of matrices provide one tool for finding eigenvalues.

Let $A \in M_{n\times n}(\mathbb{R})$. To simplify the discussion, we suppose that A has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$, with corresponding eigenvectors $\vec{v}_1, \dots, \vec{v}_n$. We suppose that $|\lambda_1| > |\lambda_i|$ for $2 \leq i \leq n$. We call λ_1 the **dominant** eigenvalue. By Theorem 6.1.3, the set $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent and hence will form a basis for $\mathbb{R}^n$ by Theorem 2.3.6. Thus, any vector $\vec{x} \in \mathbb{R}^n$ can be written

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

Multiplying both sides by A on the left gives

$$\begin{aligned} A\vec{x} &= A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \\ &= c_1A\vec{v}_1 + \cdots + c_nA\vec{v}_n \\ &= c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n \end{aligned}$$

If we multiply again by A on the left, we get

$$\begin{aligned} A^2\vec{x} &= A(c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n) \\ &= c_1\lambda_1A\vec{v}_1 + \cdots + c_n\lambda_nA\vec{v}_n \\ &= c_1\lambda_1^2\vec{v}_1 + \cdots + c_n\lambda_n^2\vec{v}_n \end{aligned}$$

Continuing this, we get

$$A^m\vec{x} = c_1\lambda_1^m\vec{v}_1 + \cdots + c_n\lambda_n^m\vec{v}_n$$

For m large, $|\lambda_1^m|$ is much greater than all other terms. Assuming $c_1 \neq 0$, if we divide by $c_1\lambda_1^m$, then all terms on the right-hand side will be negligibly small except for $\vec{v}_1$, so we will be able to identify $\vec{v}_1$. By calculating $A\vec{v}_1$, we determine λ_1.

To make this into an effective procedure, we must control the size of the vectors: if $\lambda_1 > 1$, then $\lambda_1^m \to \infty$ as m gets large, and the procedure would break down. Similarly, if all eigenvalues are between 0 and 1, then $A^m\vec{x} \to \vec{0}$, and the procedure would fail. To avoid these problems, we normalize the vector at each step (that is, convert it to a vector of length 1).

Algorithm 6.1.1 Power Method for Approximating Eigenvalues

For $A \in M_{n\times n}(\mathbb{R})$, pick an initial vector $\vec{x}_0 \in \mathbb{R}^n$ and calculate $\vec{y}_0 = \dfrac{1}{\|\vec{x}_0\|}\vec{x}_0$.

Let $\vec{x}_1 = A\vec{y}_0$ and then calculate $\vec{y}_1 = \dfrac{1}{\|\vec{x}_1\|}\vec{x}_1$.

Let $\vec{x}_2 = A\vec{y}_1$ and then calculate $\vec{y}_2 = \dfrac{1}{\|\vec{x}_2\|}\vec{x}_2$.

and so on.

We seek convergence of $\vec{y}_m$ to some limiting vector; if such a vector exists, it must be $\vec{v}_1$, a unit eigenvector for the largest eigenvalue λ_1. We can then calculate $A\vec{v}_1$ to determine λ_1.

This procedure is illustrated in the following example, which is simple enough that you can check the calculations.

EXAMPLE 6.1.11

Determine the eigenvalue of largest absolute value for the matrix $A = \begin{bmatrix} 13 & 6 \\ -12 & -5 \end{bmatrix}$ by using the power method.

Solution: We first choose any starting vector $\vec{x}_0$. We choose $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We then calculate

$$\vec{y}_0 = \frac{1}{\|\vec{x}_0\|}\vec{x}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

For our next iteration, we first define $\vec{x}_1$ by

$$\vec{x}_1 = A\vec{y}_0 \approx \begin{bmatrix} 13 & 6 \\ -12 & -5 \end{bmatrix}\begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \approx \begin{bmatrix} 13.44 \\ -12.02 \end{bmatrix}$$

and then calculate

$$\vec{y}_1 = \frac{1}{\|\vec{x}_1\|}\vec{x}_1 \approx \begin{bmatrix} 0.745 \\ -0.667 \end{bmatrix}$$

Continuing, we get

$$\vec{x}_2 = A\vec{y}_1 \approx \begin{bmatrix} 5.683 \\ -5.605 \end{bmatrix}, \qquad \vec{y}_2 \approx \begin{bmatrix} 0.712 \\ -0.702 \end{bmatrix}$$

$$\vec{x}_3 = A\vec{y}_2 \approx \begin{bmatrix} 5.044 \\ -5.034 \end{bmatrix}, \qquad \vec{y}_3 \approx \begin{bmatrix} 0.7078 \\ -0.7063 \end{bmatrix}$$

$$\vec{x}_4 = A\vec{y}_3 \approx \begin{bmatrix} 4.9636 \\ -4.9621 \end{bmatrix}, \qquad \vec{y}_4 \approx \begin{bmatrix} 0.7072 \\ -0.7070 \end{bmatrix}$$

At this point, we judge that $\vec{y}_m \to \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}$, so it seems that $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A. We then find that

$$A\vec{v}_1 = \begin{bmatrix} 7 \\ -7 \end{bmatrix} = 7\vec{v}_1$$

Hence, $\vec{v}_1$ is an eigenvector and we get that the dominant eigenvalue is $\lambda_1 = 7$.

Many questions arise with the power method. What if we poorly choose the initial vector? If we choose $\vec{x}_0$ in the subspace spanned by all eigenvectors of A *except* $\vec{v}_1$, the method will fail to give $\vec{v}_1$. Can we quickly calculate large powers of the matrix? How do we decide when to stop repeating the steps of the procedure? For a computer version of the algorithm, it would be important to have tests to decide that the procedure has converged—or that it will never converge.

Once we have determined the dominant eigenvalue of A, how can we determine other eigenvalues? If A is invertible, the dominant eigenvalue of A^{-1} would give the reciprocal of the eigenvalue of A with the smallest absolute value. Another approach is to observe that if one eigenvalue λ_1 is known, then eigenvalues of $A - \lambda_1 I$ will give us information about other eigenvalues of A. (See Section 6.2 Problem C9.)

PROBLEMS 6.1

Practice Problems

For Problems A1–A5, determine whether the given vectors are eigenvectors of A. If a vector is an eigenvector, then determine the corresponding eigenvalue. Answer without calculating the characteristic polynomial.

A1 $A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

A2 $A = \begin{bmatrix} 2 & 12 \\ -2 & -8 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

A3 $A = \begin{bmatrix} -10 & 9 & 5 \\ -10 & 9 & 5 \\ 2 & 3 & -1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

A4 $A = \begin{bmatrix} -10 & 9 & 5 \\ -10 & 9 & 5 \\ 2 & 3 & -1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

A5 $A = \begin{bmatrix} 4 & 1 & 3 \\ 8 & 6 & 1 \\ -2 & -2 & 3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

For Problems A6–A15, find all eigenvalues of the matrix and a basis for the eigenspace of each eigenvalue.

A6 $\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$ **A7** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ **A8** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

A9 $\begin{bmatrix} -26 & 10 \\ -75 & 29 \end{bmatrix}$ **A10** $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ **A11** $\begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix}$

A12 $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{bmatrix}$ **A13** $\begin{bmatrix} -4 & 6 & 6 \\ -2 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix}$

A14 $\begin{bmatrix} 3 & -1 & 2 \\ -1 & 3 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ **A15** $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

For Problems A16–A22, determine the algebraic and geometric multiplicity of each eigenvalue of the matrix.

A16 $\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}$ **A17** $\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$ **A18** $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

A19 $\begin{bmatrix} 0 & -5 & 3 \\ -2 & -6 & 6 \\ -2 & -7 & 7 \end{bmatrix}$ **A20** $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

A21 $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ **A22** $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

For Problems A23–A26, find the eigenvalue of the largest absolute value for the matrix by starting with the given $\vec{x}_0$ and using three iterations of the power method (up to y_3).

A23 $\begin{bmatrix} 27 & 84 \\ -7 & -22 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **A24** $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

A25 $\begin{bmatrix} 3.5 & 4.5 \\ 4.5 & 3.5 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **A26** $\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Homework Problems

For Problems B1–B5, determine whether the given vectors are eigenvectors of A. If a vector is an eigenvector, then determine the corresponding eigenvalue. Answer without calculating the characteristic polynomial.

B1 $A = \begin{bmatrix} 8 & 4 \\ -9 & -4 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

B2 $A = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

B3 $A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & -3 \\ 3 & -3 & 11 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

B4 $A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 9 & -3 \\ 3 & -3 & 11 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

B5 $A = \begin{bmatrix} 1 & 2 & 6 \\ 3 & -9 & 8 \\ -2 & 1 & 8 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

For Problems B6–B17, find all eigenvalues of the matrix and a basis for the eigenspace of each eigenvalue.

B6 $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ **B7** $\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$ **B8** $\begin{bmatrix} -2 & 6 \\ 1 & 3 \end{bmatrix}$

B9 $\begin{bmatrix} 5 & 0 \\ -2 & -3 \end{bmatrix}$ **B10** $\begin{bmatrix} 5 & 1 \\ -9 & -1 \end{bmatrix}$ **B11** $\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$

B12 $\begin{bmatrix} 2 & 0 & 0 \\ -8 & 4 & 0 \\ 5 & -3 & 1 \end{bmatrix}$ **B13** $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$

B14 $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ **B15** $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ -2 & -1 & 0 \end{bmatrix}$

B16 $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$ **B17** $\begin{bmatrix} 7 & -1 & -1 \\ 2 & 4 & -2 \\ -1 & 1 & 7 \end{bmatrix}$

For Problems B18–B26, determine the algebraic and geometric multiplicity of each eigenvalue of the matrix.

B18 $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

B19 $\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$

B20 $\begin{bmatrix} 7 & 2 \\ -8 & -1 \end{bmatrix}$

B21 $\begin{bmatrix} -5 & 1 & -1 \\ 0 & 2 & 5 \\ 0 & 1 & -2 \end{bmatrix}$

B22 $\begin{bmatrix} 3 & 8 & 10 \\ 1 & 4 & 4 \\ -1 & -5 & -5 \end{bmatrix}$

B23 $\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$

B24 $\begin{bmatrix} -2 & -1 & -2 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

B25 $\begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$

B26 $\begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 1 \\ -3 & -4 & -1 \end{bmatrix}$

For Problems B27–B30, find the eigenvalue of the largest absolute value for the matrix by starting with the given $\vec{x}_0$ and using three iterations of the power method (up to y_3).

B27 $\begin{bmatrix} 5 & 3 \\ 15 & 17 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

B28 $\begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B29 $\begin{bmatrix} 14 & -9 \\ 4 & -1 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B30 $\begin{bmatrix} 10 & -6 \\ 6 & -3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Conceptual Problems

C1 Invent $A \in M_{2\times 2}(\mathbb{R})$ that has eigenvalues 2 and 3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ respectively.

C2 Invent $A \in M_{3\times 3}(\mathbb{R})$ that has an eigenvalue λ such that the geometric multiplicity is less than its algebraic multiplicity. Justify your answer.

C3 Invent $A \in M_{3\times 3}(\mathbb{R})$ that has 3 distinct eigenvalues. Justify your answer.

C4 Suppose $A, B \in M_{n\times n}(\mathbb{R})$ and that $\vec{u}$ and $\vec{v}$ are eigenvectors of both A and B where $A\vec{u} = 6\vec{u}$, $B\vec{u} = 5\vec{u}$, $A\vec{v} = 10\vec{v}$, and $B\vec{v} = 3\vec{v}$.

(a) If $C = AB$, show that $5\vec{u} + 3\vec{v}$ is an eigenvector of C. Find the corresponding eigenvalue.

(b) If $n = 2$ and $\vec{w} = \begin{bmatrix} 0.2 \\ 1.4 \end{bmatrix}$, compute $AB\vec{w}$.

C5 Let $A, B \in M_{n\times n}(\mathbb{R})$ with eigenvalues λ and μ respectively. Does this imply that $\lambda\mu$ is an eigenvalue of AB? Justify your answer.

C6 Prove the following addition to the Invertible Matrix Theorem: A matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

C7 Show that if A is invertible and $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of A^{-1}. How are the corresponding eigenvalues related?

C8 Suppose that $\vec{v}$ is an eigenvector of both the matrix A and the matrix B, with corresponding eigenvalue λ for A and corresponding eigenvalue μ for B. Show that $\vec{v}$ is an eigenvector of $(A + B)$ and of AB. Determine the corresponding eigenvalues.

C9 (a) Show that if λ is an eigenvalue of a matrix A, then λ^m is an eigenvalue of A^m. How are the corresponding eigenvectors related?

(b) Give an example of $A \in M_{2\times 2}(\mathbb{R})$ such that A has no real eigenvalues, but A^3 does have real eigenvalues. (Hint: see Section 3.3 Problem C5.)

C10 (a) Let $A \in M_{n\times n}(\mathbb{R})$ with $\operatorname{rank}(A) = r < n$. Prove that 0 is an eigenvalue of A and determine its geometric multiplicity.

(b) Give an example of $A \in M_{3\times 3}(\mathbb{R})$ with $\operatorname{rank}(A) = r < n$ such that the algebraic multiplicity of the eigenvalue 0 is greater than its geometric multiplicity.

C11 Suppose that $A \in M_{n\times n}(\mathbb{R})$ such that the sum of the entries in each row is the same. That is,

$$a_{i1} + a_{i2} + \cdots + a_{in} = c$$

for all $1 \le i \le n$. Show that $\vec{v} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector of A. (Such matrices arise in probability theory.)

6.2 Diagonalization

In the last section, we found that if the k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of $A \in M_{n\times n}(\mathbb{R})$ all have the property that their geometric multiplicity is equal to their algebraic multiplicity, then, by combining the basis vectors from all k eigenspaces, we have a basis for $\mathbb{R}^n$ of eigenvectors of A. Such bases of eigenvectors are extremely useful in a wide variety of applications.

Definition
Diagonalizable

A matrix A is said to be **diagonalizable** if there exists an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. In this case, we say that the matrix P **diagonalizes** A to its **diagonal form** D.

It may be tempting to think that $P^{-1}AP = D$ implies that $A = D$ since P and P^{-1} are inverses. However, this is *not* true in general since matrix multiplication is not commutative. Not surprisingly though, if A and B are matrices such that $P^{-1}AP = B$ for some invertible matrix P, then A and B have many similarities.

Theorem 6.2.1

If $A, B \in M_{n\times n}(\mathbb{R})$ such that $P^{-1}AP = B$ for some invertible matrix P, then A and B have

(1) the same determinant,
(2) the same eigenvalues,
(3) the same rank,
(4) the same **trace**, where the trace of A is defined by

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Section 5.2 Problem C5 proved (1). The proofs of (2), (3), and (4) are left as Problems C1, C2, and C3, respectively.

This theorem motivates the following definition.

Definition
Similar Matrices

If $A, B \in M_{n\times n}(\mathbb{R})$ such that $P^{-1}AP = B$ for some invertible matrix P, then A and B are said to be **similar**.

We could now restate the definition of a matrix being diagonalizable as the matrix being similar to a diagonal matrix.

CONNECTION

If you have previously covered Section 4.6, then the equation $P^{-1}AP = B$ should seem familiar to you. In particular, Theorem 4.6.1 says that $[L]_{\mathcal{B}} = P^{-1}[L]_S P$. That is, similar matrices, and hence diagonalization, have a very nice geometric interpretation when viewed in terms of finding a natural basis for a linear mapping.

The following theorem tells us when a matrix is diagonalizable.

Theorem 6.2.2 **Diagonalization Theorem**
A matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if there exists a basis for $\mathbb{R}^n$ which consists of eigenvectors of A.

Proof: First, suppose that A is diagonalizble. Then, by definition, there exists an invertible matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ such that

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

Multiplying both sides on the left by P gives

$$\begin{aligned} AP &= P\operatorname{diag}(\lambda_1, \dots, \lambda_n) \\ A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} &= P\begin{bmatrix} \lambda_1\vec{e}_1 & \cdots & \lambda_n\vec{e}_n \end{bmatrix} \\ \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} &= \begin{bmatrix} \lambda_1 P\vec{e}_1 & \cdots & \lambda_n P\vec{e}_n \end{bmatrix} \\ \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} &= \begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} \end{aligned}$$

by Theorem 3.1.3. Thus, $A\vec{v}_i = \lambda_i\vec{v}_i$ for $1 \leq i \leq n$. Moreover, since P is invertible, $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem which also implies that $\vec{v}_i \neq \vec{0}$ for $1 \leq i \leq n$. Hence, $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ of eigenvectors of A.

On the other hand, if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ of eigenvectors, then the matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ is invertible by the Invertible Matrix Theorem, and we have

$$\begin{aligned} P^{-1}AP &= P^{-1}\begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} \\ &= P^{-1}\begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} && \text{since } A\vec{v}_i = \lambda_i\vec{v}_i \\ &= P^{-1}\begin{bmatrix} \lambda_1 P\vec{e}_1 & \cdots & \lambda_n P\vec{e}_n \end{bmatrix} && \text{by Theorem 3.1.3} \\ &= P^{-1}P\begin{bmatrix} \lambda_1\vec{e}_1 & \cdots & \lambda_n\vec{e}_n \end{bmatrix} \\ &= \operatorname{diag}(\lambda_1, \dots, \lambda_n) \end{aligned}$$

is diagonal. Therefore, A is diagonalizable. ■

Remark

It is important to observe that the proof of the Diagonalization Theorem tells us that if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of eigenvectors of A, then the matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ diagonalizes A to a diagonal matrix $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where λ_i is an eigenvalue of A corresponding to $\vec{v}_i$ for $1 \leq i \leq n$. In particular, we never actually need to multiply out $P^{-1}AP$ by hand (except to check our answer). Moreover, it shows us that we can put the eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ in any order in P as long as we order the eigenvalues in D to match.

Combining the Diagonalization Theorem, Theorem 6.1.3, and the fact that the sum of algebraic multiplicities of an $n \times n$ matrix must be n by the Fundamental Theorem of Algebra, we get the following useful corollaries.

Theorem 6.2.3 A matrix is diagonalizable if and only if every eigenvalue of the matrix has its geometric multiplicity equal to its algebraic multiplicity.

Theorem 6.2.4 If $A \in M_{n\times n}(\mathbb{R})$ has n distinct eigenvalues, then A is diagonalizable.

CONNECTION

Observe that it is possible for a matrix A with real entries to have non-real eigenvalues, which will lead to non-real eigenvectors. In this case, there cannot exist a basis for $\mathbb{R}^n$ of eigenvectors of A, and so we will say that A is not diagonalizable over $\mathbb{R}$. In Chapter 9, we will examine the case where complex eigenvalues and eigenvectors are allowed.

EXAMPLE 6.2.1 Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, where $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

Solution: We need to find a basis for $\mathbb{R}^2$ of eigenvectors of A. Hence, we need to find a basis for the eigenspace of each eigenvalue of A. The characteristic polynomial of A is

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

Hence, the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$.
For $\lambda_1 = 5$, we get

$$A - \lambda_1 I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 5$ and $\{\vec{v}_1\}$ is a basis for its eigenspace.
For $\lambda_2 = -1$, we get

$$A - \lambda_2 I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -1$ and $\{\vec{v}_2\}$ is a basis for its eigenspace.

Thus, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$, and so if we let $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, we get

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = D$$

Note that we could have instead taken $P = \begin{bmatrix} \vec{v}_2 & \vec{v}_1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, which would have given

$$P^{-1}AP = \operatorname{diag}(\lambda_2, \lambda_1) = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

EXAMPLE 6.2.2

Determine whether $A = \begin{bmatrix} 0 & 3 & -2 \\ -2 & 5 & -2 \\ -2 & 3 & 0 \end{bmatrix}$ is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution: To evaluate $\det(A - \lambda I)$, we use $C_1 - C_3$ and then $R_3 + R_1$ to get

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} 0-\lambda & 3 & -2 \\ -2 & 5-\lambda & -2 \\ -2 & 3 & 0-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 3 & -2 \\ 0 & 5-\lambda & -2 \\ -2+\lambda & 3 & 0-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 2-\lambda & 3 & -2 \\ 0 & 5-\lambda & -2 \\ 0 & 6 & -2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 3\lambda + 2) \\ &= -(\lambda-2)(\lambda-2)(\lambda-1) \end{aligned}$$

Hence, $\lambda_1 = 2$ is an eigenvalue with algebraic multiplicity 2, and $\lambda_2 = 1$ is an eigenvalue with algebraic multiplicity 1. By Theorem 6.1.2, the geometric multiplicity of $\lambda_2 = 1$ must equal 1. Thus, A is diagonalizable if and only if the geometric multiplicity of $\lambda_1 = 2$ is 2.

For $\lambda_1 = 2$, we get

$$A - \lambda_1 I = \begin{bmatrix} -2 & 3 & -2 \\ -2 & 3 & -2 \\ -2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence, the geometric multiplicity of λ_1 equals its algebraic multiplicity.

Therefore, by Theorem 6.2.3, A is diagonalizable. To diagonalize A we still need to find a basis for the eigenspace of λ_2.

For $\lambda_2 = 1$, we get

$$A - \lambda_2 I = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 4 & -2 \\ -2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of λ_2.

So, we can take $P = \begin{bmatrix} 3/2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and get $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

EXERCISE 6.2.1

Diagonalize $A = \begin{bmatrix} 1 & 0 & -1 \\ 11 & -4 & -7 \\ -7 & 3 & 4 \end{bmatrix}$.

EXAMPLE 6.2.3

Is the matrix $A = \begin{bmatrix} -1 & 7 & -5 \\ -4 & 11 & -6 \\ -4 & 8 & -3 \end{bmatrix}$ diagonalizable?

Solution: To evaluate $\det(A - \lambda I)$, we first use $R_3 - R_2$ and then $C_2 + C_3$ to get

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} -1-\lambda & 7 & -5 \\ -4 & 11-\lambda & -6 \\ -4 & 8 & -3-\lambda \end{vmatrix} = \begin{vmatrix} -1-\lambda & 7 & -5 \\ -4 & 11-\lambda & -6 \\ 0 & -3+\lambda & 3-\lambda \end{vmatrix} \\ &= \begin{vmatrix} -1-\lambda & 2 & -5 \\ -4 & 5-\lambda & -6 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 4\lambda + 3) \\ &= -(\lambda-3)(\lambda-3)(\lambda-1) \end{aligned}$$

Thus, $\lambda_1 = 3$ is an eigenvalue with algebraic multiplicity 2, and $\lambda_2 = 1$ is an eigenvalue with algebraic multiplicity 1.

For $\lambda_1 = 3$, we get

$$A - \lambda_1 I = \begin{bmatrix} -4 & 7 & -5 \\ -4 & 8 & -6 \\ -4 & 8 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$. Hence, the geometric multiplicity of λ_1 is 1, which is less than its algebraic multiplicity. So, A is not diagonalizable by Theorem 6.2.3.

EXERCISE 6.2.2

Show that $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not diagonalizable.

Applications

Graphing Quadratic Forms A geometrical application of diagonalization occurs when we try to picture the graph of a quadratic equation in two variables, such as $ax^2 + 2bxy + cy^2 = d$. It turns out that we should consider the associated matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. By diagonalizing this matrix, we can easily recognize the graph as an ellipse, a hyperbola, or perhaps some degenerate case. This application will be discussed in Section 8.3.

Deformation of Solids Imagine, for example, a small steel block that experiences a small deformation when some forces are applied. The change of shape in the block can be described in terms of a 3×3 *strain matrix*. This matrix can always be diagonalized, so it turns out that we can identify the change of shape as the composition of three stretches along mutually orthogonal directions. This application is discussed in Section 8.4.

Finding a Geometrically Natural Basis In Section 4.6 we saw that if $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then its matrix with respect to the basis $\mathcal{B}$ is determined from its standard matrix by the equation

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P$$

where $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ is the change of coordinates matrix. Example 4.6.5 and Example 4.6.7 show that we can more easily give a geometrical interpretation of a linear mapping L if there is a basis $\mathcal{B}$ such that $[L]_{\mathcal{B}}$ is in diagonal form. Hence, our diagonalization process is a method for finding such a geometrically natural basis. In particular, if the standard matrix of L is diagonalizable, then the basis for $\mathbb{R}^n$ of eigenvectors forms the geometrically natural basis.

Systems of Linear Difference Equations If $A \in M_{n\times n}(\mathbb{R})$ and $\vec{s}(m)$ is a vector for each positive integer m, then the matrix vector equation

$$\vec{s}(m+1) = A\vec{s}(m)$$

may be regarded as a system of n linear first-order difference equations, describing the coordinates $s_1, s_2, \dots, s_n$ at times $m+1$ in terms of those at time m. They are "first-order difference" equations because they involve only one time difference from m to $m+1$; the Fibonacci equation $s(m+1) = s(m) + s(m-1)$ is a second-order difference equation.

Linear difference equations arise in many settings. Consider, for example, a population that is divided into two groups; we count these two groups at regular intervals (say, once a month) so that at every time n, we have a vector $\vec{p} = \begin{bmatrix} p_1(n) \\ p_2(n) \end{bmatrix}$ that tells us how many are in each group. For some situations, the change from month to month can be described by saying that the vector $\vec{p}$ changes according to the rule

$$\vec{p}(n+1) = A\vec{p}(n)$$

where A is some known 2×2 matrix. It follows that $p(n) = A^n p(0)$. We are often interested in understanding what happens to the population "in the long run." This requires us to calculate A^n for n large.

Markov processes form a special class of this large class of systems of linear difference equations, but there are applications that do not fit the Markov assumptions. For example, in population models, we might wish to consider deaths (so that some column sums of A would be less than 1) or births (so that some entries in A would be greater than 1). Similar considerations apply to some economic models, which are represented by matrix models. A proper discussion of such models requires more theory than is discussed in this book.

PROBLEMS 6.2

Practice Problems

For Problems A1–A6, determine whether P diagonalizes A by checking whether the columns of P are eigenvectors of A. If so, determine P^{-1} and check that $P^{-1}AP$ is diagonal.

A1 $A = \begin{bmatrix} 11 & 6 \\ 9 & -4 \end{bmatrix}$, $P = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

A2 $A = \begin{bmatrix} 6 & 5 \\ 3 & -7 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

A3 $A = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$

A4 $A = \begin{bmatrix} 5 & -8 \\ 4 & -7 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

A5 $A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$, $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

A6 $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 0 & -1 \\ 4 & -1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$

For Problems A7–A23, either diagonalize the matrix or show that the matrix is not diagonalizable.

A7 $\begin{bmatrix} 7 & 3 \\ 0 & -8 \end{bmatrix}$ **A8** $\begin{bmatrix} 5 & 2 \\ 0 & 5 \end{bmatrix}$ **A9** $\begin{bmatrix} 1 & 9 \\ 4 & -4 \end{bmatrix}$

A10 $\begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$ **A11** $\begin{bmatrix} -2 & 3 \\ 4 & -3 \end{bmatrix}$ **A12** $\begin{bmatrix} 3 & 6 \\ -5 & -3 \end{bmatrix}$

A13 $\begin{bmatrix} 3 & 0 \\ -3 & 3 \end{bmatrix}$ **A14** $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ **A15** $\begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$

A16 $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **A17** $\begin{bmatrix} 6 & -9 & -5 \\ -4 & 9 & 4 \\ 9 & -17 & -8 \end{bmatrix}$

A18 $\begin{bmatrix} -2 & 7 & 3 \\ -1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ **A19** $\begin{bmatrix} -1 & 6 & 3 \\ 3 & -4 & -3 \\ -6 & 12 & 8 \end{bmatrix}$

A20 $\begin{bmatrix} 0 & 6 & -8 \\ -2 & 4 & -4 \\ -2 & 2 & -2 \end{bmatrix}$ **A21** $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

A22 $\begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ **A23** $\begin{bmatrix} -3 & -3 & 5 \\ 13 & 10 & -13 \\ 3 & 2 & -1 \end{bmatrix}$

Homework Problems

For Problems B1–B7, determine whether P diagonalizes A by checking whether the columns of P are eigenvectors of A. If so, determine P^{-1} and check that $P^{-1}AP$ is diagonal.

B1 $A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

B2 $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

B3 $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

B4 $A = \begin{bmatrix} 6 & 2 \\ -2 & 1 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

B5 $A = \begin{bmatrix} -5 & 8 & 18 \\ 2 & 1 & -6 \\ -4 & 4 & 13 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

B6 $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$

B7 $A = \begin{bmatrix} 7 & -4 & -4 \\ 4 & -1 & -4 \\ 4 & -4 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

For Problems B8–B23, either diagonalize the matrix or show that the matrix is not diagonalizable.

B8 $\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ **B9** $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ **B10** $\begin{bmatrix} -2 & 8 \\ 1 & -4 \end{bmatrix}$

B11 $\begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$ **B12** $\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$ **B13** $\begin{bmatrix} 8 & 4 \\ -1 & 4 \end{bmatrix}$

B14 $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ **B15** $\begin{bmatrix} 2 & -2 & -5 \\ -2 & -5 & -2 \\ -5 & 2 & 2 \end{bmatrix}$

B16 $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ **B17** $\begin{bmatrix} 1 & 2 & 1 \\ -2 & 5 & 1 \\ 2 & -2 & 2 \end{bmatrix}$

B18 $\begin{bmatrix} -4 & -3 & 7 \\ -2 & -9 & 14 \\ -1 & -3 & 4 \end{bmatrix}$ **B19** $\begin{bmatrix} 1 & 2 & -4 \\ 2 & -2 & -2 \\ -4 & -2 & 1 \end{bmatrix}$

B20 $\begin{bmatrix} 5 & 4 & -12 \\ 3 & 4 & -9 \\ 2 & 2 & -5 \end{bmatrix}$ **B21** $\begin{bmatrix} 3 & 1 & -4 \\ 3 & 2 & -5 \\ 2 & 1 & -3 \end{bmatrix}$

B22 $\begin{bmatrix} -5 & 6 & 5 \\ -2 & 0 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ **B23** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

Conceptual Problems

C1 Prove that if A and B are similar, then A and B have the same eigenvalues.

C2 Prove that if A and B are similar, then A and B have the same rank.

C3 (a) Let $A, B \in M_{n\times n}(\mathbb{R})$. Prove that $\text{tr}(AB) = \text{tr}(BA)$.

(b) Use the result of part (a) to prove that if A and B are similar, then $\text{tr}(A) = \text{tr}(B)$.

C4 (a) Suppose that P diagonalizes A and that the diagonal form is D. Show that $A = PDP^{-1}$.

(b) Use the result of part (a) and properties of eigenvectors to calculate a matrix that has eigenvalues 2 and 3 with corresponding eigenvectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$, respectively.

(c) Determine a matrix that has eigenvalues 2, -2, and 3, with corresponding eigenvectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\-1\\2\end{bmatrix}$, respectively.

C5 (a) Suppose that P diagonalizes A and that the diagonal form is D. Show that $A^k = PD^kP^{-1}$.

(b) Use the result of part (a) to calculate A^5, where $A = \begin{bmatrix}-1 & 6 & 3\\3 & -4 & -3\\-6 & 12 & 8\end{bmatrix}$ is the matrix from Problem A19.

C6 (a) Suppose that A is diagonalizable. Prove that $\text{tr}(A)$ is equal to the sum of the eigenvalues of A (including repeated eigenvalues) by using Theorem 6.2.1.

(b) Use the result of part (a) to determine, by inspection, the algebraic and geometric multiplicities of all of the eigenvalues of
$$A = \begin{bmatrix}a+b & a & a\\a & a+b & a\\a & a & a+b\end{bmatrix}.$$

C7 (a) Suppose that A is diagonalizable. Prove that $\det A$ is equal to the product of the eigenvalues of A (repeated according to their multiplicity) by considering $P^{-1}AP$.

(b) Show that the constant term in the characteristic polynomial is $\det A$. (Hint: how do you find the constant term in any polynomial $p(\lambda)$?)

(c) Without assuming that A is diagonalizable, show that $\det A$ is equal to the product of the roots of the characteristic equation of A (including any repeated roots and complex roots). (Hint: consider the constant term in the characteristic equation and the factored version of that equation.)

C8 Let $A \in M_{n\times n}(\mathbb{R})$. Prove that A is invertible if and only if A does not have 0 as an eigenvalue. (Hint: see Problem C7.)

C9 Suppose that A is diagonalized by the matrix P and that the eigenvalues of A are $\lambda_1, \dots, \lambda_n$. Show that the eigenvalues of $(A - \lambda_1 I)$ are $0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1$. (Hint: $A - \lambda_1 I$ is diagonalized by P.)

6.3 Applications of Diagonalization

Powers of Matrices

In some applications of linear algebra, it is necessary to calculate large powers of a matrix. For example, say we wanted to calculate A^{1000} where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. We certainly would not want to multiply A by itself 1000 times (especially if we were doing this by hand)! Instead, we use the benefits of diagonalization.

Observe that multiplying a diagonal matrix D by itself is easy. In particular, we have if $D = \text{diag}(d_1, \dots, d_n)$, then $D^k = \text{diag}(d_1^k, \dots, d_n^k)$. See Section 3.1 Problem C9. The following theorem, shows us how we can use this fact with diagonalization to quickly take powers of a diagonalizable matrix A.

Theorem 6.3.1 Let $A \in M_{n\times n}(\mathbb{R})$. If there exists an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$, then

$$A^k = PD^kP^{-1}$$

Proof: We prove the result by induction on k. If $k = 1$, then $P^{-1}AP = D$ implies $A = PDP^{-1}$ and so the result holds. Assume the result is true for some $k \geq 1$. We then have

$$A^{k+1} = A^kA = (PD^kP^{-1})(PDP^{-1}) = PD^kP^{-1}PDP^{-1} = PD^kIDP^{-1} = PD^{k+1}P^{-1}$$

as required. ■

We now demonstrate finding large powers of matrices with a couple of examples.

EXAMPLE 6.3.1 Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Show that $A^{1000} = \begin{bmatrix} 2^{1001} - 3^{1000} & -2^{1001} + 2\cdot 3^{1000} \\ 2^{1000} - 3^{1000} & -2^{1000} + 2 \cdot 3^{1000} \end{bmatrix}$.

Solution: We first diagonalize A. We have

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

Thus, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3$.
For $\lambda_1 = 2$ we have

$$A - \lambda_1 I = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

EXAMPLE 6.3.1 (continued)

For $\lambda_2 = 3$ we have

$$A - \lambda_2 I = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Therefore, $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. We find that $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Hence,

$$\begin{aligned} A^{1000} &= PD^{1000}P^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{1000} & 0 \\ 0 & 3^{1000} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^{1001} & 3^{1000} \\ 2^{1000} & 3^{1000} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^{1001} - 3^{1000} & -2^{1001} + 2 \cdot 3^{1000} \\ 2^{1000} - 3^{1000} & -2^{1000} + 2 \cdot 3^{1000} \end{bmatrix} \end{aligned}$$

EXAMPLE 6.3.2

Let $A = \begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix}$. Calculate A^{200}.

Solution: We have

$$0 = \det(A - \lambda I) = \begin{vmatrix} -2 - \lambda & 2 \\ -3 & 5 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4).$$

Thus, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 4$.

For $\lambda_1 = -1$ we have $A - \lambda_1 I = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. So, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 4$ we have $A - \lambda_2 I = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$. So, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

Therefore, $P = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$. We find that $P^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. Hence,

$$\begin{aligned} A^{200} = PD^{200}P^{-1} &= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4^{200} \end{bmatrix} \left(\frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \right) \\ &= \frac{1}{5} \begin{bmatrix} 6 - 4^{200} & -2 + 2 \cdot 4^{200} \\ 3 - 3 \cdot 4^{200} & -1 + 6 \cdot 4^{200} \end{bmatrix} \end{aligned}$$

EXERCISE 6.3.1

Let $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$. Calculate A^{100}.

Markov Processes

A Markov process, also commonly called a **Markov chain**, is a mathematical model of a system in which the system at any time occupies one of a finite number of states, and how the system transitions to the next state is dependent only on its current state.

Here is just a small sample of how Markov processes are used.

Chemical Reactions A chemical reaction network can be modelled as a Markov process where the states are the number of molecules of each species and the transitions are calculated from the rate of the chemical reactions.

Population Genetics Physical characteristics of animals are inherited from their parents based on their parents' genes. We can model the transition of characteristics from parent to child by a Markov process. For example, our possible states could be blue eyes or brown eyes, and the transition would be determined by considering the dominant and recessive genes.

Random Walks Many real world situations can be represented as randomly moving through a graph. For example, current passing through an electric network. These can be modelled by a Markov process where the states are the nodes of the graph and the transitions are based on the probability of moving from one node to the next.

We begin our look at Markov processes with an example.

EXAMPLE 6.3.3

Smith and Jones are the only competing suppliers of communication services in their community. At present, they each have a 50% share of the market. However, Smith has recently upgraded his service, and a survey indicates that from one month to the next, 90% of Smith's customers remain loyal, while 10% switch to Jones. On the other hand, 70% of Jones's customers remain loyal and 30% switch to Smith. If this goes on for six months, how large are their market shares? If this goes on for a long time, how big will Smith's share become?

Solution: Let S_m be Smith's market share (as a decimal) at the end of the m-th month and let J_m be Jones's share. Then $S_m + J_m = 1$, since between them they have 100% of the market. At the end of the $(m + 1)$-st month, Smith has 90% of his previous customers and 30% of Jones's previous customers, so

$$S_{m+1} = 0.9S_m + 0.3J_m$$

Similarly,

$$J_{m+1} = 0.1S_m + 0.7J_m$$

We can rewrite these equations in matrix-vector form:

$$\begin{bmatrix} S_{m+1} \\ J_{m+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} S_m \\ J_m \end{bmatrix}$$

The matrix $T = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}$ is called the **transition matrix** for this problem: it describes the transition (change) from the **state** $\begin{bmatrix} S_m \\ J_m \end{bmatrix}$ at time m to the state $\begin{bmatrix} S_{m+1} \\ J_{m+1} \end{bmatrix}$ at time $m + 1$.

EXAMPLE 6.3.3
(continued)

To answer the questions, we need to determine $T^6 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and $T^m \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ for m large.

We could compute T^6 directly, but this approach is not reasonable for calculating T^m for large values of m. So, we diagonalize T. We find that $\lambda_1 = 1$ is an eigenvalue of T with eigenvector $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\lambda_2 = 0.6$ is the other eigenvalue, with eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus,

$$P = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

It follows that

$$T^m = PD^mP^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1^m & 0 \\ 0 & (0.6)^m \end{bmatrix}\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

We could now answer our question directly, but we get a simpler calculation if we observe that the eigenvectors form a basis, so we can write

$$\begin{bmatrix} S_0 \\ J_0 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\begin{bmatrix} S_0 \\ J_0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} S_0 + J_0 \\ S_0 - 3J_0 \end{bmatrix}$$

Hence,

$$\begin{aligned}
T^m\begin{bmatrix} S_0 \\ J_0 \end{bmatrix} &= T^m\left(c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\
&= c_1T^m\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2T^m\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= c_1\lambda_1^m\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\lambda_2^m\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{by Section 6.1 Problem C9 (a)} \\
&= \frac{1}{4}(S_0 + J_0)\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{1}{4}(S_0 - 3J_0)(0.6)^m\begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

Now $S_0 = J_0 = 0.5$. When $m = 6$,

$$\begin{aligned}
\begin{bmatrix} S_6 \\ J_6 \end{bmatrix} &= \frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{1}{4}(0.6)^6\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&\approx \frac{1}{4}\begin{bmatrix} 3 - 0.0467 \\ 1 + 0.0467 \end{bmatrix} \\
&\approx \begin{bmatrix} 0.738 \\ 0.262 \end{bmatrix}
\end{aligned}$$

Thus, after six months, Smith has approximately 73.8% of the market.

When m is very large, $(0.6)^m$ is nearly zero, so for m large enough ($m \to \infty$), we have $S_\infty = 0.75$ and $J_\infty = 0.25$.

Thus, in this problem, Smith's share approaches 75% as m gets large, but it never gets larger than 75%. Now look carefully: we get the same answer in the long run, no matter what the initial value of S_0 and J_0 are because $(0.6)^m \to 0$ and $S_0 + J_0 = 1$.

By emphasizing some features of Example 6.3.3, we will be led to an important definition and several general properties:

(1) Each column of T has sum 1. This means that all of Smith's customers show up a month later as customers of Smith or Jones; the same is true for Jones's customers. No customers are lost from the system and none are added after the process begins.

(2) It is natural to interpret the entries t_{ij} as **probabilities**. For example, $t_{11} = 0.9$ is the probability that a Smith customer remains a Smith customer, with $t_{21} = 0.1$ as the probability that a Smith customer becomes a Jones customer. If we consider "Smith customer" as "state 1" and "Jones customer" as "state 2," then t_{ij} is the probability of **transition** from state j to state i between time m and time $m + 1$.

(3) The "initial state vector" is $\begin{bmatrix} S_0 \\ J_0 \end{bmatrix}$. The state vector at time m is $\begin{bmatrix} S_m \\ J_m \end{bmatrix} = T^m \begin{bmatrix} S_0 \\ J_0 \end{bmatrix}$.

(4) Note that

$$\begin{bmatrix} S_1 \\ J_1 \end{bmatrix} = T \begin{bmatrix} S_0 \\ J_0 \end{bmatrix} = S_0 \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} + J_0 \begin{bmatrix} t_{12} \\ t_{22} \end{bmatrix}$$

Since $t_{11} + t_{21} = 1$ and $t_{12} + t_{22} = 1$, it follows that

$$S_1 + J_1 = S_0 + J_0$$

Thus, it follows from (1) that each state vector has the same column sum. In our example, S_0 and J_0 are decimal fractions, so $S_0 + J_0 = 1$, but we could consider a process whose states have some other constant column sum.

(5) Note that 1 is an eigenvalue of T with eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To get a state vector with the appropriate sum, we take the eigenvector to be $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$. Thus,

$$T \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

and the state vector $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$ is **fixed** or **invariant** under the transformation with matrix T. Moreover, this fixed vector is the limiting state approached by $T^m \begin{bmatrix} S_0 \\ J_0 \end{bmatrix}$ for any $\begin{bmatrix} S_0 \\ J_0 \end{bmatrix}$.

The following definition captures the essential properties of this example.

Definition

Markov Matrix

Markov Process

A matrix $T \in M_{n\times n}(\mathbb{R})$ is the **Markov matrix** (or transition matrix) of an n-state **Markov process** if

(1) $t_{ij} \geq 0$, for each i and j.
(2) Each column sum is 1. That is, $t_{1j} + \cdots + t_{nj} = 1$ for each j.

We take possible states of the process to be the vectors $S = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ such that $s_i \geq 0$ for each i, and $s_1 + \cdots + s_n = 1$.

Remark

With minor changes, we could develop the theory with $s_1 + \cdots + s_n = \text{constant}$.

EXAMPLE 6.3.4

The matrix $\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.8 \end{bmatrix}$ is not a Markov matrix since the sum of the entries in the second column does not equal 1.

EXAMPLE 6.3.5

Find the fixed-state vector for the Markov matrix $A = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$.

Solution: We know the fixed-state vector is an eigenvector for the eigenvalue $\lambda = 1$. We have

$$A - I = \begin{bmatrix} -0.9 & 0.3 \\ 0.9 & -0.3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

Therefore, an eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The components in the state vector must sum to 1, so the invariant state is $\begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$.

It is easy to verify that

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}\begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

EXERCISE 6.3.2

Determine which of the following matrices is a Markov matrix. Find the fixed-state vector of the Markov matrix.

(a) $A = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$

The goal with the Markov process is to establish the behaviour of a sequence with states $\vec{s}, T\vec{s}, T^2\vec{s}, \ldots, T^m\vec{s}$. If possible, we want to say something about the limit of $T^m\vec{s}$ as $m \to \infty$. As we saw in Example 1, diagonalization of T is a key to solving the problem. It is beyond the scope of this book to establish all the properties of the Markov process, but some of the properties are easy to prove, and others are easy to illustrate if we make extra assumptions.

PROPERTY 1. One eigenvalue of a Markov matrix is $\lambda_1 = 1$.

Proof: Since each column of T has sum 1, each column of $(T - 1I)$ has sum 0. Hence, the sum of the rows of $(T - 1I)$ is the zero vector. Thus the rows are linearly dependent, and $(T - 1I)$ has rank less than n, so $\det(T - 1I) = 0$. Therefore, 1 is an eigenvalue of T. ■

PROPERTY 2. The eigenvector $\vec{s}^*$ for $\lambda_1 = 1$ has $s_j^* \geq 0$ for $1 \leq j \leq n$.
This property is important because it means that the eigenvector $\vec{s}^*$ is a real state of the process. In fact, it is a fixed, or invariant, state:

$$T\vec{s}^* = \lambda_1 \vec{s}^* = \vec{s}^*$$

PROPERTY 3. All other eigenvalues satisfy $|\lambda_i| \leq 1$.
To see why we expect this, let us assume that T is diagonalizable, with distinct eigenvalues $1, \lambda_2, \dots, \lambda_n$ and corresponding eigenvectors $\vec{s}^*, \vec{s}_2, \dots, \vec{s}_n$. Then any initial state $\vec{s}$ can be written

$$\vec{s} = c_1\vec{s}^* + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

It follows that

$$T^m\vec{s} = c_1 1^m \vec{s}^* + c_2\lambda_2^m\vec{s}_2 + \cdots + c_n\lambda_n^m\vec{s}_n$$

If any $|\lambda_i| > 1$, then the term $|\lambda_i^m|$ would become much larger than the other terms when m is large; it would follow that $T^m\vec{s}$ has some coordinates with magnitude greater than 1. This is impossible because state coordinates satisfy $0 \leq s_i \leq 1$, so we must have $|\lambda_i| \leq 1$.

PROPERTY 4. Suppose that for some m all the entries in T^m are not zero. Then all the eigenvalues of T except for $\lambda_1 = 1$ satisfy $|\lambda_i| < 1$. In this case, for any initial state $\vec{s}$, $T^m\vec{s} \to \vec{s}^*$ as $m \to \infty$: all states tend to the invariant state $\vec{s}^*$ under the process.

The proof of Property 4 is omitted. Notice that in the diagonalizable case, the fact that $T^m\vec{s} \to \vec{s}^*$ follows from the expression for $T^m\vec{s}$ given under Property 3.

EXERCISE 6.3.3

The Markov matrix $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1; it does not satisfy the conclusion of Property 4. However, it also does not satisfy the extra assumption of Property 4. It is worthwhile to explore this "bad" case.

Let $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$. Determine the behaviour of the sequence $\vec{s}, T\vec{s}, T^2\vec{s}, \dots$. What is the fixed-state vector for T?

Differential Equations

We demonstrate how diagonalization can be used to help us solve systems of differential equations with an example.

EXAMPLE 6.3.6

Consider two tanks, Y and Z, each containing 1000 litres of a salt solution. At an initial time, $t = 0$ (in hours), the concentration of salt in tank Y is different from the concentration in tank Z. In each tank the solution is well stirred, so that the concentration is constant throughout the tank. The two tanks are joined by pipes; through one pipe, solution is pumped from Y to Z at a rate of 20 L/h; through the other, solution is pumped from Z to Y at the same rate. Determine the amount of salt in each tank at time t.

Solution: Let $y(t)$ be the amount of salt (in kilograms) in tank Y at time t, and let $z(t)$ be the amount of salt (in kilograms) in the tank Z at time t. Then the concentration in Y at time t is $(y(t)/1000)$ kg/L. Similarly, $(z(t)/1000)$ kg/L is the concentration in Z. Then for tank Y, salt is flowing out through one pipe at a rate of $(20)(y/1000)$ kg/h and in through the other pipe at a rate of $(20)(z/1000)$ kg/h. Since the rate of change is measured by the derivative, we have $\frac{dy}{dt} = -0.02y + 0.02z$. By consideration of Z, we get a second differential equation, so y and z are the solutions of the **system of linear ordinary differential equations**:

$$\frac{dy}{dt} = -0.02y + 0.02z$$
$$\frac{dz}{dt} = 0.02y - 0.02z$$

It is convenient to rewrite this system in the form $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$.

How can we solve this system? Well, it might be easier if we could change variables so that the 2×2 matrix is diagonalized. By standard methods, one eigenvalue of $A = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$ is $\lambda_1 = 0$, with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The other eigenvalue is $\lambda_2 = -0.04$, with corresponding eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence, A is diagonalized by $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 0 & 0 \\ 0 & -0.04 \end{bmatrix}$.

Introduce new coordinates $\begin{bmatrix} y^* \\ z^* \end{bmatrix}$ by the change of coordinates equation $\begin{bmatrix} y \\ z \end{bmatrix} = P\begin{bmatrix} y^* \\ z^* \end{bmatrix}$. See Section 4.4. Substitute this for $\begin{bmatrix} y \\ z \end{bmatrix}$ on both sides of the system to obtain

$$\frac{d}{dt}P\begin{bmatrix} y^* \\ z^* \end{bmatrix} = AP\begin{bmatrix} y^* \\ z^* \end{bmatrix}$$

Since the entries in P are constants, it is easy to check that

$$\frac{d}{dt}P\begin{bmatrix} y^* \\ z^* \end{bmatrix} = P\frac{d}{dt}\begin{bmatrix} y^* \\ z^* \end{bmatrix}$$

EXAMPLE 6.3.6
(continued)

Multiply both sides of the system of equations (on the left) by P^{-1}. Since P diagonalizes A, we get

$$\frac{d}{dt}\begin{bmatrix} y^* \\ z^* \end{bmatrix} = P^{-1}AP\begin{bmatrix} y^* \\ z^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -0.04 \end{bmatrix}\begin{bmatrix} y^* \\ z^* \end{bmatrix}$$

Now write the pair of equations:

$$\frac{dy^*}{dt} = 0 \text{ and } \frac{dz^*}{dt} = -0.04z^*$$

These equations are "decoupled," and we can easily solve each of them by using simple one-variable calculus.

The only functions satisfying $\frac{dy^*}{dt} = 0$ are constants: we write $y^*(t) = a$. The only functions satisfying an equation of the form $\frac{dx}{dt} = kx$ are exponentials of the form $x(t) = ce^{kt}$ for a constant c. So, from $\frac{dz^*}{dt} = -0.04z^*$, we obtain $z^*(t) = be^{-0.04t}$, where b is a constant.

Now we need to express the solution in terms of the original variables y and z:

$$\begin{bmatrix} y \\ z \end{bmatrix} = P\begin{bmatrix} y^* \\ z^* \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y^* \\ z^* \end{bmatrix} = \begin{bmatrix} y^* - z^* \\ y^* + z^* \end{bmatrix} = \begin{bmatrix} a - be^{-0.04t} \\ a + be^{-0.04t} \end{bmatrix}$$

For later use, it is helpful to rewrite this as $\begin{bmatrix} y \\ z \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{-0.04t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. This is the general solution of the problem. To determine the constants a and b, we would need to know the amounts $y(0)$ and $z(0)$ at the initial time $t = 0$. Then we would know y and z for all t. Note that as $t \to \infty$, y and z tend to a common value a, as we might expect.

The usual solution procedure takes advantage of the understanding obtained from this diagonalization argument, but it takes a major shortcut. Now that the expected form of the solution is known, we simply look for a solution of the form $\begin{bmatrix} y \\ z \end{bmatrix} = ce^{\lambda t}\vec{v}$. Substitute this into the original system and use the fact that $\frac{d}{dt}ce^{\lambda t}\vec{v} = \lambda ce^{\lambda t}\vec{v}$ to get

$$\lambda ce^{\lambda t}\vec{v} = Ace^{\lambda t}\vec{v}$$

After the common factor $ce^{\lambda t}$ is cancelled, this tells us that $\vec{v}$ is an eigenvector of A, with eigenvalue λ. We find the two eigenvalues λ_1 and λ_2 and the corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$, as above. Observe that since our problem is a linear homogeneous problem, the general solution will be of the form

$$\begin{bmatrix} y \\ z \end{bmatrix} = ae^{\lambda_1 t}\vec{v}_1 + be^{\lambda_2 t}\vec{v}_2$$

General Discussion

There are many other problems that give rise to systems of linear homogeneous ordinary differential equations (for example, electrical circuits or a mechanical system consisting of springs). Many of these systems are much larger than the example we considered. Methods for solving these systems make extensive use of eigenvectors and eigenvalues, and they require methods for dealing with cases where the characteristic equation has complex roots.

PROBLEMS 6.3

Practice Problems

A1 Let $A = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$. Use diagonalization to calculate A^3. Verify your answer by computing A^3 directly.

A2 Calculate A^{100} where $A = \begin{bmatrix} -6 & -10 \\ 4 & 7 \end{bmatrix}$.

A3 Calculate A^{100} where $A = \begin{bmatrix} 2 & 2 \\ -3 & -5 \end{bmatrix}$.

A4 Calculate A^{200} where $A = \begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix}$.

A5 Calculate A^{200} where $A = \begin{bmatrix} 6 & -6 \\ 2 & -1 \end{bmatrix}$.

A6 Calculate A^{100} where $A = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 1 \end{bmatrix}$.

A7 Calculate A^{100} where $A = \begin{bmatrix} 7 & -3 & 2 \\ 8 & -4 & 2 \\ -10 & 4 & -3 \end{bmatrix}$.

A8 Calculate A^{100} where $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 1 \end{bmatrix}$.

For Problems A9–A14, determine whether the matrix is a Markov matrix. If so, determine the invariant or fixed state (corresponding to the eigenvalue $\lambda = 1$).

A9 $\begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.3 \end{bmatrix}$

A10 $\begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}$

A11 $\begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{bmatrix}$

A12 $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$

A13 $\begin{bmatrix} 0.7 & 0.3 & 0.0 \\ 0.1 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.9 \end{bmatrix}$

A14 $\begin{bmatrix} 0.9 & 0.1 & 0.0 \\ 0.0 & 0.9 & 0.1 \\ 0.1 & 0.0 & 0.9 \end{bmatrix}$

A15 Suppose that census data show that every decade, 15% of people dwelling in rural areas move into towns and cities, while 5% of urban dwellers move into rural areas.
(a) What would be the eventual steady-state population distribution?
(b) If the population were 50% urban, 50% rural at some census, what would be the distribution after 50 years?

A16 A car rental company serving one city has three locations: the airport, the train station, and the city centre. Of the cars rented at the airport, 8/10 are returned to the airport, 1/10 are left at the train station, and 1/10 are left at the city centre. Of the cars rented at the train station, 3/10 are left at the airport, 6/10 are returned to the train station, and 1/10 are left at the city centre. Of the cars rented at the city centre, 3/10 go to the airport, 1/10 go to the train station, and 6/10 are returned to the city centre. Model this as a Markov process and determine the steady-state distribution for the cars.

A17 The town of Markov Centre has only two suppliers of widgets—Johnson and Thomson. All inhabitants buy their supply on the first day of each month. Neither supplier is very successful at keeping customers. 70% of the customers who deal with Johnson decide that they will "try the other guy" next time. Thomson does even worse: only 20% of his customers come back the next month, and the rest go to Johnson.
(a) Model this as a Markov process and determine the steady-state distribution of customers.
(b) Determine a general expression for Johnson and Thomson's shares of the customers, given an initial state where Johnson has 25% and Thomson has 75%.

For Problems A18–A27, find the general solution of the system of linear differential equations.

A18 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A19 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0.2 & 0.7 \\ 0.1 & -0.4 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A20 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 8 & -5 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A21 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ 0.1 & 0.3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A22 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A23 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A24 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A25 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

A26 $\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ -13 & 3 & 8 \\ 11 & -5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

A27 $\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 5 & -5 \\ 6 & -3 & 6 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Homework Problems

B1 Calculate A^{100} where $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

B2 Calculate A^{100} where $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$.

B3 Calculate A^{100} where $A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$.

B4 Calculate A^{100} where $A = \begin{bmatrix} -8 & 4 \\ -15 & 8 \end{bmatrix}$.

B5 Calculate A^{100} where $A = \begin{bmatrix} 2 & -6 & -6 \\ 4 & -12 & -12 \\ -4 & 12 & 12 \end{bmatrix}$.

B6 Calculate A^{100} where $A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$.

For Problems B7–B12, determine whether the matrix is a Markov matrix. If so, determine the invariant or fixed state (corresponding to the eigenvalue $\lambda = 1$).

B7 $\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}$

B8 $\begin{bmatrix} 0.6 & 0.5 \\ 0.5 & 0.4 \end{bmatrix}$

B9 $\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$

B10 $\begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{bmatrix}$

B11 $\begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.0 & 0.6 & 0.2 \\ 0.2 & 0.1 & 0.6 \end{bmatrix}$

B12 $\begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0.6 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$

B13 Suppose that the only competing suppliers of Internet services are NewServices and RealWest. At present, they each have 50% of the market share in their community. However, NewServices is upgrading their connection speeds but at a higher cost. Because of this, each month 30% of RealWest's customers will switch to NewServices, while 10% of NewServices customers will switch to RealWest.

(a) Model this situation as a Markov process.

(b) Calculate the market share of each company for the first two months.

(c) Determine the steady-state distribution.

B14 Consider the population-migration system described in Section 3.1 Problem B43 on page 170.

(a) Given initial populations of n_0 and m_0, use diagonalization to verify that the populations in year t are given by

$$\begin{bmatrix} n_t \\ m_t \end{bmatrix} = A^t \begin{bmatrix} n_0 \\ m_0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 + \left(\frac{7}{10}\right)^t & 2 - 2\left(\frac{7}{10}\right)^t \\ 1 - \left(\frac{7}{10}\right)^t & 1 + 2\left(\frac{7}{10}\right)^t \end{bmatrix}\begin{bmatrix} n_0 \\ m_0 \end{bmatrix}$$

(b) Determine the steady-state distribution for the population.

B15 A student society at a university campus decides to create a pool of bicycles that can be used by its members. Borrowed bicycles can be returned to the residence, the library, or the athletic centre. On the first day, 200 marked bicycles are left at each location. At the end of the day, at the residence, there are 160 bicycles that started at the residence, 40 that started at the library, and 60 that started at the athletic centre. At the library, there are 20 that started at the residence, 140 that started at the library, and 40 that started at the athletic centre. At the athletic centre, there are 20 that started at the residence, 20 that started at the library, and 100 that started at the athletic centre. If this pattern is repeated every day, what is the steady-state distribution of bicycles?

For Problems B16–B24, find the general solution of the system of linear differential equations.

B16 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

B17 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

B18 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

B19 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

B20 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

B21 $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 5 & -9 \\ 6 & -10 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

B22 $\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

B23 $\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 & 5 & 4 \\ -4 & 2 & 4 \\ -5 & 13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

B24 $\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ -2 & 10 & -6 \\ -1 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Conceptual Problems

C1 Assume that there exists an invertible matrix P such that $P^{-1}AP = B$. Show that $A^3 = PB^3P^{-1}$.

C2 (a) Let T be the transition matrix for a two-state Markov process. Show that the eigenvalue that is not 1 is $\lambda_2 = t_{11} + t_{22} - 1$.

(b) For a two-state Markov process with $t_{21} = a$ and $t_{12} = b$, show that the fixed state is $\frac{1}{a+b}\begin{bmatrix} b \\ a \end{bmatrix}$.

C3 Suppose that $T \in M_{n\times n}(\mathbb{R})$ is a Markov matrix.

(a) Show that for any state $\vec{x}$,

$$(T\vec{x})_1 + \cdots + (T\vec{x})_n = x_1 + \cdots + x_n$$

(b) Show that if $\vec{v}$ is an eigenvector of T with eigenvalue $\lambda \neq 1$, then $v_1 + \cdots + v_n = 0$.

CHAPTER REVIEW

Suggestions for Student Review

1 At the beginning of this chapter, we mentioned that we are synthesizing many of the concepts from Chapters 1 – 5. Make a list of these concepts and indicate which section they are from. (Sections 6.1 – 6.3)

2 Define eigenvalues and eigenvectors of a matrix A. Explain why A must be a square matrix to have eigenvectors. Also explain the connection between the statement that λ is an eigenvalue of A with eigenvector $\vec{v}$ and the condition $\det(A - \lambda I) = 0$. (Section 6.1)

3 Define the algebraic and geometric multiplicity of an eigenvalue of a matrix A. What is the relationship between these quantities? Give some examples showing the different possibilities. (Section 6.1)

4 Describe the geometric meaning of eigenvalues and eigenvectors. Create a few examples of eigenvalues and eigenvectors of linear mappings to support your description. (Section 6.1)

5 Is there a relationship between a matrix being diagonalizable and the matrix being invertible? (Section 6.1)

6 What does it mean to say that matrices A and B are similar? What properties do similar matrices have in common? (Section 6.2)

7 Suppose that $A \in M_{n\times n}(\mathbb{R})$ has eigenvalues $\lambda_1, \dots, \lambda_n$ (repeated according to multiplicity). (Section 6.2)

(a) What conditions on these eigenvalues guarantee that A is diagonalizable over $\mathbb{R}$?

(b) Is there any case where you can tell just from the eigenvalues that A is diagonalizable over $\mathbb{R}$?

(c) Assume A is diagonalizable. Can we choose the eigenvalues in any order? If so, how does this affect the matrix P which diagonalizes A? Demonstrate with some examples.

8 Use the idea suggested in Section 6.2 Problem C4 to create matrices for your classmates to diagonalize. (Section 6.2)

9 Explain the procedure for calculating large powers of a diagonalizable matrix. Will the procedure work if the matrix is not diagonalizable? (Section 6.3)

10 Describe what a Markov process is. Search the Internet for applications of Markov processes related to your field(s) of interest. (Section 6.3)

11 Suppose that $P^{-1}AP = D$, where D is a diagonal matrix with distinct diagonal entries $\lambda_1, \dots, \lambda_n$. How can we use this information to solve the system of linear differential equations $\frac{d}{dt}\vec{x} = A\vec{x}$? (Section 6.3)

Chapter Quiz

E1 Let $A = \begin{bmatrix} 5 & -16 & -4 \\ 2 & -7 & -2 \\ -2 & 8 & 3 \end{bmatrix}$.

(a) Determine whether the following are eigenvectors of A. State the corresponding eigenvalue of any eigenvector.

(i) $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

(b) Use the results of part (a) to diagonalize A.

For Problems E2 and E3, determine the algebraic and geometric multiplicity of each eigenvalue.

E2 $\begin{bmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$ **E3** $\begin{bmatrix} 2 & 4 & -5 \\ 1 & 4 & -4 \\ 1 & 3 & -3 \end{bmatrix}$

For Problems E4–E7, either diagonalize the matrix or show that the matrix is not diagonalizable.

E4 $\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$ **E5** $\begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}$

E6 $\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$ **E7** $\begin{bmatrix} -3 & 1 & 0 \\ 13 & -7 & -8 \\ -11 & 5 & 4 \end{bmatrix}$

E8 Use the power method to determine the eigenvalue of $A = \begin{bmatrix} 2 & 9 \\ -4 & 22 \end{bmatrix}$ of the largest absolute value.

E9 Let $A = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$. Calculate A^{100}.

E10 Suppose that $A \in M_{3\times 3}(\mathbb{R})$ such that

$$\det A = 0, \quad \det(A + 2I) = 0, \quad \det(A - 3I) = 0$$

Answer the following questions and give a brief explanation in each case.

(a) What is the dimension of the solution space of $A\vec{x} = \vec{0}$?

(b) What is the dimension of the nullspace of the matrix $B = A - 2I$?

(c) What is the rank of A?

E11 Let $A = \begin{bmatrix} 0.9 & 0.1 & 0.0 \\ 0.0 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.9 \end{bmatrix}$. Verify that A is a Markov matrix and determine its invariant state $\vec{x}$ such that $x_1 + x_2 + x_3 = 1$.

E12 Find the general solution of the system of differential equations

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

E13 If λ is an eigenvalue of the invertible matrix A, prove that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

E14 Let $A \in M_{n\times n}(\mathbb{R})$ be a diagonalizable matrix such that λ_1 is an eigenvalue with algebraic multiplicity n. Prove that $A = \lambda_1 I$.

E15 Prove that if $\vec{v}$ is an eigenvector of a matrix A, then any non-zero scalar multiple of $\vec{v}$ is also an eigenvector of A.

E16 Prove if A is diagonalizable and B is similar to A, then B is also diagonalizable.

For Problems E17–E22, determine whether the statement is true or false. Justify your answer.

E17 Every invertible matrix is diagonalizable.

E18 Every diagonalizable matrix is invertible.

E19 If $A \in M_{n\times n}(\mathbb{R})$ has n distinct eigenvalues, then A is diagonalizable.

E20 If $A\vec{v} = \lambda\vec{v}$, then $\vec{v}$ is an eigenvector of A.

E21 If the reduced row echelon form of $A - \lambda I$ is I, then λ is not an eigenvalue of A.

E22 If $\vec{v}$ is an eigenvector of a linear mapping L, then $\vec{v}$ is an eigenvector of $[L]$.

Further Problems

These exercises are intended to be challenging.

F1 Diagonalize $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Such a matrix A is called **symmetric** since $A^T = A$. It is important to observe that the columns of the diagonalizing matrix P are orthogonal to each other. We will look at the diagonalization of symmetric matrices in Chapter 8.

F2 (a) Suppose that A and B are square matrices such that $AB = BA$. Suppose that the eigenvalues of A all have algebraic multiplicity 1. Prove that any eigenvector of A is also an eigenvector of B.

(b) Give an example to illustrate that the result in part (a) may not be true if A has eigenvalues with algebraic multiplicity greater than 1.

F3 Let $A, B \in M_{n\times n}(\mathbb{R})$. If $\det B \neq 0$, prove that AB and BA have the same eigenvalues.

F4 Let $A \in M_{n\times n}(\mathbb{R})$. Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of A with corresponding eigenvectors $\vec{v}_1, \dots, \vec{v}_k$ respectively. Use the following steps to prove that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

(a) Show that for any i and j we have

$$(A - \lambda_i I)\vec{v}_j = (\lambda_j - \lambda_i)\vec{v}_j$$

(b) Use a proof by induction on k and the result of (a) to prove that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

F5 If $A \in M_{n\times n}(\mathbb{R})$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ respectively, then, by representing $\vec{x}$ with respect to the basis of eigenvectors, show that

$$(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)\vec{x} = \vec{0}$$

for every $\vec{x} \in \mathbb{R}^n$, and hence conclude that "A is a root of its characteristic polynomial." That is, if the characteristic polynomial is

$$C(\lambda) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

then

$$(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = O_{n,n}$$

(Hint: write the characteristic polynomial in factored form.) This result, called the Cayley-Hamilton Theorem, is true for any square matrix A.

F6 If $A \in M_{n\times n}(\mathbb{R})$ is invertible, then use the Cayley-Hamilton Theorem to show that A^{-1} can be written as a polynomial of degree less than or equal to $n - 1$ in A (that is, a linear combination of $\{A^{n-1}, \dots, A^2, A, I\}$).

CHAPTER 7

Inner Products and Projections

CHAPTER OUTLINE

In Section 1.5 we saw that we can use a projection to find a point P *in a plane that is closest to some other point* Q. *We can view this as finding the point in the plane that best approximates* Q. *Since there are many other situations in which we want to find a best approximation, it is very useful to generalize our work with projections from Section 1.5 not only to subspaces of* $\mathbb{R}^n$, *but also to general vector spaces. You may find it helpful to review Section 1.5 carefully before proceeding with this chapter.*

7.1 Orthogonal Bases in $\mathbb{R}^n$

Most of our intuition about coordinate geometry is based on our experience with the standard basis for $\mathbb{R}^n$. *It is therefore a little uncomfortable for many beginners to deal with the arbitrary bases that arise in Chapter 1. Fortunately, for many problems, it is possible to work with bases that have the most essential property of the standard basis: the basis vectors are mutually orthogonal.*

Recall from Section 1.5 that two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are said to be orthogonal if $\vec{x} \cdot \vec{y} = 0$. We now extend the definition of orthogonality to sets.

Definition
Orthogonal Set

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is called an **orthogonal set** if $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$.

EXAMPLE 7.1.1

The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is an orthogonal set of vectors in $\mathbb{R}^4$.

The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is also an orthogonal set of vectors in $\mathbb{R}^4$.

Geometrically, it seems clear that an orthogonal set will be linearly independent. However, since the zero vector is orthogonal to all other vectors, we must ensure that the zero vector is excluded from the set to ensure that this is true.

Theorem 7.1.1 If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then it is linearly independent.

Proof: Consider

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

Take the dot product of $\vec{v}_i$ with each side to get

$$\begin{aligned} \vec{v}_i \cdot (c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) &= \vec{v}_i \cdot \vec{0} \\ c_1(\vec{v}_i \cdot \vec{v}_1) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_k(\vec{v}_i \cdot \vec{v}_k) &= 0 \\ 0 + \cdots + 0 + c_i\|\vec{v}_i\|^2 + 0 + \cdots + 0 &= 0 \end{aligned}$$

since $\vec{v}_i \cdot \vec{v}_j = 0$ unless $i = j$. Moreover, $\vec{v}_i \neq \vec{0}$, so $\|\vec{v}_i\| \neq 0$ and hence $c_i = 0$. Since this works for all $1 \leq i \leq k$, it follows that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent. ■

Theorem 7.1.1 implies that an orthogonal set of non-zero vectors is a basis for the subspace it spans.

Definition
Orthogonal Basis

If $\mathcal{B}$ is an orthogonal set and $\mathcal{B}$ is a basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, then $\mathcal{B}$ is called an **orthogonal basis** for $\mathbb{S}$.

Because the vectors in the basis are all orthogonal to each other, our geometric intuition tells us that it should be quite easy to determine how to write any vector in $\mathbb{S}$ as a linear combination of these basis vectors. The following theorem demonstrates this.

Theorem 7.1.2 If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, then the coefficients c_i, $1 \leq i \leq k$, when any vector $\vec{x} \in \mathbb{S}$ is written as a linear combination of the vectors in $\mathcal{B}$ are given by

$$c_i = \frac{\vec{v}_i \cdot \vec{x}}{\|\vec{v}_i\|^2}$$

Proof: Since $\mathcal{B}$ is a basis, there exists unique coefficients $c_1, \dots, c_k$ such that

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

Take the dot product of $\vec{v}_i$ with each side to get

$$\begin{aligned} \vec{v}_i \cdot \vec{x} &= \vec{v}_i \cdot (c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) \\ \vec{v}_i \cdot \vec{x} &= c_1(\vec{v}_i \cdot \vec{v}_1) + \cdots + c_k(\vec{v}_i \cdot \vec{v}_k) \\ \vec{v}_i \cdot \vec{x} &= 0 + \cdots + 0 + c_i\|\vec{v}_i\|^2 + 0 + \cdots + 0 \end{aligned}$$

since $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$. Because $\mathcal{B}$ is linearly independent, $\vec{v}_i$ cannot be the zero vector and hence $\|\vec{v}_i\| \neq 0$. Therefore, we can solve for c_i to get

$$c_i = \frac{\vec{v}_i \cdot \vec{x}}{\|\vec{v}_i\|^2}$$

■

EXAMPLE 7.1.2

Given that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^2$, write $\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in $\mathcal{B}$ and illustrate with a sketch.

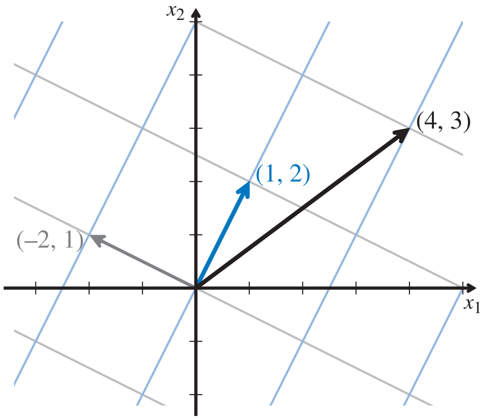
Solution: To simplify the notation, we denote the vectors in $\mathcal{B}$ by $\vec{v}_1$ and $\vec{v}_2$ respectively. By Theorem 7.1.2, we have

$$c_1 = \frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2} = \frac{10}{5} = 2$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{x}}{\|\vec{v}_2\|^2} = \frac{-5}{5} = -1$$

Thus,

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Another useful property of orthogonal sets is that they satisfy the multidimensional Pythagorean Theorem.

Theorem 7.1.3 If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal set in $\mathbb{R}^n$, then

$$\|\vec{v}_1 + \cdots + \vec{v}_k\|^2 = \|\vec{v}_1\|^2 + \cdots + \|\vec{v}_k\|^2$$

Orthonormal Bases

Observe that the formula in Theorem 7.1.2 would be even simpler if all the basis vectors had length 1.

Definition
Orthonormal Set

A set $\{\vec{v}_1, \dots, \vec{v}_k\}$ of vectors in $\mathbb{R}^n$ is called an **orthonormal set** if it is an orthogonal set and each vector $\vec{v}_i$ is a unit vector (that is, $\|\vec{v}_i\| = 1$ for all $1 \leq i \leq k$).

Remark

Recall from Section 1.5 that we find a unit vector $\hat{x}$ in the direction of a vector $\vec{x}$ with the formula

$$\hat{x} = \frac{1}{\|\vec{x}\|}\vec{x}$$

This process is called **normalizing the vector**. Thus, the term "orthonormal" indicates an orthogonal set of non-zero vectors where all the vectors have been normalized.

EXAMPLE 7.1.3

Any subset of the standard basis vectors in $\mathbb{R}^n$ is an orthonormal set. For example, in $\mathbb{R}^6$, $\{\vec{e}_1, \vec{e}_2, \vec{e}_5, \vec{e}_6\}$ is an orthonormal set of four vectors (where, as usual, $\vec{e}_i$ is the i-th standard basis vector).

EXAMPLE 7.1.4

The set $\left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\right\}$ is an orthonormal set in $\mathbb{R}^4$. The vectors are multiples of the vectors in the first set in Example 7.1.1, so they are certainly mutually orthogonal. They have been normalized so that each vector has length 1.

EXERCISE 7.1.1

Verify that $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-5\\2\end{bmatrix}\right\}$ is an orthogonal set and then normalize the vectors to produce the corresponding orthonormal set.

By Theorem 7.1.1 an orthonormal set is necessarily linearly independent. Hence, an orthonormal set is a basis for the subspace it spans.

Definition
Orthonormal Basis

If $\mathcal{B}$ is an orthonormal set and $\mathcal{B}$ is a basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, then $\mathcal{B}$ is called an **orthonormal basis** for $\mathbb{S}$.

EXAMPLE 7.1.5

Given that $\mathcal{B} = \left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\right\}$ is an orthonormal basis for $\mathbb{R}^4$, write $\vec{x} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ as a linear combination of the vectors in $\mathcal{B}$.

Solution: Denote the vectors in $\mathcal{B}$ by $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ respectively. Since each of these vectors has length 1, by Theorem 7.1.2, the coefficients when $\vec{x}$ is written as a linear combination of these vectors are

$$\begin{aligned}
c_1 &= \vec{v}_1 \cdot \vec{x} = \frac{1}{2}(1+2+3+4) = 5\\
c_2 &= \vec{v}_2 \cdot \vec{x} = \frac{1}{2}(1-2+3-4) = -1\\
c_3 &= \vec{v}_3 \cdot \vec{x} = \frac{1}{\sqrt{2}}(-1+0+3+0) = \sqrt{2}\\
c_4 &= \vec{v}_4 \cdot \vec{x} = \frac{1}{\sqrt{2}}(0+2+0-4) = -\sqrt{2}
\end{aligned}$$

Hence,

$$\vec{x} = 5\vec{v}_1 - 1\vec{v}_2 + \sqrt{2}\vec{v}_3 - \sqrt{2}\vec{v}_4$$

EXERCISE 7.1.2

Let $\mathcal{B} = \left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix} \right\}$. Verify that $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^3$ and then write $\vec{x} = \begin{bmatrix}3\\1\\4\end{bmatrix}$ as a linear combination of the vectors in $\mathcal{B}$.

CONNECTION

Dot products and orthonormal bases in $\mathbb{R}^n$ have important generalizations to **inner products** and orthonormal bases in general vector spaces. These will be considered in Section 7.4.

Orthogonal Matrices

In Section 3.3 we saw that the standard matrix of a rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ about the origin through an angle θ is

$$[R_\theta] = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

We also saw in Section 3.5 that it satisfies

$$[R_\theta]^{-1} = [R_\theta]^T$$

We now show that this remarkable property, that the inverse of the matrix equals its transpose, holds whenever the columns or rows of the matrix form an orthonormal basis for $\mathbb{R}^n$.

Theorem 7.1.4 For $P \in M_{n\times n}(\mathbb{R})$, the following are equivalent:

(1) The columns of P form an orthonormal basis for $\mathbb{R}^n$
(2) $P^T = P^{-1}$
(3) The rows of P form an orthonormal basis for $\mathbb{R}^n$

Proof: Let $P = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}$. By definition of matrix-matrix multiplication we have that

$$(P^TP)_{ij} = \vec{v}_i \cdot \vec{v}_j$$

Hence, $P^TP = I$ if and only if $\vec{v}_i \cdot \vec{v}_i = 1$ for all i, and $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. This is true if and only if the columns of P form an orthonormal set of n vectors in $\mathbb{R}^n$. But, by Theorem 7.1.1 and Theorem 2.3.6, such a set of n vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

The result for the rows of P follows from consideration of the product $PP^T = I$. You are asked to show this in Problem C5. ■

Since such matrices are extremely useful, we make the following definition.

Definition

Orthogonal Matrix

A matrix $P \in M_{n\times n}(\mathbb{R})$ such that $P^{-1} = P^T$ is called an **orthogonal matrix**.

Remark

Observe that such matrices should probably be called orthonormal matrices, but the name *orthogonal matrix* is the generally accepted name. Be sure that you remember that an orthogonal matrix has *orthonormal* columns and rows.

EXAMPLE 7.1.6

The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal set (verify this). If the vectors are normalized, the resulting set is orthonormal, so the following matrix P is orthogonal:

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$$

Thus, P is invertible and

$$P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

Moreover, observe that P^{-1} is also orthogonal.

EXERCISE 7.1.3

Verify that $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$ is orthogonal by showing that $PP^T = I$.

Orthogonal matrices have the following useful properties.

Theorem 7.1.5

If $P, Q \in M_{n\times n}(\mathbb{R})$ are orthogonal matrices and $\vec{x}, \vec{y} \in \mathbb{R}^n$, then

(1) $(P\vec{x}) \cdot (P\vec{y}) = \vec{x} \cdot \vec{y}$
(2) $\|P\vec{x}\| = \|\vec{x}\|$
(3) $\det P = \pm 1$
(4) All real eigenvalues of P are 1 or -1
(5) PQ is an orthogonal matrix

Proof: We will prove (1) and leave the others as Problems C1, C2, C3, and C4 respectively.

Recall from page 160 the formula $\vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$. Using this formula we get

$$(P\vec{x}) \cdot (P\vec{y}) = (P\vec{x})^T(P\vec{y}) = \vec{x}^T P^T P\vec{y} = \vec{x}^T I\vec{y} = \vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$$

■

The most important applications of orthogonal matrices considered in this book are the diagonalization of symmetric matrices and the singular value decomposition in Chapter 8. Of course, there are many other applications of orthogonal matrices, such as for 3D computer graphics and Lorenz transformations in physics.

PROBLEMS 7.1

Practice Problems

For Problems A1–A8, determine whether the given set is an orthogonal set. If so, normalize the vectors to produce the corresponding orthonormal set.

A1 $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\-1\end{bmatrix}\right\}$

A2 $\left\{\begin{bmatrix}1\\-3\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$

A3 $\left\{\begin{bmatrix}4\\-1\\1\end{bmatrix},\begin{bmatrix}-1\\0\\4\end{bmatrix},\begin{bmatrix}4\\17\\1\end{bmatrix}\right\}$

A4 $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$

A5 $\left\{\begin{bmatrix}1\\3\\-1\end{bmatrix},\begin{bmatrix}2\\0\\2\end{bmatrix},\begin{bmatrix}3\\-2\\-3\end{bmatrix}\right\}$

A6 $\left\{\begin{bmatrix}1\\-2\\2\end{bmatrix},\begin{bmatrix}-2\\1\\2\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$

A7 $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}-2\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\right\}$

A8 $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}2\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\-3\\-1\\0\end{bmatrix}\right\}$

A9 Given that $\mathcal{B} = \left\{\begin{bmatrix}1\\-2\\2\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix},\begin{bmatrix}-2\\1\\2\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^3$, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{w} = \begin{bmatrix}4\\3\\5\end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ (c) $\vec{y} = \begin{bmatrix}-2\\2\\-1\end{bmatrix}$

A10 Given that $\mathcal{B} = \left\{\begin{bmatrix}1/3\\-2/3\\2/3\end{bmatrix},\begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix},\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}\right\}$ is an orthonormal basis for $\mathbb{R}^3$, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{w} = \begin{bmatrix}4\\3\\5\end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix}3\\-7\\2\end{bmatrix}$ (c) $\vec{y} = \begin{bmatrix}2\\-4\\6\end{bmatrix}$

A11 Given that $\mathcal{B} = \left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix},\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^4$, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{x} = \begin{bmatrix}2\\4\\-3\\5\end{bmatrix}$ (b) $\vec{y} = \begin{bmatrix}-4\\1\\3\\-5\end{bmatrix}$ (c) $\vec{w} = \begin{bmatrix}3\\1\\0\\1\end{bmatrix}$

For Problems A12–A17, decide whether the matrix is orthogonal. If not, indicate how the columns fail to form an orthonormal set (for example, "the second and third columns are not orthogonal").

A12 $\begin{bmatrix}5/13 & 12/13\\12/13 & -5/13\end{bmatrix}$

A13 $\begin{bmatrix}3/5 & 4/5\\-4/5 & -3/5\end{bmatrix}$

A14 $\begin{bmatrix}2/5 & -1/5\\1/5 & 2/5\end{bmatrix}$

A15 $\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

A16 $\begin{bmatrix}1/3 & 2/3 & -2/3\\2/3 & -2/3 & 1/3\\2/3 & 1/3 & 2/3\end{bmatrix}$

A17 $\begin{bmatrix}1/3 & 2/3 & 2/3\\2/3 & -2/3 & 1/3\\2/3 & 1/3 & -2/3\end{bmatrix}$

A18 Given that $\mathcal{B} = \left\{\begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix},\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix},\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}\right\}$ is an orthonormal basis for $\mathbb{R}^3$. Determine another orthonormal basis for $\mathbb{R}^3$ that includes the vector $\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{3}\\1/\sqrt{2}\end{bmatrix}$ and briefly explain why your basis is orthonormal.

A19 Prove that an orthonormal set $\mathcal{B}$ of n vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Homework Problems

For Problems B1–B6, determine whether the given set is an orthogonal set. If so, normalize the vectors to produce the corresponding orthonormal set.

B1 $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\-2\end{bmatrix}\right\}$

B2 $\left\{\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$

B3 $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

B4 $\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\-1\end{bmatrix}\right\}$

B5 $\left\{\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}3\\1\\0\\-2\end{bmatrix}\right\}$

B6 $\left\{\begin{bmatrix}0\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\1\end{bmatrix}, \begin{bmatrix}-3\\0\\2\\1\end{bmatrix}\right\}$

B7 Given that $\mathcal{B} = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^2$, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{w} = \begin{bmatrix}1\\0\end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix}2\\7\end{bmatrix}$ (c) $\vec{y} = \begin{bmatrix}-3\\4\end{bmatrix}$

B8 Given that $\mathcal{B} = \left\{\begin{bmatrix}3\\-4\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^2$, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{w} = \begin{bmatrix}1\\1\end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix}-2\\3\end{bmatrix}$ (c) $\vec{y} = \begin{bmatrix}2\\3\end{bmatrix}$

B9 Given that $\mathcal{B} = \left\{\begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\end{bmatrix}, \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}\right\}$ is an orthonormal basis for $\mathbb{R}^2$, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{w} = \begin{bmatrix}0\\1\end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix}3\\-3\end{bmatrix}$ (c) $\vec{y} = \begin{bmatrix}-2\\-3\end{bmatrix}$

B10 Given that $\mathcal{B} = \left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-3\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-4\\7\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^3$, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{w} = \begin{bmatrix}2\\1\\1\end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ (c) $\vec{y} = \begin{bmatrix}1\\-1\\-3\end{bmatrix}$

B11 Given that $\mathcal{B} = \left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}-2\\-4\\5\\5\end{bmatrix}\right\}$ is an orthogonal basis for the subspace of $\mathbb{R}^4$ it spans, use Theorem 7.1.2 to write the following vectors as a linear combination of the vectors in $\mathcal{B}$.

(a) $\vec{x} = \begin{bmatrix}7\\9\\-1\\-3\end{bmatrix}$ (b) $\vec{y} = \begin{bmatrix}4\\-2\\9\\5\end{bmatrix}$ (c) $\vec{w} = \begin{bmatrix}5\\10\\4\\4\end{bmatrix}$

For Problems B12–B16, decide whether the matrix is orthogonal. If not, indicate how the columns fail to form an orthonormal set.

B12 $\begin{bmatrix}1/\sqrt{5} & 0 & 1/\sqrt{3}\\ 2/\sqrt{5} & 1/\sqrt{2} & 1/\sqrt{3}\\ 0 & 1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}$

B13 $\begin{bmatrix}8/17 & -15/17\\ 15/17 & 8/17\end{bmatrix}$

B14 $\begin{bmatrix}1/3 & -4/\sqrt{18} & 0\\ 2/3 & 1/\sqrt{18} & 1/\sqrt{2}\\ 2/3 & 1/\sqrt{18} & -1/\sqrt{2}\end{bmatrix}$

B15 $\begin{bmatrix}3/\sqrt{24} & 5/\sqrt{24}\\ -5/\sqrt{24} & 3/\sqrt{24}\end{bmatrix}$

B16 $\begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5} & 0\\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6}\\ 2/\sqrt{30} & -1/\sqrt{30} & 5/\sqrt{30}\end{bmatrix}$

Conceptual Problems

C1 Prove Theorem 7.1.5 (2).

C2 (a) Prove Theorem 7.1.5 (3).
(b) Give an example of a 2×2 matrix A such that $\det A = 1$, but A is not orthogonal.

C3 Prove Theorem 7.1.5 (4).

C4 Prove Theorem 7.1.5 (5).

C5 Prove that $P \in M_{n\times n}(\mathbb{R})$ is orthogonal if and only if the rows of P form an orthonormal set.

7.2 Projections and the Gram-Schmidt Procedure

Projections onto a Subspace

The projection of a vector $\vec{y}$ onto another vector $\vec{x}$ was defined in Chapter 1 by finding a scalar multiple of $\vec{x}$, denoted $\operatorname{proj}_{\vec{x}}(\vec{y})$, and a vector perpendicular to $\vec{x}$, denoted $\operatorname{perp}_{\vec{x}}(\vec{y})$, such that

$$\vec{y} = \operatorname{proj}_{\vec{x}}(\vec{y}) + \operatorname{perp}_{\vec{x}}(\vec{y})$$

Since we were just trying to find a scalar multiple of $\vec{x}$, the projection of $\vec{y}$ onto $\vec{x}$ can be viewed as projecting $\vec{y}$ onto the subspace spanned by $\vec{x}$. Similarly, we saw how to find the projection of $\vec{y}$ onto a plane, which is just a 2-dimensional subspace. It is natural and useful to define the projection of vectors onto more general subspaces.

Let $\vec{y} \in \mathbb{R}^n$ and let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$. To match what we did in Chapter 1, we want to write $\vec{y}$ as

$$\vec{y} = \operatorname{proj}_{\mathbb{S}}(\vec{y}) + \operatorname{perp}_{\mathbb{S}}(\vec{y})$$

where $\operatorname{proj}_{\mathbb{S}}(\vec{y})$ is a vector in $\mathbb{S}$ and $\operatorname{perp}_{\mathbb{S}}(\vec{y})$ is a vector orthogonal to $\mathbb{S}$. To do this, we first observe that we need to define precisely what we mean by a vector orthogonal to a subspace.

Definition
Orthogonal
Orthogonal Complement

Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$. We say that a vector $\vec{x} \in \mathbb{R}^n$ is **orthogonal** to $\mathbb{S}$ if

$$\vec{x} \cdot \vec{s} = 0 \qquad \text{for all } \vec{s} \in \mathbb{S}$$

We call the set of all vectors orthogonal to $\mathbb{S}$ the **orthogonal complement** of $\mathbb{S}$ and denote it by $\mathbb{S}^\perp$. That is,

$$\mathbb{S}^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{s} = 0 \text{ for all } \vec{s} \in \mathbb{S}\}$$

EXAMPLE 7.2.1

Let $\mathbb{S}$ be a plane passing through the origin in $\mathbb{R}^3$ with normal vector $\vec{n}$. By definition, $\vec{n}$ is orthogonal to every vector in the plane. So, we say that $\vec{n}$ is orthogonal to the plane. Moreover, we know that any scalar multiple of $\vec{n}$ is also orthogonal to $\mathbb{S}$, so $\mathbb{S}^\perp = \operatorname{Span}\{\vec{n}\}$.

On the other hand, we saw in Chapter 1 that the plane is the set of all vectors orthogonal to $\vec{n}$ (or any scalar multiple of $\vec{n}$), so the orthogonal complement of the subspace $\operatorname{Span}\{\vec{n}\}$ is $\mathbb{S}$.

Instead of having to show that a vector $\vec{x} \in \mathbb{R}^n$ is orthogonal to every vector $\vec{s} \in \mathbb{S}$ to prove that $\vec{x} \in \mathbb{S}^\perp$, the following theorem tells us that we only need to show that $\vec{x}$ is orthogonal to every vector in a spanning set for $\mathbb{S}$.

Theorem 7.2.1

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a spanning set for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, and let $\vec{x} \in \mathbb{R}^n$. We have that $\vec{x} \in \mathbb{S}^\perp$ if and only if $\vec{x} \cdot \vec{v}_i = 0$ for all $1 \leq i \leq k$.

EXAMPLE 7.2.2

Let $\mathbb{W} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\}$. Find $\mathbb{W}^\perp$.

Solution: By Theorem 7.2.1, we just need to find all $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \in \mathbb{R}^4$ such that

$$0 = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\1\end{bmatrix} = x_1 + x_4$$

$$0 = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\0\end{bmatrix} = x_1 + x_3$$

The solution space of this homogeneous system is $\operatorname{Span}\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}\right\}$. Hence,

$$\mathbb{W}^\perp = \operatorname{Span}\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}\right\}$$

EXERCISE 7.2.1

Let $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right\}$. Find $\mathbb{S}^\perp$.

We get the following important facts about $\mathbb{S}$ and $\mathbb{S}^\perp$.

Theorem 7.2.2

If $\mathbb{S}$ is a k-dimensional subspace of $\mathbb{R}^n$, then

(1) $\mathbb{S}^\perp$ is a subspace of $\mathbb{R}^n$.
(2) $\dim(\mathbb{S}^\perp) = n - k$
(3) $\mathbb{S} \cap \mathbb{S}^\perp = \{\vec{0}\}$
(4) If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis for $\mathbb{S}$ and $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{S}^\perp$, then $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$.

You are asked to prove (2), (3), and (4) in Problems C1, C2, and C3.

We are now able to return to our goal of defining the projection of a vector $\vec{x} \in \mathbb{R}^n$ onto a subspace $\mathbb{S}$ of $\mathbb{R}^n$. Assume that we have an orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for $\mathbb{S}$ and an orthogonal basis $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ for $\mathbb{S}^\perp$. Then, by Theorem 7.2.2 (4), we know that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$. Therefore, Theorem 7.1.2 gives

$$\vec{x} = \frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\vec{v}_k \cdot \vec{x}}{\|\vec{v}_k\|^2}\vec{v}_k + \frac{\vec{v}_{k+1} \cdot \vec{x}}{\|\vec{v}_{k+1}\|^2}\vec{v}_{k+1} + \cdots + \frac{\vec{v}_n \cdot \vec{x}}{\|\vec{v}_n\|^2}\vec{v}_n$$

But this is exactly what we have been looking for! In particular, we have written $\vec{x}$ as a sum of

$$\frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\vec{v}_k \cdot \vec{x}}{\|\vec{v}_k\|^2}\vec{v}_k \in \mathbb{S}$$

and

$$\frac{\vec{v}_{k+1} \cdot \vec{x}}{\|\vec{v}_{k+1}\|^2}\vec{v}_{k+1} + \cdots + \frac{\vec{v}_n \cdot \vec{x}}{\|\vec{v}_n\|^2}\vec{v}_n \in \mathbb{S}^\perp$$

Thus, we can make the following definition.

Definition

Projection onto a Subspace

Perpendicular of a Projection onto a Subspace

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be an *orthogonal basis* of a k-dimensional subspace $\mathbb{S}$ of $\mathbb{R}^n$. For any $\vec{x} \in \mathbb{R}^n$, the **projection** of $\vec{x}$ onto $\mathbb{S}$ is defined to be

$$\operatorname{proj}_{\mathbb{S}}(\vec{x}) = \frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\vec{v}_k \cdot \vec{x}}{\|\vec{v}_k\|^2}\vec{v}_k$$

The **projection of $\vec{x}$ perpendicular to** $\mathbb{S}$ is defined to be

$$\operatorname{perp}_{\mathbb{S}}(\vec{x}) = \vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x})$$

Remark

A key component for this definition is that we must have an orthogonal basis for the subspace $\mathbb{S}$. If you try to use the given formula on a basis that is not an orthogonal basis, you generally will not get the correct answer.

We have defined $\operatorname{perp}_{\mathbb{S}}(\vec{x})$ so that we do not require an orthogonal basis for $\mathbb{S}^\perp$. To ensure that this is valid, we do need to verify that $\operatorname{perp}_{\mathbb{S}}(\vec{x}) \in \mathbb{S}^\perp$. For any $1 \le i \le k$, we have

$$\begin{aligned}
\vec{v}_i \cdot \operatorname{perp}_{\mathbb{S}}(\vec{x}) &= \vec{v}_i \cdot \left[\vec{x} - \left(\frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\vec{v}_k \cdot \vec{x}}{\|\vec{v}_k\|^2}\vec{v}_k\right)\right] \\
&= \vec{v}_i \cdot \vec{x} - \vec{v}_i \cdot \left(\frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\vec{v}_k \cdot \vec{x}}{\|\vec{v}_k\|^2}\vec{v}_k\right) \\
&= \vec{v}_i \cdot \vec{x} - \left(0 + \cdots + 0 + \frac{\vec{v}_i \cdot \vec{x}}{\|\vec{v}_i\|^2}(\vec{v}_i \cdot \vec{v}_i) + 0 + \cdots + 0\right) \\
&= \vec{v}_i \cdot \vec{x} - \vec{v}_i \cdot \vec{x} \\
&= 0
\end{aligned}$$

since $\mathcal{B}$ is an orthogonal basis. Hence, $\operatorname{perp}_{\mathbb{S}}(\vec{x})$ is orthogonal to every vector in the orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ of $\mathbb{S}$. Consequently, $\operatorname{perp}_{\mathbb{S}}(\vec{x}) \in \mathbb{S}^\perp$ by Theorem 7.2.1.

EXAMPLE 7.2.3

Let $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right\}$ and let $\vec{x} = \begin{bmatrix}2\\5\\-7\\3\end{bmatrix}$. Determine $\operatorname{proj}_{\mathbb{S}}(\vec{x})$ and $\operatorname{perp}_{\mathbb{S}}(\vec{x})$.

Solution: Observe that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{S}$. Thus,

$$\operatorname{proj}_{\mathbb{S}}(\vec{x}) = \frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{x}}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{-13}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}-5/2\\4\\-5/2\\4\end{bmatrix}$$

$$\operatorname{perp}_{\mathbb{S}}(\vec{x}) = \vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix}2\\5\\-7\\3\end{bmatrix} - \begin{bmatrix}-5/2\\4\\-5/2\\4\end{bmatrix} = \begin{bmatrix}9/2\\1\\-9/2\\-1\end{bmatrix}$$

EXERCISE 7.2.2

Let $\mathcal{B} = \left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}\right\}$ and let $\vec{x} = \begin{bmatrix}2\\1\\3\end{bmatrix}$. Show that $\mathcal{B}$ is an orthogonal basis for $\mathbb{S} = \operatorname{Span}\mathcal{B}$, and determine $\operatorname{proj}_{\mathbb{S}}(\vec{x})$ and $\operatorname{perp}_{\mathbb{S}}(\vec{x})$.

Recall that we showed in Chapter 1 that the projection of a vector $\vec{x} \in \mathbb{R}^3$ onto a plane in $\mathbb{R}^3$ is the vector in the plane that is closest to $\vec{x}$. We now prove that the projection of $\vec{x} \in \mathbb{R}^n$ onto a subspace $\mathbb{S}$ of $\mathbb{R}^n$ is the vector in $\mathbb{S}$ that is closest to $\vec{x}$.

Theorem 7.2.3

Approximation Theorem

Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$. If $\vec{x} \in \mathbb{R}^n$, then the vector in $\mathbb{S}$ that is closest to $\vec{x}$ is $\operatorname{proj}_{\mathbb{S}}(\vec{x})$. That is,

$$\|\vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x})\| < \|\vec{x} - \vec{v}\|$$

for all $\vec{v} \in \mathbb{S}$, $\vec{v} \neq \operatorname{proj}_{\mathbb{S}}(\vec{x})$.

Proof: Consider $\vec{x} - \vec{v} = (\vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x})) + (\operatorname{proj}_{\mathbb{S}}(\vec{x}) - \vec{v})$. Now, observe that $\{\vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x}), \operatorname{proj}_{\mathbb{S}}(\vec{x}) - \vec{v}\}$ is an orthogonal set since $\vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x}) = \operatorname{perp}_{\mathbb{S}}(\vec{x}) \in \mathbb{S}^{\perp}$ and $\operatorname{proj}_{\mathbb{S}}(\vec{x}) - \vec{v} \in \mathbb{S}$. Therefore, by Theorem 7.1.3, we get

$$\begin{aligned}\|\vec{x} - \vec{v}\|^2 &= \|(\vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x})) + (\operatorname{proj}_{\mathbb{S}}(\vec{x}) - \vec{v})\|^2 \\ &= \|\vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x})\|^2 + \|\operatorname{proj}_{\mathbb{S}}(\vec{x}) - \vec{v}\|^2 \\ &> \|\vec{x} - \operatorname{proj}_{\mathbb{S}}(\vec{x})\|^2\end{aligned}$$

since $\|\operatorname{proj}_{\mathbb{S}}(\vec{x}) - \vec{v}\|^2 > 0$ if $\vec{v} \neq \operatorname{proj}_{\mathbb{S}}(\vec{x})$. The result follows. ■

The Gram-Schmidt Procedure

For many of the calculations in this chapter, we need an orthogonal (or orthonormal) basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$. If $\mathbb{S}$ is a k-dimensional subspace of $\mathbb{R}^n$, it is certainly possible to use the methods of Section 4.3 to produce some basis $\{\vec{w}_1, \dots, \vec{w}_k\}$ for $\mathbb{S}$. We will now show that we can convert any such basis for $\mathbb{S}$ into an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for $\mathbb{S}$.

The construction is recursive. We first take $\vec{v}_1 = \vec{w}_1$ and see that $\{\vec{v}_1\}$ is an orthogonal basis for Span$\{\vec{w}_1\}$. Then, whenever we have an orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_{i-1}\}$ for Span$\{\vec{w}_1, \dots, \vec{w}_{i-1}\}$, we want to find a vector $\vec{v}_i$ such that $\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_i\}$ is an orthogonal basis for Span$\{\vec{w}_1, \dots, \vec{w}_i\}$. We will repeat this procedure, called the **Gram-Schmidt Procedure**, until we have the desired orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for Span$\{\vec{w}_1, \dots, \vec{w}_k\}$. We will use the following theorem.

Theorem 7.2.4 If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then for any $t_1, \dots, t_{k-1} \in \mathbb{R}$, we have

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k + t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}\}$$

We first demonstrate the **Gram-Schmidt Procedure** with an example.

EXAMPLE 7.2.4 Let $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$. Find an orthogonal basis for $\mathbb{S} = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

Solution: First step: Let $\vec{v}_1 = \vec{w}_1$ and let $\mathbb{S}_1 = \text{Span}\{\vec{v}_1\}$.

Second step: We want to find a vector $\vec{v}_2$ that is orthogonal to $\vec{v}_1$, and such that Span$\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{w}_1, \vec{w}_2\}$. We know that

$$\text{perp}_{\mathbb{S}_1}(\vec{w}_2) = \vec{w}_2 - \frac{\vec{v}_1 \cdot \vec{w}_2}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

is orthogonal to $\vec{v}_1$. We also get that Span$\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{w}_1, \vec{w}_2\}$ by Theorem 7.2.4. Hence, we take $\vec{v}_2 = \text{perp}_{\mathbb{S}_1}(\vec{w}_2)$ and define $\mathbb{S}_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third step: We want to find a vector $\vec{v}_3$ that it is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$, and such that Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$. Again, we see that we can use $\text{perp}_{\mathbb{S}_2}(\vec{w}_3)$. We let

$$\vec{v}_3 = \text{perp}_{\mathbb{S}_2}(\vec{w}_3) = \vec{w}_3 - \frac{\vec{v}_1 \cdot \vec{w}_3}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_3}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} -1 \\ 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

We now have that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{S}$.

Algorithm 7.2.1

The Gram-Schmidt Procedure:
Let $\{\vec{w}_1, \dots, \vec{w}_k\}$ be a basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$.
First step: Let $\vec{v}_1 = \vec{w}_1$ and let $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.
Second step: Let

$$\vec{v}_2 = \operatorname{perp}_{\mathbb{S}_1}(\vec{w}_2) = \vec{w}_2 - \frac{\vec{v}_1 \cdot \vec{w}_2}{\|\vec{v}_1\|^2}\vec{v}_1$$

and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.
***i*-th step:** Suppose that $i-1$ steps have been carried out so that $\{\vec{v}_1, \dots, \vec{v}_{i-1}\}$ is an orthogonal set, and $\mathbb{S}_{i-1} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}\} = \operatorname{Span}\{\vec{w}_1, \dots, \vec{w}_{i-1}\}$. Let

$$\vec{v}_i = \operatorname{perp}_{\mathbb{S}_{i-1}}(\vec{w}_i) = \vec{w}_i - \frac{\vec{v}_1 \cdot \vec{w}_i}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_i}{\|\vec{v}_2\|^2}\vec{v}_2 - \cdots - \frac{\vec{v}_{i-1} \cdot \vec{w}_i}{\|\vec{v}_{i-1}\|^2}\vec{v}_{i-1}$$

After the k-th step is completed, we will have an orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for $\mathbb{S}$.

Remarks

1. It is an important feature of the construction that $\mathbb{S}_{i-1}$ is a subspace of the next $\mathbb{S}_i$ and that $\{\vec{v}_1, \dots, \vec{v}_i\}$ is an orthogonal basis for $\mathbb{S}_i$.
2. Since it is really only the direction of $\vec{v}_i$ that is important in this procedure, we can rescale each $\vec{v}_i$ in any convenient fashion to simplify the calculations.
3. The order of the vectors in the original basis has an effect on the calculations because each step takes the perpendicular part of the next vector. That is, if the original vectors were given in a different order, the procedure might produce a different orthogonal basis.
4. The Gram-Schmidt Procedure does not require that we start with a basis $\mathbb{S}$; only a spanning set is required. In particular, if we find that $\operatorname{perp}_{\mathbb{S}_{i-1}}(\vec{w}_i) = \vec{0}$, then $\vec{w}_i$ is a linear combination of $\{\vec{w}_1, \dots, \vec{w}_{i-1}\}$. In this case, we omit $\vec{w}_i$ from the spanning set and continue the procedure with the next vector. This is demonstrated in Example 7.2.6.

EXAMPLE 7.2.5

Use the Gram-Schmidt Procedure on the set $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ to find an orthonormal basis for $\mathbb{R}^3$.

Solution: Call the vectors in the basis $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$, respectively.
First step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.
Second step: Determine $\operatorname{perp}_{\mathbb{S}_1}(\vec{w}_2)$:

$$\operatorname{perp}_{\mathbb{S}_1}(\vec{w}_2) = \vec{w}_2 - \frac{\vec{v}_1 \cdot \vec{w}_2}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix}-3/2\\3/2\\1\end{bmatrix}$$

(It is wise to check your arithmetic by verifying that $\vec{v}_1 \cdot \operatorname{perp}_{\mathbb{S}_1}(\vec{w}_2) = 0$.)
As mentioned above, we can take any non-zero scalar multiple of $\operatorname{perp}_{\mathbb{S}_1}(\vec{w}_2)$, so we take $\vec{v}_2 = \begin{bmatrix}-3\\3\\2\end{bmatrix}$ and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

EXAMPLE 7.2.5 (continued)

Third step: Determine $\text{perp}_{\mathbb{S}_2}(\vec{w}_3)$:

$$\text{perp}_{\mathbb{S}_2}(\vec{w}_3) = \vec{w}_3 - \frac{\vec{v}_1 \cdot \vec{w}_3}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_3}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} 2/11 \\ -2/11 \\ 6/11 \end{bmatrix}$$

(Again, it is wise to check that $\text{perp}_{\mathbb{S}_2}(\vec{w}_3)$ *is orthogonal to both* $\vec{v}_1$ *and* $\vec{v}_2$*.)*

We now see that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{S}$. To obtain an orthonormal basis for $\mathbb{S}$, we divide each vector in this basis by its length. Thus, an orthonormal basis for $\mathbb{S}$ is

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -3/\sqrt{22} \\ 3/\sqrt{22} \\ 2/\sqrt{22} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{11} \\ -1/\sqrt{11} \\ 3/\sqrt{11} \end{bmatrix} \right\}$$

EXAMPLE 7.2.6

Use the Gram-Schmidt Procedure to find an orthogonal basis for the subspace

$$\mathbb{S} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\} \text{ of } \mathbb{R}^4.$$

Solution: **First step:** Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\{\vec{v}_1\}$.

Second step: Determine $\text{perp}_{\mathbb{S}_1}(\vec{w}_2)$:

$$\text{perp}_{\mathbb{S}_1}(\vec{w}_2) = \vec{w}_2 - \frac{\vec{v}_1 \cdot \vec{w}_2}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 1/3 \end{bmatrix}$$

We take $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbb{S}_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third step: Determine $\text{perp}_{\mathbb{S}_2}(\vec{w}_3)$:

$$\text{perp}_{\mathbb{S}_2}(\vec{w}_3) = \vec{w}_3 - \frac{\vec{v}_1 \cdot \vec{w}_3}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_3}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $\vec{w}_3 \in \text{Span}\{\vec{w}_1, \vec{w}_2\}$. Therefore, we ignore $\vec{w}_3$ and instead calculate $\text{perp}_{\mathbb{S}_2}(\vec{w}_4)$.

$$\vec{v}_3 = \text{perp}_{\mathbb{S}_2}(\vec{w}_4) = \vec{w}_4 - \frac{\vec{v}_1 \cdot \vec{w}_4}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{w}_4}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} 1/5 \\ -2/5 \\ -2/5 \\ 1/5 \end{bmatrix}$$

We get that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{S}$.

EXAMPLE 7.2.7

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and let $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ -2 \\ 5 \end{bmatrix}$. Find the projection of $\vec{x}$ onto the subspace $\mathbb{S} = \operatorname{Span} \mathcal{B}$ of $\mathbb{R}^4$.

Solution: Observe that $\mathcal{B}$ is not orthogonal. Therefore, our first step must be to perform the Gram-Schmidt Procedure on $\mathcal{B}$ to create an orthogonal basis for $\mathbb{S}$.

First step: Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second step: Determine $\operatorname{perp}_{\mathbb{S}_1}(\vec{b}_2)$:

$$\operatorname{perp}_{\mathbb{S}_1}(\vec{b}_2) = \vec{b}_2 - \frac{\vec{v}_1 \cdot \vec{b}_2}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

To simplify calculations we use $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$. Let $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third step: Determine $\operatorname{perp}_{\mathbb{S}_2}(\vec{b}_3)$:

$$\operatorname{perp}_{\mathbb{S}_2}(\vec{b}_3) = \vec{b}_3 - \frac{\vec{v}_1 \cdot \vec{b}_3}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{b}_3}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{15}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}$$

We take $\vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}$. Thus, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{S}$.

We can now determine the projection.

$$\operatorname{proj}_{\mathbb{S}}(\vec{x}) = \frac{\vec{v}_1 \cdot \vec{x}}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{x}}{\|\vec{v}_2\|^2}\vec{v}_2 + \frac{\vec{v}_3 \cdot \vec{x}}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{11}{3}\vec{v}_1 + \frac{14}{15}\vec{v}_2 + \frac{1}{10}\vec{v}_3 = \begin{bmatrix} 9/2 \\ 3 \\ -2 \\ 9/2 \end{bmatrix}$$

PROBLEMS 7.2

Practice Problems

For Problems A1–A8, let $\mathbb{S}$ be the subspace spanned by the given orthogonal set. Determine $\text{proj}_{\mathbb{S}}(\vec{x})$ and $\text{perp}_{\mathbb{S}}(\vec{x})$.

A1 $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}-1\\2\end{bmatrix}$

A2 $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}6\\9\\5\end{bmatrix}$

A3 $\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}2\\\sqrt{5}\\3\end{bmatrix}$

A4 $\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}7\\4\\1\end{bmatrix}$

A5 $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

A6 $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}1\\0\\5\end{bmatrix}$

A7 $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}2\\3\\5\\6\end{bmatrix}$

A8 $\left\{\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\\2\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}2\\3\\5\\6\end{bmatrix}$

For Problems A9–13, find a basis for the orthogonal complement of the subspace spanned by the given set.

A9 $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$

A10 $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}$

A11 $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$

A12 $\left\{\begin{bmatrix}2\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\3\\1\end{bmatrix}\right\}$

A13 $\left\{\begin{bmatrix}1\\1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\4\end{bmatrix}\right\}$

For Problems A14–A20, use the Gram-Schmidt Procedure to produce an orthogonal basis for the subspace spanned by the given set.

A14 $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

A15 $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

A16 $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$

A17 $\left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

A18 $\left\{\begin{bmatrix}4\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\-4\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$

A19 $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}$

A20 $\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$

For Problems A21–A24, use the Gram-Schmidt Procedure to produce an orthonormal basis for the subspace spanned by the given set.

A21 $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

A22 $\left\{\begin{bmatrix}2\\1\\-2\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$

A23 $\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}\right\}$

A24 $\left\{\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right\}$

For Problems A25–A30, let $\mathbb{S}$ be the subspace spanned by the given set. Find the vector $\vec{y}$ in $\mathbb{S}$ that is closest to $\vec{x}$.

A25 $\left\{\begin{bmatrix}2\\3\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}-1\\2\end{bmatrix}$

A26 $\left\{\begin{bmatrix}3\\-4\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}5\\0\\2\end{bmatrix}$

A27 $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}-1\\2\end{bmatrix}$

A28 $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}6\\0\\3\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}5\\4\\5\end{bmatrix}$

A29 $\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\\1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}-1\\3\\-2\\3\end{bmatrix}$

A30 $\left\{\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-2\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$

Homework Problems

For Problems B1–B8, let $\mathbb{S}$ be the subspace spanned by the given orthogonal set. Determine $\text{proj}_{\mathbb{S}}(\vec{x})$ and $\text{perp}_{\mathbb{S}}(\vec{x})$.

B1 $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}3\\1\\2\end{bmatrix}$

B2 $\left\{\begin{bmatrix}3\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-5\\2\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}1\\5\\3\end{bmatrix}$

B3 $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}2\\-3\\6\end{bmatrix}$

B4 $\left\{\begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-4\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}3\\0\\3\end{bmatrix}$

B5 $\left\{\begin{bmatrix}3\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}3\\5\\7\end{bmatrix}$

B6 $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\-3\\-2\\1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}1\\1\\2\\0\end{bmatrix}$

B7 $\left\{\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}2\\1\\2\\3\end{bmatrix}$

B8 $\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\0\\2\end{bmatrix}\right\}, \vec{x} = \begin{bmatrix}3\\1\\4\\7\end{bmatrix}$

For Problems B9–B16, find a basis for the orthogonal complement of the subspace spanned by the given set.

B9 $\left\{\begin{bmatrix}1\\3\\3\end{bmatrix}\right\}$

B10 $\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$

B11 $\left\{\begin{bmatrix}3\\1\\-1\end{bmatrix},\begin{bmatrix}4\\4\\1\end{bmatrix}\right\}$

B12 $\left\{\begin{bmatrix}1\\-1\\1\\2\end{bmatrix},\begin{bmatrix}2\\1\\0\\-1\end{bmatrix}\right\}$

B13 $\left\{\begin{bmatrix}2\\3\\1\\-3\end{bmatrix},\begin{bmatrix}1\\2\\3\\2\end{bmatrix}\right\}$

B14 $\left\{\begin{bmatrix}1\\2\\1\\2\end{bmatrix},\begin{bmatrix}2\\3\\1\\0\end{bmatrix},\begin{bmatrix}3\\3\\1\\2\end{bmatrix}\right\}$

B15 $\left\{\begin{bmatrix}3\\6\\-2\\3\end{bmatrix},\begin{bmatrix}6\\8\\-3\\1\end{bmatrix},\begin{bmatrix}-3\\2\\0\\7\end{bmatrix}\right\}$

B16 $\left\{\begin{bmatrix}1\\2\\3\\1\end{bmatrix},\begin{bmatrix}2\\4\\6\\2\end{bmatrix},\begin{bmatrix}-2\\-4\\1\\1\end{bmatrix},\begin{bmatrix}-3\\-6\\-2\\0\end{bmatrix}\right\}$

For Problems B17–B23, use the Gram-Schmidt Procedure to produce an orthogonal basis for the subspace spanned by the given set.

B17 $\left\{\begin{bmatrix}2\\1\\-1\end{bmatrix},\begin{bmatrix}1\\2\\7\end{bmatrix}\right\}$

B18 $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$

B19 $\left\{\begin{bmatrix}2\\1\\2\end{bmatrix},\begin{bmatrix}5\\-2\\-4\end{bmatrix}\right\}$

B20 $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix},\begin{bmatrix}0\\-2\\-5\end{bmatrix}\right\}$

B21 $\left\{\begin{bmatrix}1\\2\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-2\\1\\6\end{bmatrix},\begin{bmatrix}1\\-5\\6\\-3\end{bmatrix}\right\}$

B22 $\left\{\begin{bmatrix}-1\\2\\-1\\1\end{bmatrix},\begin{bmatrix}3\\-6\\3\\-3\end{bmatrix},\begin{bmatrix}1\\0\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$

B23 $\left\{\begin{bmatrix}1\\2\\0\\2\end{bmatrix},\begin{bmatrix}1\\3\\-3\\1\end{bmatrix},\begin{bmatrix}2\\3\\5\\-1\end{bmatrix}\right\}$

For Problems B24–B30, use the Gram-Schmidt Procedure to produce an orthonormal basis for the subspace spanned by the given set.

B24 $\left\{\begin{bmatrix}2\\2\\1\end{bmatrix},\begin{bmatrix}1\\3\\1\end{bmatrix}\right\}$

B25 $\left\{\begin{bmatrix}2\\2\\4\end{bmatrix},\begin{bmatrix}1\\3\\3\end{bmatrix}\right\}$

B26 $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\-2\\3\end{bmatrix}\right\}$

B27 $\left\{\begin{bmatrix}1\\-1\\1\\2\end{bmatrix},\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right\}$

B28 $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}2\\1\\2\\2\end{bmatrix}\right\}$

B29 $\left\{\begin{bmatrix}0\\1\\2\\2\end{bmatrix},\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix},\begin{bmatrix}4\\5\\1\\1\end{bmatrix}\right\}$

B30 $\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\1\\1\end{bmatrix},\begin{bmatrix}1\\3\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right\}$

For Problems B31–B36, let $\mathbb{S}$ be the subspace spanned by the given set. Find the vector $\vec{y}$ in $\mathbb{S}$ that is closest to $\vec{x}$.

B31 $\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}3\\1\\3\end{bmatrix}$

B32 $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}4\\3\end{bmatrix}$

B33 $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}-4\\2\\1\end{bmatrix}$

B34 $\left\{\begin{bmatrix}-1\\1\\-1\end{bmatrix},\begin{bmatrix}3\\1\\1\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}0\\-2\\-5\end{bmatrix}$

B35 $\left\{\begin{bmatrix}1\\2\\2\end{bmatrix},\begin{bmatrix}2\\3\\2\end{bmatrix},\begin{bmatrix}4\\5\\2\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}1\\3\\1\end{bmatrix}$

B36 $\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix},\begin{bmatrix}2\\-3\\-2\\1\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}4\\4\\0\\4\end{bmatrix}$

Conceptual Problems

C1 Prove that if $\mathbb{S}$ is a k-dimensional subspace of $\mathbb{R}^n$, then $\mathbb{S}^\perp$ is an $(n-k)$-dimensional subspace.

C2 Prove that if $\mathbb{S}$ is a k-dimensional subspace of $\mathbb{R}^n$, then $\mathbb{S}\cap\mathbb{S}^\perp=\{\vec{0}\}$.

C3 Prove that if $\{\vec{v}_1,\dots,\vec{v}_k\}$ is an orthonormal basis for $\mathbb{S}$ and $\{\vec{v}_{k+1},\dots,\vec{v}_n\}$ is an orthonormal basis for $\mathbb{S}^\perp$, then $\{\vec{v}_1,\dots,\vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

C4 Prove that if $\mathbb{S}$ is a k-dimensional subspace of $\mathbb{R}^n$, then $(\mathbb{S}^\perp)^\perp=\mathbb{S}$.

C5 Suppose that $\mathbb{S}$ is a k-dimensional subspace of $\mathbb{R}^n$. Prove that for any $\vec{x}\in\mathbb{R}^n$ we have

$$\text{proj}_{\mathbb{S}^\perp}(\vec{x})=\text{perp}_{\mathbb{S}}(\vec{x})$$

C6 Assume $\mathcal{B}=\{\vec{a}_1,\vec{a}_2\}$ is a linearly independent set in $\mathbb{R}^2$. Let $\vec{q}_1,\vec{q}_2$ denote the vectors that result from applying the Gram-Schmidt Procedure to $\{\vec{a}_1,\vec{a}_2\}$ (in order) and then normalizing. Prove that the matrix $R=\begin{bmatrix}\vec{a}_1\cdot\vec{q}_1 & \vec{a}_2\cdot\vec{q}_1\\ 0 & \vec{a}_2\cdot\vec{q}_2\end{bmatrix}$ is invertible.

C7 Prove Theorem 7.2.4.

C8 If $\{\vec{v}_1,\dots,\vec{v}_k\}$ is an orthonormal basis for a subspace $\mathbb{S}$, verify that the standard matrix of $\text{proj}_{\mathbb{S}}$ can be written in the form

$$[\text{proj}_{\mathbb{S}}]=\vec{v}_1\vec{v}_1^T+\vec{v}_2\vec{v}_2^T+\cdots+\vec{v}_k\vec{v}_k^T$$

7.3 Method of Least Squares

In the sciences one often tries to find a correlation between quantities, say y and t, by collecting data from repeated experimentation. For example, y might be the position of a particle or the temperature of some body fluid at time t. Optimally, the experimenter would like to find a function f such that $y = f(t)$ for all t. However, it is unlikely that there is a simple function which perfectly matches the data. So, the experimenter tries to find a simple function which best fits the data. We now look at one common method for doing this.

Suppose that an experimenter would like to find an equation which best predicts the value of a quantity y at any time t. Further suppose that the scientific theory indicates that this can be done by an equation of the form $y = a + bt + ct^2$. The experimenter will perform an experiment in which they measure the value of y at times $t_1, t_2, \dots, t_m$ and obtains the values $y_1, y_2, \dots, y_m$.

For each data point (t_i, y_i) we get a linear equation

$$y_i = a + bt_i + ct_i^2$$

Note that t_i is a fixed number, so this equation is actually linear in variables a, b, and c. Thus, we have a system of m linear equations in 3 variables

$$\begin{aligned} y_1 &= a + bt_1 + ct_1^2 \\ \vdots &= \qquad \vdots \\ y_m &= a + bt_m + ct_m^2 \end{aligned}$$

Due to experimentation error, the system is very likely to be inconsistent. So, the experimenter needs to find the values of a, b, and c which best approximates the data they collected.

To solve this problem, we observe that we can write the system in the form

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a + bt_1 + ct_1^2 \\ \vdots \\ a + bt_m + ct_m^2 \end{bmatrix} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

To write this more compactly we define $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, $X = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix}$, and $\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to get

$$\vec{y} = X\vec{a}$$

Since we are assuming this system is inconsistent, we cannot find $\vec{a}$ such that $X\vec{a} = \vec{y}$. So, instead, we try to find $\vec{a}$ such that $X\vec{a}$ is as close as possible to $\vec{y}$. In particular, we want to minimize

$$\|\vec{y} - X\vec{a}\|$$

The Approximation Theorem tells us how to do this. In particular, since $X\vec{a}$ is a vector in the columnspace of X, the Approximation Theorem says that the vector in Col(X) that is closet to $\vec{y}$ is $\text{proj}_{\text{Col}(X)}(\vec{y})$. Hence, to determine the best $\vec{a}$ we just need to solve the system

$$X\vec{a} = \text{proj}_{\text{Col}(X)}(\vec{y}) \tag{7.1}$$

Notice that this is guaranteed to be a consistent system since $\text{proj}_{\text{Col}(X)}(\vec{y}) \in \text{Col}(X)$, so we now can find our desired vector $\vec{a}$.

Although this works, it is not particularly nice. To find the projection, we first need to find a basis for the columnspace of X, and then apply the Gram-Schmidt Procedure to make it orthogonal. After all of that work, we still need to find the projection, and then solve the system. Consequently, we try to further manipulate equation (7.1) to create an easier method.

Since we are really trying to minimize $\|\vec{y} - X\vec{a}\|$, we subtract both sides of equation (7.1) from $\vec{y}$ to get

$$\vec{y} - X\vec{a} = \vec{y} - \text{proj}_{\text{Col}(X)}(\vec{y})$$

But,

$$\vec{y} - \text{proj}_{\text{Col}(X)}(\vec{y}) = \text{perp}_{\text{Col}(X)}(\vec{y}) \in (\text{Col}(X))^{\perp}$$

By the Fundamental Theorem of Linear Algebra, the orthogonal complement of the columnspace is the left nullspace. Hence, we have that $\vec{y} - X\vec{a} \in \text{Null}(X^T)$. Therefore, by definition of the left nullspace, we have that

$$X^T(\vec{y} - X\vec{a}) = \vec{0}$$

Rearranging this gives

$$X^T X\vec{a} = X^T\vec{y}$$

This is called the **normal system**. This system will be consistent by construction. However, it need not have a unique solution. If it does have infinitely many solutions, then each of the solutions will minimize $\|\vec{y} - X\vec{a}\|$.

Wait! This is amazing! We have started with the inconsistent system $X\vec{a} = \vec{y}$, and by simply multiplying both sides by X^T, we not only get a consistent system, but the solution of the new system best approximates a solution to the original inconsistent system!

This method of finding an approximate solution is called the **method of least squares**. It is called this because we are minimizing

$$\|\vec{y} - X\vec{a}\|^2 = \|\vec{y} - (a\vec{1} + b\vec{t} + c\vec{t}^2)\|^2 = \sum_{i=1}^{m} (y_i - (a + bt_i + ct_i^2))^2$$

where

$$\text{err}_i = y_i - (a + bt_i + ct_i^2)$$

measures the error between each data point y_i and the curve as shown in Figure 7.3.1.

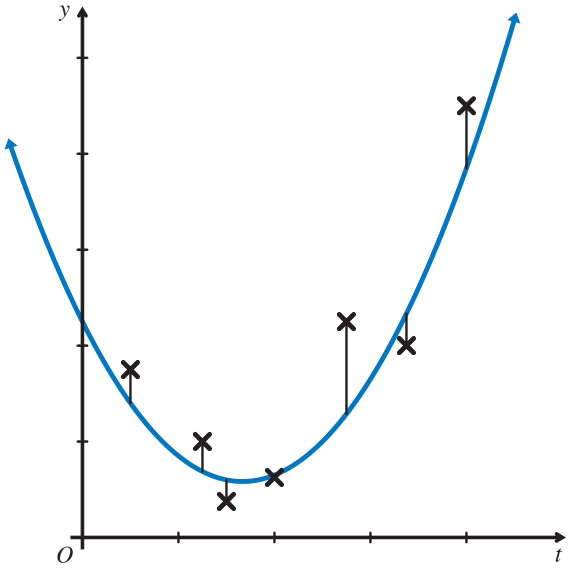


Figure 7.3.1 Some data points and a curve $y = a + bt + ct^2$. Vertical line segments measure the error err_i in the fit at each t_i.

We now demonstrate the method of least squares with a couple of examples.

EXAMPLE 7.3.1

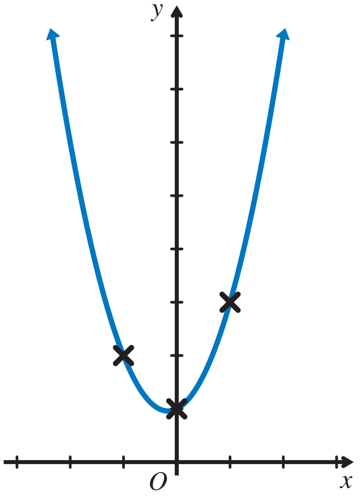
Find a, b, c to obtain the best fitting equation of the form $y = a + bt + ct^2$ for the data points $(-1, 2), (0, 1), (1, 3)$.

Solution: We let $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $X = \begin{bmatrix} 1 & -1 & (-1)^2 \\ 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

The normal system gives

$$X^T X \vec{a} = X^T \vec{y}$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \vec{a} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$



Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 6 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$$

So, the best fitting parabola is

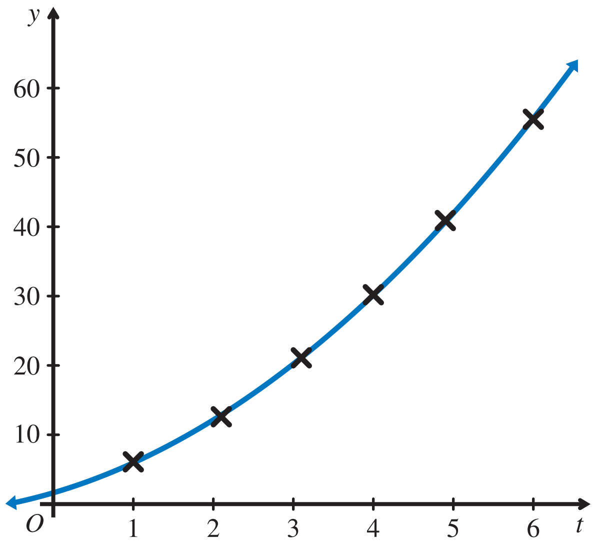
$$p(x) = 1 + \frac{1}{2}t + \frac{3}{2}t^2$$

EXAMPLE 7.3.2

Suppose that the experimenter's data points are $(1.0, 6.1)$, $(2.1, 12.6)$, $(3.1, 21.1)$, $(4.0, 30.2)$, $(4.9, 40.9)$, $(6.0, 55.5)$. Find the values of a, b, c so that the equation $y = a + bt + ct^2$ best fits the data.

Solution: We let $\vec{y} = \begin{bmatrix} 6.1 \\ 12.6 \\ 21.1 \\ 30.2 \\ 40.9 \\ 55.5 \end{bmatrix}$, $\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and $X = \begin{bmatrix} 1 & 1.0 & (1.0)^2 \\ 1 & 2.1 & (2.1)^2 \\ 1 & 3.1 & (3.1)^2 \\ 1 & 4.0 & (4.0)^2 \\ 1 & 4.9 & (4.9)^2 \\ 1 & 6.0 & (6.0)^2 \end{bmatrix} = \begin{bmatrix} 1 & 1.0 & 1.00 \\ 1 & 2.1 & 4.41 \\ 1 & 3.1 & 9.61 \\ 1 & 4.0 & 16.00 \\ 1 & 4.9 & 24.01 \\ 1 & 6.0 & 36.00 \end{bmatrix}$.

Using a computer, we can find that the solution for the system $(X^T X)\vec{a} = X^T\vec{y}$ is $\vec{a} = \begin{bmatrix} 1.63175 \\ 3.38382 \\ 0.93608 \end{bmatrix}$. The data does not justify retaining so many decimal places, so we take the best-fitting quadratic curve to be $y = 1.6 + 3.4t + 0.9t^2$.



In both of these examples we find that $X^T X$ is invertible. It can be proven (see Problem C5) that $X^T X$ is invertible whenever the number of distinct t_i values is greater than or equal to the number of unknown coefficients. In this case, the method of least squares always has the unique solution

$$\vec{a} = (X^T X)^{-1} X^T \vec{y}$$

If $X^T X$ is not invertible, then the normal system will have infinitely many solutions. Each solution will be an equally good approximation. In this case, we typically take the solution with the smallest length.

So far, we have just been finding a parabola of best fit. For a more general situation, we use a similar construction. The matrix X, called the **design matrix**, depends on the desired model curve and the way the data is collected. This is demonstrated in the next example.

EXAMPLE 7.3.3

Find a and b to obtain the best-fitting equation of the form $y = at^2 + bt$ for the data points $(-1, 4), (0, 1), (1, 1)$.

Solution: We substitute each data point into the equation $y = at^2 + bt$ to get the system of linear equations

$$\begin{aligned} 4 &= a(-1)^2 + b(-1) \\ 1 &= a(0)^2 + b(0) \\ 1 &= a(1)^2 + b(1) \end{aligned}$$

We then rewrite this system into matrix form.

$$\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a(-1)^2 + b(-1) \\ a(0)^2 + b(0) \\ a(1)^2 + b(1) \end{bmatrix} = \begin{bmatrix} (-1)^2 & -1 \\ 0^2 & 0 \\ 1^2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

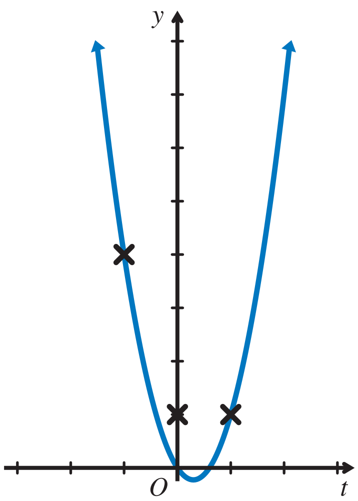
Thus, we let $\vec{y} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$, $\vec{a} = \begin{bmatrix} a \\ b \end{bmatrix}$, and $X = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

We find that $X^T X = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is easy to invert, and hence the method of least squares gives

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= (X^T X)^{-1} X^T \vec{y} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 5/2 \\ -3/2 \end{bmatrix} \end{aligned}$$

So, the equation of best fit for the given data is

$$y = \frac{5}{2}t^2 - \frac{3}{2}t$$



Overdetermined Systems

To get the best possible mathematical model, an experimenter will always perform their experiment many more times than the number of unknowns. This will result in a system of linear equations which has more equations than unknowns. Such a system of equations is called an **overdetermined system**.

The problem of finding the best-fitting curve can be viewed as a special case of the problem of "solving" an overdetermined system. Suppose that $A\vec{x} = \vec{b}$ is a system of m equations in n variables, where m is greater than n. With more equations than variables, we expect the system to be inconsistent unless $\vec{b}$ has some special properties.

Note that the problem in Example 7.3.2 of finding the best-fitting quadratic curve was of this form: we needed to solve $X\vec{a} = \vec{y}$ for the three variables a, b, and c, where there were 6 equations.

If there is no $\vec{x}$ such that $A\vec{x} = \vec{b}$, the next best "solution" is to find a vector $\vec{x}$ that minimizes the "error" $\|A\vec{x} - \vec{b}\|$. Using an argument analogous to that in the special case above, it can be shown that this vector $\vec{x}$ must also satisfy the normal system

$$A^T A\vec{x} = A^T\vec{b}$$

EXAMPLE 7.3.4 Determine the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ for the system

$$\begin{aligned} 3x_1 - x_2 &= 4 \\ x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 1 \end{aligned}$$

Solution: We have $\vec{b} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solving the normal system gives

$$\begin{aligned} A^T A\vec{x} &= A^T\vec{b} \\ \begin{bmatrix} 14 & 1 \\ 1 & 6 \end{bmatrix}\vec{x} &= \begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \\ \vec{x} &= \begin{bmatrix} 14 & 1 \\ 1 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 14 \\ -3 \end{bmatrix} \\ \vec{x} &= \begin{bmatrix} 87/83 \\ -56/83 \end{bmatrix} \end{aligned}$$

So, $\vec{x} = \begin{bmatrix} 87/83 \\ -56/83 \end{bmatrix}$ is the vector that minimizes $\|A\vec{x} - \vec{b}\|$.

Geometry of the Method of Least Squares

Using our work in Section 3.4 and the Fundamental Theorem of Linear Algebra, we can get a geometric view of the Method of Least Squares.

Let $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ is inconsistent. This implies that there exists $\vec{c}_0 \in \mathrm{Col}(A)$ and $\vec{l}_0 \in \mathrm{Null}(A^T)$ with $\vec{l}_0 \neq \vec{0}$ such that

$$\vec{b} = \vec{c}_0 + \vec{l}_0$$

Figure 7.3.2 makes it clear that the vector in $\mathrm{Col}(A)$ that is closest to $\vec{b}$ is

$$\mathrm{proj}_{\mathrm{Col}(A)}(\vec{b}) = \vec{c}_0$$

as indicated by the Approximation Theorem. Thus, as derived above, to find the $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ we just need to solve

$$A\vec{x} = \vec{c}_0$$

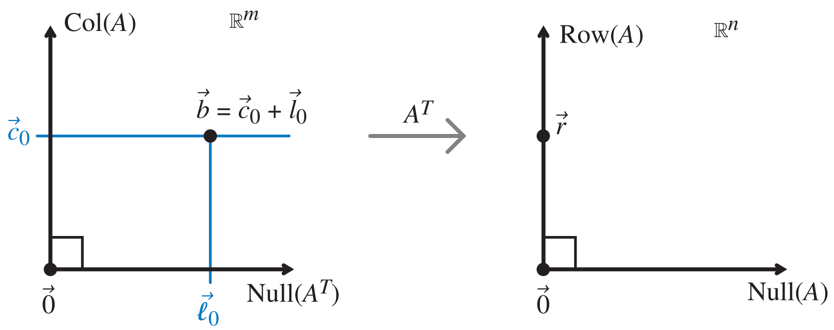


Figure 7.3.2 Geometry of the Method of Least Squares.

As we saw in Section 3.4, if we multiply any vector on the horizontal line $\vec{x} = \vec{c}_0 + \vec{l}$ for any $\vec{l} \in \mathrm{Null}(A^T)$, by A^T, then it will be mapped to the same vector $\vec{r} \in \mathrm{Row}(A)$. That is,

$$A^T\vec{c}_0 = A^T\vec{b}$$

But, $\vec{c}_0 = A\vec{x}$, so this gives

$$A^TA\vec{x} = A^T\vec{b}$$

which is the normal system.

If A^TA is not invertible, then the normal system will have infinitely many solutions. In this case, we want to find the solution of the normal system with the smallest length. Observe that any solution $\vec{x}$ of the normal system is a vector in $\mathbb{R}^n$. Thus, by the Fundamental Theorem of Linear Algebra, every solution has the form

$$\vec{x} = \vec{a} + \vec{n}$$

where $\vec{a} \in \mathrm{Row}(A)$ and $\vec{n} \in \mathrm{Null}(A)$. Since, $\vec{a}$ and $\vec{n}$ are orthogonal, we get by Theorem 7.1.3, that

$$\|\vec{x}\|^2 = \|\vec{a} + \vec{n}\|^2 = \|\vec{a}\|^2 + \|\vec{n}\|^2$$

Thus, the solution with the smallest length will be a vector in $\mathrm{Row}(A)$. We will look at a formula to find the solution with the smallest length in Section 8.5.

PROBLEMS 7.3

Practice Problems

For Problems A1 and A2, find a and b to obtain the best-fitting equation of the form $y = a + bt$ for the given data. Graph the data points and the best-fitting line.

A1 $(1, 9), (2, 6), (3, 5), (4, 3), (5, 1)$

A2 $(-2, 2), (-1, 2), (0, 4), (1, 4), (2, 5)$

A3 Find a, b, and c to obtain the best-fitting equation of the form $y = a + bt + ct^2$ for the data points $(-2, 3)$, $(-1, 2), (0, 0), (1, 2), (2, 8)$. Graph the data points and the best-fitting curve.

For Problems A4–A7, find a and b to obtain the best-fitting equation of the form $y = a + bt$ for the given data.

A4 $(-1, 4), (0, 1), (1, 1)$

A5 $(-2, 2), (1, -1), (1, -2)$

A6 $(-2, 1), (-1, 1), (0, 2), (1, 3)$

A7 $(-1, 1), (0, 2), (1, 2), (2, 3)$

For Problems A8–A11, find a and b to obtain the best-fitting equation of the form $y = a + bt + ct^2$ for the given data.

A8 $(-1, 4), (0, 1), (1, 1)$

A9 $(-2, 2), (0, -2), (2, 3)$

A10 $(-1, 3), (1, 0), (2, -2)$

A11 $(-1, 2), (-1, 3), (1, 1), (1, 2)$

For Problems A12–A15, find the best-fitting equation of the given form for the given data points.

A12 $y = bt + ct^2$: $(-1, 4), (0, 1), (1, 1)$

A13 $y = bt + dt^3$: $(-1, 1), (0, 0), (1, -1), (2, -20)$

A14 $y = a + ct^2$: $(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$

A15 $y = a + bt + c2^t$: $(-1, 2), (-1, 3), (1, 1), (2, 2)$

For Problems A16–A23, verify that the system $A\vec{x} = \vec{b}$ is inconsistent and then determine for the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$.

A16
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 - 3x_2 &= 5 \\ x_1 - 5x_2 &= -4 \end{aligned}$$

A17
$$\begin{aligned} 2x_1 - 2x_2 &= 2 \\ 2x_1 - 3x_2 &= 1 \\ x_1 + x_2 &= 8 \end{aligned}$$

A18
$$\begin{aligned} 2x_1 + x_2 &= 5 \\ 2x_1 - x_2 &= -1 \\ 3x_1 + 2x_2 &= 3 \end{aligned}$$

A19
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 &= 3 \\ -x_1 + x_2 &= 2 \end{aligned}$$

A20
$$\begin{aligned} x_1 + x_3 &= 2 \\ x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 1 \\ -x_1 - x_2 - x_3 &= 2 \end{aligned}$$

A21
$$\begin{aligned} x_1 + 3x_2 &= 1 \\ -x_1 + x_2 &= 7 \\ -2x_1 + x_2 &= 6 \end{aligned}$$

A22
$$\begin{aligned} 2x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ x_1 - x_2 + 3x_3 &= 0 \\ x_2 &= -4 \end{aligned}$$

A23
$$\begin{aligned} x_1 + 2x_2 &= 2 \\ x_1 + 2x_2 &= 3 \\ x_1 + 3x_2 &= 2 \\ x_1 + 3x_2 &= 3 \end{aligned}$$

Homework Problems

For Problems B1 and B2, find a and b to obtain the best-fitting equation of the form $y = a + bt$ for the given data. Graph the data points and the best-fitting line.

B1 $(-2, 9), (-1, 8), (0, 5), (1, 3), (2, 1)$

B2 $(1, 4), (2, 3), (3, 4), (4, 5), (5, 5)$

For Problems B3–B6, find a and b to obtain the best-fitting equation of the form $y = a + bt$ for the given data.

B3 $(-2, 5), (-1, 3), (1, -1), (2, -3)$

B4 $(-2, 2), (-1, 4), (1, 6)$

B5 $(-2, -1), (1, -2), (2, -3)$

B6 $(-2, 5), (0, 2), (1, 1), (2, -3)$

For Problems B7–B10, find a and b to obtain the best-fitting equation of the form $y = a + bt + ct^2$ for the given data.

B7 $(-1, 2), (0, 3), (1, 5)$

B8 $(-2, 2), (-1, 2), (-1, 3), (0, 3)$

B9 $(-1, -2), (0, 1), (1, 1), (2, 3)$

B10 $(-2, -3), (-1, -1), (1, -3), (2, -5)$

For Problems B11–B19, find the best-fitting equation of the given form for the given data points.

B11 $y = a + ct^2$: $(-1, 2), (0, 3), (1, 5)$

B12 $y = bt + ct^2$: $(-1, 3), (0, 2), (1, 2)$

B13 $y = a + ct^2$: $(-2, 3), (-1, 3), (0, 4), (1, 6)$

B14 $y = bt + dt^3$: $(-2, -1), (-1, 4), (0, 2), (1, 6)$

B15 $y = bt + ct^2$: $(-1, 6), (0, 1), (1, -1), (2, 3)$

B16 $y = a + bt + dt^3$: $(-1, 6), (0, 2), (1, -2), (2, -24)$

B17 $y = a + b2^t$: $(-1, 2), (0, 1), (1, -1)$

B18 $y = a\sin t + b\cos t$: $(-\pi/2, -4), (0, 5), (\pi/2, 3)$

B19 $y = a\sin t + b\cos t$: $(-\pi/2, -2), (0, -1), (\pi/2, 1)$

For Problems B20–B25, verify that the system $A\vec{x} = \vec{b}$ is inconsistent and then determine for the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$.

B20
$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 + 2x_2 &= 5 \\ x_1 + x_2 &= 1 \end{aligned}$$

B21
$$\begin{aligned} x_1 + x_2 &= 7 \\ 2x_1 - x_2 &= -8 \\ x_1 + x_2 &= -4 \end{aligned}$$

B22
$$\begin{aligned} x_1 + 4x_2 &= 4 \\ 2x_1 - x_2 &= 2 \\ 2x_1 - x_2 &= 5 \end{aligned}$$

B23
$$\begin{aligned} x_2 &= -3 \\ x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 2 \end{aligned}$$

B24
$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 - x_2 &= 5 \\ 4x_1 + x_2 &= 2 \\ 2x_1 - 2x_2 &= 3 \end{aligned}$$

B25
$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 + 2x_2 &= 2 \\ x_2 &= 5 \\ -x_1 + x_2 &= 3 \end{aligned}$$

Conceptual Problems

C1 Let $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ is a consistent system. Prove that if $\vec{y}$ is a vector in $\mathbb{R}^n$ such that $A^TA\vec{y} = A^T\vec{b}$, then $A\vec{y} = \vec{b}$.

C2 Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in M_{m\times n}(\mathbb{R})$. Prove if $\vec{b} \in \mathbb{R}^m$ is orthogonal to each $\vec{a}_i$, then $\vec{x} = \vec{0}$ is a solution of the normal system.

C3 Let $A \in M_{m\times n}(\mathbb{R})$ such that A^TA is invertible. Prove that for any $\vec{y} \in \mathbb{R}^n$ we have

$$\operatorname{proj}_{\operatorname{Col}(A)}(\vec{y}) = A(A^TA)^{-1}A^T\vec{y}$$

C4 Let $A \in M_{m\times n}(\mathbb{R})$ whose columns form a linearly independent set. We define the **pseudoinverse** of A by

$$A^+ = (A^TA)^{-1}A^T$$

Prove that A^+ has the following properties.

(a) $AA^+A = A$
(b) $A^+AA^+ = A^+$
(c) $(AA^+)^T = AA^+$
(d) $(A^+A)^T = A^+A$

In Section 8.5, we will see that the unique least squares with minimal length of $A\vec{x} = \vec{b}$ is given by $\vec{x} = A^+\vec{b}$.

C5 Let $X = \begin{bmatrix} \vec{1} & \vec{t} & \vec{t}^2 \end{bmatrix}$, where $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ and $\vec{t}^2 = \begin{bmatrix} t_1^2 \\ \vdots \\ t_n^2 \end{bmatrix}$.

Show that $X^TX = \begin{bmatrix} n & \sum_{i=1}^n t_i & \sum_{i=1}^n t_i^2 \\ \sum_{i=1}^n t_i & \sum_{i=1}^n t_i^2 & \sum_{i=1}^n t_i^3 \\ \sum_{i=1}^n t_i^2 & \sum_{i=1}^n t_i^3 & \sum_{i=1}^n t_i^4 \end{bmatrix}$.

C6 Let $X = \begin{bmatrix} \vec{1} & \vec{t} & \vec{t}^2 & \cdots & \vec{t}^m \end{bmatrix}$, where $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ and

$\vec{t}^i = \begin{bmatrix} t_1^i \\ \vdots \\ t_n^i \end{bmatrix}$ for $1 \leq i \leq n$. Assume that at least $m+1$ of the numbers $t_1, \dots, t_n$ are distinct.

(a) Prove that the columns of X form a linearly independent set by showing that the only solution to $c_0\vec{1} + c_1\vec{t} + \cdots + c_m\vec{t}^m = \vec{0}$ is $c_0 = \cdots = c_m = 0$. (Hint: let $p(t) = c_0 + c_1t + \cdots + c_mt^m$ and show that if $c_0\vec{1} + c_1\vec{t} + \cdots + c_m\vec{t}^m = \vec{0}$, $p(t)$ must be the zero polynomial.)

(b) Use the result from part (a) to prove that X^TX is invertible. (Hint: show that the only solution to $X^TX\vec{v} = \vec{0}$ is $\vec{v} = \vec{0}$ by considering $\|X\vec{v}\|^2$.)

7.4 Inner Product Spaces

In Sections 1.3, 1.5, and 7.2, we saw that the dot product plays an essential role in the discussion of lengths, distances, and projections in $\mathbb{R}^n$. In Chapter 4, we saw that the ideas of vector spaces and linear mappings apply to more general sets, including some function spaces. If ideas such as projections are going to be used in these more general spaces, it will be necessary to have a generalization of the dot product to general vector spaces.

Inner Product Spaces

Consideration of the most essential properties of the dot product in Theorem 1.5.1 on page 60 leads to the following definition.

Definition
Inner Product
Inner Product Space

Let $\mathbb{V}$ be a vector space. An **inner product** on $\mathbb{V}$ is a function $\langle\, ,\rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ such that for all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{V}$ and $s, t \in \mathbb{R}$ we have

(1) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positive definite)
(2) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ (symmetric)
(3) $\langle \mathbf{v}, s\mathbf{w} + t\mathbf{z} \rangle = s\langle \mathbf{v}, \mathbf{w} \rangle + t\langle \mathbf{v}, \mathbf{z} \rangle$ (right linear)

A vector space $\mathbb{V}$ with an inner product is called an **inner product space**.

Remarks

1. The notation $\langle\, ,\rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ indicates that an inner product is a binary operation just like the dot product. In particular, it takes two vectors from $\mathbb{V}$ as inputs and it outputs a real number.

2. Since an inner product is right linear and symmetric, it is also **left linear**:

$$\langle s\mathbf{w} + t\mathbf{z}, \mathbf{v} \rangle = s\langle \mathbf{w}, \mathbf{v} \rangle + t\langle \mathbf{z}, \mathbf{v} \rangle$$

   Thus, we say that an inner product is **bilinear**.

In the same way that a vector space is dependent on the definitions of addition and scalar multiplication, an inner product space is dependent on the definitions of addition, scalar multiplication, and the inner product. This will be demonstrated in the examples below.

EXAMPLE 7.4.1 The dot product is an inner product on $\mathbb{R}^n$, called the **standard inner product** on $\mathbb{R}^n$.

EXAMPLE 7.4.2 Show that the function defined by

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 3x_2y_2$$

is an inner product on $\mathbb{R}^2$.

Solution: We verify that $\langle \ , \ \rangle$ satisfies the three properties of an inner product:

1. $\langle \vec{x}, \vec{x} \rangle = 2x_1^2 + 3x_2^2 \geq 0$. From this we also see that $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$. Thus, it is positive definite.
2. $\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 3x_2y_2 = 2y_1x_1 + 3y_2x_2 = \langle \vec{y}, \vec{x} \rangle$. Thus, it is symmetric.
3. For any $\vec{x}, \vec{w}, \vec{z} \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$,
$$\begin{aligned} \langle \vec{x}, s\vec{w} + t\vec{z} \rangle &= 2x_1(sw_1 + tz_1) + 3x_2(sw_2 + tz_2) \\ &= s(2x_1w_1 + 3x_2w_2) + t(2x_1z_1 + 3x_2z_2) \\ &= s\langle \vec{x}, \vec{w} \rangle + t\langle \vec{x}, \vec{z} \rangle \end{aligned}$$

So, $\langle \ , \ \rangle$ is bilinear. Thus, $\langle \ , \ \rangle$ is an inner product on $\mathbb{R}^2$.

Note that $\mathbb{R}^2$ with the inner product defined in Example 7.4.2 is a different inner product space than $\mathbb{R}^2$ with the dot product. However, although there are infinitely many inner products on $\mathbb{R}^n$, it can be proven that for any inner product on $\mathbb{R}^n$ there exists an orthonormal basis such that the inner product is just the dot product on $\mathbb{R}^n$ with respect to this basis. See Problem C11.

EXAMPLE 7.4.3 Verify that $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)$ defines an inner product on the vector space $P_2(\mathbb{R})$ and determine $\langle 1 + x, 2 - 3x^2 \rangle$.

Solution: We first verify that $\langle \ , \ \rangle$ satisfies the three properties of an inner product:

(1) $\langle \mathbf{p}, \mathbf{p} \rangle = (\mathbf{p}(0))^2 + (\mathbf{p}(1))^2 + (\mathbf{p}(2))^2 \geq 0$ for all $\mathbf{p} \in P_2(\mathbb{R})$. Moreover, $\langle \mathbf{p}, \mathbf{p} \rangle = 0$ if and only if $\mathbf{p}(0) = \mathbf{p}(1) = \mathbf{p}(2) = 0$. But, the only $\mathbf{p} \in P_2(\mathbb{R})$ that is zero for three values of x is the zero polynomial, $\mathbf{p}(x) = 0$. Thus $\langle \ , \ \rangle$ is positive definite.

(2) $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2) = \mathbf{q}(0)\mathbf{p}(0) + \mathbf{q}(1)\mathbf{p}(1) + \mathbf{q}(2)\mathbf{p}(2) = \langle \mathbf{q}, \mathbf{p} \rangle$. So, $\langle \ , \ \rangle$ is symmetric.

(3) For any $\mathbf{p}, \mathbf{q}, \mathbf{r} \in P_2(\mathbb{R})$ and $s, t \in \mathbb{R}$,

$$\begin{aligned} \langle \mathbf{p}, s\mathbf{q} + t\mathbf{r} \rangle &= \mathbf{p}(0)(s\mathbf{q}(0) + t\mathbf{r}(0)) + \mathbf{p}(1)(s\mathbf{q}(1) + t\mathbf{r}(1)) + \mathbf{p}(2)(s\mathbf{q}(2) + t\mathbf{r}(2)) \\ &= s(\mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)) + t(\mathbf{p}(0)\mathbf{r}(0) + \mathbf{p}(1)\mathbf{r}(1) \\ &\quad + \mathbf{p}(2)\mathbf{r}(2)) = s\langle \mathbf{p}, \mathbf{q} \rangle + t\langle \mathbf{p}, \mathbf{r} \rangle \end{aligned}$$

So, $\langle \ , \ \rangle$ is bilinear. Thus, $\langle \ , \ \rangle$ is an inner product on $P_2(\mathbb{R})$. That is, $P_2(\mathbb{R})$ is an inner product space under the inner product $\langle \ , \ \rangle$.

In this inner product space, we have

$$\begin{aligned} \langle 1 + x, 2 - 3x^2 \rangle &= (1 + 0)(2 - 3(0)^2) + (1 + 1)(2 - 3(1)^2) + (1 + 2)(2 - 3(2)^2) \\ &= 2 - 2 - 30 = -30 \end{aligned}$$

EXAMPLE 7.4.4 Let $\operatorname{tr}(C)$ represent the trace of a matrix (the definition of $\operatorname{tr}(C)$ is in property (4) of Theorem 6.2.1 on page 361). Then, $M_{2\times 2}(\mathbb{R})$ is an inner product space under the inner product defined by $\langle A, B\rangle = \operatorname{tr}(A^T B)$. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$, then under this inner product, we have

$$\left\langle \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} \right\rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} \right) = \operatorname{tr}\left(\begin{bmatrix} 4 & 23 \\ 8 & 4 \end{bmatrix} \right) = 4 + 4 = 8$$

EXERCISE 7.4.1 Verify that $\langle A, B\rangle = \operatorname{tr}(A^T B)$ is an inner product for $M_{2\times 2}(\mathbb{R})$. Do you notice a relationship between this inner product and the dot product on $\mathbb{R}^4$?

EXAMPLE 7.4.5 An extremely important inner product in applied mathematics, physics, and engineering is the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

on the vector space $C[-\pi, \pi]$ of continuous functions defined on the closed interval from $-\pi$ to π. This is the foundation for Fourier Series. See Section 7.5.

Orthogonality and Length

The concepts of length and orthogonality are fundamental in geometry and have many real-world applications. Thus, we now extend these concepts to general inner product spaces. Since the definition of an inner product mimics the properties of the dot product, we can define length and orthogonality in an inner product space to match exactly with what we did in $\mathbb{R}^n$.

Definition
Norm
Unit Vector

Let $\mathbb{V}$ be an inner product space. For any $\mathbf{v} \in \mathbb{V}$, we define the **norm** (or **length**) of $\mathbf{v}$ to be

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

A vector $\mathbf{v}$ in an inner product space $\mathbb{V}$ is called a **unit vector** if $\|\mathbf{v}\| = 1$.

EXAMPLE 7.4.6 Find the norm of $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$ under the inner product $\langle A, B\rangle = \operatorname{tr}(A^T B)$.

Solution: Using the result of Exercise 7.4.1 we get

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\operatorname{tr}(A^T A)} = \sqrt{1^2 + 0^2 + 2^2 + 1^2} = \sqrt{6}$$

EXAMPLE 7.4.7

Find the norm of $\mathbf{p}(x) = 1 - 2x - x^2$ in $P_2(\mathbb{R})$ under the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)$$

Solution: We have

$$\|\mathbf{p}\| = \sqrt{(\mathbf{p}(0))^2 + (\mathbf{p}(1))^2 + (\mathbf{p}(2))^2} = \sqrt{1^2 + (1-2-1)^2 + (1-4-4)^2} = \sqrt{54}$$

EXERCISE 7.4.2

Find the norm of $\mathbf{p}(x) = 1$ and $\mathbf{q}(x) = x$ in $P_2(\mathbb{R})$ under each inner product

(a) $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)$

(b) $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$

Of course, we have the usual properties of length in an inner product space.

Theorem 7.4.1

If $\mathbb{V}$ is an inner product space, $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $t \in \mathbb{R}$, then

(1) $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$

(2) $\|t\mathbf{x}\| = |t| \, \|\mathbf{x}\|$

(3) $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$, with equality if and only if $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent

(4) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

We now look at orthogonality in inner product spaces.

Definition

Orthogonal

Orthogonal Set

Orthonormal

Let $\mathbb{V}$ be an inner product space with inner product $\langle\ ,\ \rangle$.
Two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{V}$ are said to be **orthogonal** if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.
The set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in $\mathbb{V}$ is said to be an **orthogonal set** if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$.
An orthogonal set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is called an **orthonormal set** if we also have $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$ for $1 \leq i \leq k$.

With this definition, we can now repeat our arguments from Sections 7.1 and 7.2 for coordinates with respect to an orthogonal basis and projections. In particular, we get that if $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for a subspace $\mathbb{S}$ of an inner product space $\mathbb{V}$ with inner product $\langle\ ,\ \rangle$, then for any $\mathbf{x} \in \mathbb{V}$ we have

$$\text{proj}_{\mathbb{S}}(\mathbf{x}) = \frac{\langle \mathbf{v}_1, \mathbf{x} \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \cdots + \frac{\langle \mathbf{v}_k, \mathbf{x} \rangle}{\|\mathbf{v}_k\|^2}\mathbf{v}_k$$

Additionally, the Gram-Schmidt Procedure is also identical. If we have a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ for an inner product space $\mathbb{V}$ with inner product $\langle\ ,\ \rangle$, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ defined by

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{w}_1 \\
\mathbf{v}_2 &= \mathbf{w}_2 - \frac{\langle \mathbf{v}_1, \mathbf{w}_2 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\
&\vdots \\
\mathbf{v}_n &= \mathbf{w}_n - \frac{\langle \mathbf{v}_1, \mathbf{w}_n \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{v}_{n-1}, \mathbf{w}_n \rangle}{\|\mathbf{v}_{n-1}\|^2}\mathbf{v}_{n-1}
\end{aligned}$$

is an orthogonal basis for $\mathbb{V}$.

EXAMPLE 7.4.8

Use the Gram-Schmidt Procedure to determine an orthonormal basis for $\mathbb{S} = \text{Span}\{1, x\}$ of $P_2(\mathbb{R})$ under the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)$$

Use this basis to determine $\text{proj}_{\mathbb{S}}(x^2)$.

Solution: Denote the basis vectors of $\mathbb{S}$ by $\mathbf{p}_1(x) = 1$ and $\mathbf{p}_2(x) = x$. We want to find an orthogonal basis $\{\mathbf{q}_1(x), \mathbf{q}_2(x)\}$ for $\mathbb{S}$. By using the Gram-Schmidt Procedure, we take $\mathbf{q}_1(x) = \mathbf{p}_1(x) = 1$ and then let

$$\mathbf{q}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{q}_1, \mathbf{p}_2 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 = x - \frac{1(0) + 1(1) + 1(2)}{1^2 + 1^2 + 1^2}1 = x - 1$$

Therefore, our orthogonal basis is $\{\mathbf{q}_1, \mathbf{q}_2\} = \{1, x - 1\}$. Hence, we have

$$\begin{aligned}
\text{proj}_{\mathbb{S}}(x^2) &= \frac{\langle 1, x^2 \rangle}{\|1\|^2}1 + \frac{\langle x - 1, x^2 \rangle}{\|x - 1\|^2}(x - 1) \\
&= \frac{1(0) + 1(1) + 1(4)}{1^2 + 1^2 + 1^2}1 + \frac{(-1)0 + 0(1) + 1(4)}{(-1)^2 + 0^2 + 1^2}(x - 1) \\
&= \frac{5}{3}1 + 2(x - 1) = 2x - \frac{1}{3}
\end{aligned}$$

PROBLEMS 7.4

Practice Problems

For Problems A1–A5, evaluate the expression in $M_{2\times2}(\mathbb{R})$ with inner product $\langle A, B \rangle = \text{tr}(A^T B)$.

A1 $\left\langle \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \right\rangle$

A2 $\left\langle \begin{bmatrix} 3 & 6 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ -3 & -2 \end{bmatrix} \right\rangle$

A3 $\left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|$

A4 $\left\| \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right\|$

A5 $\left\| \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \right\|$

For Problems A6–A9, evaluate the expression in $P_2(\mathbb{R})$ with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)$.

A6 $\langle x - 2x^2, 1 + 3x \rangle$

A7 $\langle 2 - x + 3x^2, 4 - 3x^2 \rangle$

A8 $\|3 - 2x + x^2\|$

A9 $\|9 + 9x + 9x^2\|$

For Problems A10–A14, determine whether $\langle\ ,\ \rangle$ defines an inner product on $P_2(\mathbb{R})$.

A10 $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$

A11 $\langle \mathbf{p}, \mathbf{q} \rangle = |\mathbf{p}(0)\mathbf{q}(0)| + |\mathbf{p}(1)\mathbf{q}(1)| + |\mathbf{p}(2)\mathbf{q}(2)|$

A12 $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + 2\mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$

A13 $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(1) + 2\mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(-1)$

A14 $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{p}(0) + \mathbf{p}(1)\mathbf{p}(1) + \mathbf{q}(0)\mathbf{q}(0) + \mathbf{q}(1)\mathbf{q}(1)$

For Problems A15–A18, assume $\mathbb{S}$ is a subspace of $M_{2\times2}(\mathbb{R})$ with inner product $\langle A, B \rangle = \text{tr}(A^T B)$.

(a) Use the Gram-Schmidt Procedure to determine an orthogonal basis for the following subspaces of $M_{2\times2}(\mathbb{R})$.

(b) Use the orthogonal basis you found in part (a) to determine $\text{proj}_{\mathbb{S}}\left(\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}\right)$.

A15 $\mathbb{S} = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\right\}$

A16 $\mathbb{S} = \text{Span}\left\{\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}$

A17 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1 & 0\\2 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\3 & 1\end{bmatrix}, \begin{bmatrix}3 & -1\\2 & 1\end{bmatrix}\right\}$

A18 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}3 & 1\\-1 & -2\end{bmatrix}, \begin{bmatrix}3 & 1\\0 & -1\end{bmatrix}, \begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right\}$

For Problems A19–A22, assume $\mathbb{S}$ is a subspace of $\mathbb{R}^3$ with inner product $\langle \vec{x}, \vec{y}\rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3$.

(a) Use the Gram-Schmidt Procedure to determine an orthogonal basis for the following subspaces of $\mathbb{R}^3$.

(b) Use the orthogonal basis you found in part (a) to determine $\operatorname{proj}_{\mathbb{S}}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$.

A19 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ **A20** $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}0\\4\\1\end{bmatrix}\right\}$

A21 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$ **A22** $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\5\\2\end{bmatrix}\right\}$

For Problems A23–A26, find the projection of the given polynomials onto the given subspace $\mathbb{S}$ in $P_2(\mathbb{R})$ with inner product $\langle \mathbf{p}, \mathbf{q}\rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$.

A23 $\mathbf{p}(x) = x^2$, $\mathbf{q}(x) = 3 + 2x - x^2$, $\mathbb{S} = \operatorname{Span}\{1 + 2x - x^2\}$

A24 $\mathbf{p}(x) = x$, $\mathbf{q}(x) = 1 - 3x$, $\mathbb{S} = \operatorname{Span}\{1 + 3x\}$

A25 $\mathbf{p}(x) = 1 + x + x^2$, $\mathbf{q}(x) = 1 - x^2$, $\mathbb{S} = \operatorname{Span}\{1, x\}$

A26 $\mathbf{p}(x) = 1$, $\mathbf{q}(x) = 2x + 3x^2$, $\mathbb{S} = \operatorname{Span}\{1 + x, 1 - x^2\}$

A27 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be an orthogonal set in an inner product space $\mathbb{V}$. Prove that

$$\|\mathbf{v}_1 + \cdots + \mathbf{v}_k\|^2 = \|\mathbf{v}_1\|^2 + \cdots + \|\mathbf{v}_k\|^2$$

Homework Problems

For Problems B1–B5, evaluate the expression in $M_{2\times 2}(\mathbb{R})$ with inner product $\langle A, B\rangle = \operatorname{tr}(A^TB)$.

B1 $\left\langle \begin{bmatrix}2 & 1\\-1 & 3\end{bmatrix}, \begin{bmatrix}4 & 6\\-7 & -3\end{bmatrix}\right\rangle$ **B2** $\left\langle \begin{bmatrix}2 & 5\\-8 & 4\end{bmatrix}, \begin{bmatrix}1 & -1\\0 & -4\end{bmatrix}\right\rangle$

B3 $\left\|\begin{bmatrix}1 & -1\\0 & -4\end{bmatrix}\right\|$ **B4** $\left\|\begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}\right\|$ **B5** $\left\|\begin{bmatrix}2 & 3\\-5 & 4\end{bmatrix}\right\|$

For Problems B6–B11, evaluate the expression in $P_2(\mathbb{R})$ with inner product $\langle \mathbf{p}, \mathbf{q}\rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)$.

B6 $\langle 1 + x - x^2, -6 - 2x^2\rangle$ **B7** $\langle 2 + 3x, 1 + x - x^2\rangle$

B8 $\langle 1 + x^2, 1 - x^2\rangle$ **B9** $\|2 - x^2\|$

B10 $\|x^2\|$ **B11** $\|1 + x - x^2\|$

For Problems B12–B15, determine whether $\langle\ ,\ \rangle$ defines an inner product on $P_2(\mathbb{R})$.

B12 $\langle \mathbf{p}, \mathbf{q}\rangle = \mathbf{p}(-2)\mathbf{q}(-2) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(2)\mathbf{q}(2)$

B13 $\langle \mathbf{p}, \mathbf{q}\rangle = 2\mathbf{p}(-2)\mathbf{q}(-2) + \mathbf{p}(0)\mathbf{q}(0) + 2\mathbf{p}(1)\mathbf{q}(1)$

B14 $\langle \mathbf{p}, \mathbf{q}\rangle = \mathbf{p}(-1)\mathbf{q}(-1) - \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$

B15 $\langle \mathbf{p}, \mathbf{q}\rangle = \mathbf{p}(0)\mathbf{q}(2) + 2\mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(0)$

For Problems B16 and B17, assume $\mathbb{S}$ is a subspace of $M_{2\times 2}(\mathbb{R})$ with inner product $\langle A, B\rangle = \operatorname{tr}(A^TB)$.

(a) Use the Gram-Schmidt Procedure to determine an orthogonal basis for the following subspaces of $M_{2\times 2}(\mathbb{R})$.

(b) Use the orthogonal basis you found in part (a) to determine $\operatorname{proj}_{\mathbb{S}}\left(\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\right)$.

B16 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\right\}$

B17 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1 & 2\\2 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\4 & 1\end{bmatrix}, \begin{bmatrix}2 & 5\\3 & 1\end{bmatrix}\right\}$

For Problems B18–B21, assume $\mathbb{S}$ is a subspace of $\mathbb{R}^3$ with inner product $\langle \vec{x}, \vec{y}\rangle = 3x_1y_1 + 2x_2y_2 + x_3y_3$.

(a) Use the Gram-Schmidt Procedure to determine an orthogonal basis for the following subspaces of $\mathbb{R}^3$.

(b) Use the orthogonal basis you found in part (a) to determine $\operatorname{proj}_{\mathbb{S}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

B18 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ **B19** $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$

B20 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right\}$ **B21** $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\5\end{bmatrix}\right\}$

For Problems B22–B27, find the projection of the given polynomials onto the given subspace $\mathbb{S}$ in $P_2(\mathbb{R})$ with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$.

B22 $\mathbf{p}(x) = 1, \mathbf{q}(x) = 1 + 2x^2, \mathbb{S} = \text{Span}\left\{1 + x + x^2\right\}$

B23 $\mathbf{p}(x) = x, \mathbf{q}(x) = x^2, \mathbb{S} = \text{Span}\{1 - x\}$

B24 $\mathbf{p}(x) = 1 - x, \mathbf{q}(x) = 1 + x - x^2, \mathbb{S} = \text{Span}\left\{x, 1 - x^2\right\}$

B25 $\mathbf{p}(x) = x, \mathbf{q}(x) = 2 + x + x^2, \mathbb{S} = \text{Span}\left\{x^2, 1 + x^2\right\}$

B26 $\mathbf{p}(x) = x, \mathbf{q}(x) = 2 + x + x^2, \mathbb{S} = \text{Span}\left\{x^2, 1 - x\right\}$

B27 $\mathbf{p}(x) = 1 - x^2, \mathbf{q}(x) = x^2, \mathbb{S} = \text{Span}\left\{1 + x, x - x^2\right\}$

Conceptual Problems

For Problems C1–C3, determine if the following statements are true or false. Justify your answer.

C1 If there exists $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ such that $\langle \mathbf{x}, \mathbf{y} \rangle < 0$, then $\langle\, , \rangle$ is not an inner product on $\mathbb{V}$.

C2 For any $\mathbf{x}$ in an inner product space $\mathbb{V}$, we have $\langle \mathbf{x}, \mathbf{0} \rangle = 0$.

C3 If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal in an inner product space $\mathbb{V}$, then $s\mathbf{x}$ and $t\mathbf{y}$ are also orthogonal for any $s, t \in \mathbb{R}$.

For Problems C4–C9, define the distance between two vectors $\mathbf{x}, \mathbf{y}$ in an inner product space $\mathbb{V}$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

C4 Prove $d(\mathbf{x}, \mathbf{y}) \geq 0$.

C5 Prove $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.

C6 Prove $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.

C7 Prove $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

C8 Prove $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x} + \mathbf{z}, \mathbf{y} + \mathbf{z})$.

C9 Prove $d(t\mathbf{x}, t\mathbf{y}) = |t|\, d(\mathbf{x}, \mathbf{y})$ for all $t \in \mathbb{R}$.

C10 Suppose that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for an inner product space $\mathbb{V}$ with inner product $\langle\, , \rangle$. Define $G \in M_{3\times 3}(\mathbb{R})$ by

$$g_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle, \qquad \text{for } i, j = 1, 2, 3$$

(a) Prove that G is symmetric ($G^T = G$).

(b) Show that if $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then

$$\langle \vec{x}, \vec{y} \rangle = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} G \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(c) Determine the matrix G of the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2)$ for P_2 with respect to the basis $\{1, x, x^2\}$.

C11 (a) Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and suppose that $\langle\, , \rangle$ is an inner product on $\mathbb{R}^2$. Show that if $\vec{x}, \vec{y} \in \mathbb{R}^2$,

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle = {} & x_1y_1\langle \vec{e}_1, \vec{e}_1 \rangle + x_1y_2\langle \vec{e}_1, \vec{e}_2 \rangle \\ & + x_2y_1\langle \vec{e}_2, \vec{e}_1 \rangle + x_2y_2\langle \vec{e}_2, \vec{e}_2 \rangle \end{aligned}$$

(b) For the inner product in part (a), define a matrix G, called the **standard matrix of the inner product** $\langle\, , \rangle$, by $g_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$ for $i, j = 1, 2$. Show that G is symmetric and that

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i,j=1}^{2} g_{ij}x_iy_j = \vec{x}^T G \vec{y}$$

(c) Apply the Gram-Schmidt Procedure, using the inner product $\langle\, , \rangle$ and the corresponding norm, to produce an orthonormal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$.

(d) Define $\tilde{G}$, the $\mathcal{B}$-matrix of the inner product $\langle\, , \rangle$, by $\tilde{g}_{ij} = \langle \vec{v}_i, \vec{v}_j \rangle$ for $i, j = 1, 2$. Show that $\tilde{G} = I$ and that for $\vec{x} = \tilde{x}_1\vec{v}_1 + \tilde{x}_2\vec{v}_2$ and $\vec{y} = \tilde{y}_1\vec{v}_1 + \tilde{y}_2\vec{v}_2$,

$$\langle \vec{x}, \vec{y} \rangle = \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_2$$

Conclusion. For an arbitrary inner product $\langle\, , \rangle$ on $\mathbb{R}^2$, there exists a basis for $\mathbb{R}^2$ that is orthonormal with respect to this inner product. Moreover, when $\vec{x}$ and $\vec{y}$ are expressed in terms of this basis, $\langle \vec{x}, \vec{y} \rangle$ looks just like the standard inner product in $\mathbb{R}^2$. This argument generalizes in a straightforward way to $\mathbb{R}^n$; see Section 8.2 Problem C6.

7.5 Fourier Series

The Inner Product $\int_a^b f(x)g(x)\,dx$

Let $C[a,b]$ be the space of functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous on the interval $[a,b]$. Then, for any $f, g \in C[a,b]$ we have that the product fg is also continuous on $[a,b]$ and hence integrable on $[a,b]$. Therefore, it makes sense to define an inner product as follows.

The inner product $\langle\ ,\ \rangle$ is defined on $C[a,b]$ by

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

The three properties of an inner product are satisfied because

(1) $\langle f, f\rangle = \int_a^b f(x)f(x)\,dx \geq 0$ for all $f \in C[a,b]$ and $\langle f, f\rangle = \int_a^b f(x)f(x)\,dx = 0$ if and only if $f(x) = 0$ for all $x \in [a,b]$.

(2) $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g, f\rangle$

(3) $\langle f, sg + th\rangle = \int_a^b f(x)(sg(x) + th(x))\,dx = s\int_a^b f(x)g(x)\,dx + t\int_a^b f(x)h(x)\,dx = s\langle f, g\rangle + t\langle f, h\rangle$ for any $s, t \in \mathbb{R}$.

Since an integral is the limit of sums, this inner product defined as the *integral of the product of the values of* f *and* g *at each* x is a fairly natural generalization of the dot product in $\mathbb{R}^n$ defined as a *sum of the product of the* i*-th components of* $\vec{x}$ *and* $\vec{y}$ *for each* i.

One interesting consequence is that the norm of a function f with respect to this inner product is

$$\|f\| = \left(\int_a^b f^2(x)\,dx\right)^{1/2}$$

Intuitively, this is quite satisfactory as a measure of how far the function is from the zero function.

One of the most interesting and important applications of this inner product involves Fourier series.

Fourier Series

Let $CP_{2\pi}$ denote the space of continuous real-valued functions of a real variable that are periodic with period 2π. Such functions satisfy $f(x+2\pi) = f(x)$ for all x. Examples of such functions are $f(x) = c$ for any constant c, $\cos x$, $\sin x$, $\cos 2x$, $\sin 3x$, etc. (Note that the function $\cos 2x$ is periodic with period 2π because $\cos(2(x + 2\pi)) = \cos 2x$. However, its "fundamental (smallest) period" is π.)

In some electrical engineering applications, it is of interest to consider a signal described by functions such as the function

$$f(x) = \begin{cases} -\pi - x & \text{if } -\pi \leq x \leq -\pi/2 \\ x & \text{if } -\pi/2 < x \leq \pi/2 \\ \pi - x & \text{if } \pi/2 < x \leq \pi \end{cases}$$

This function is shown in Figure 7.5.1.

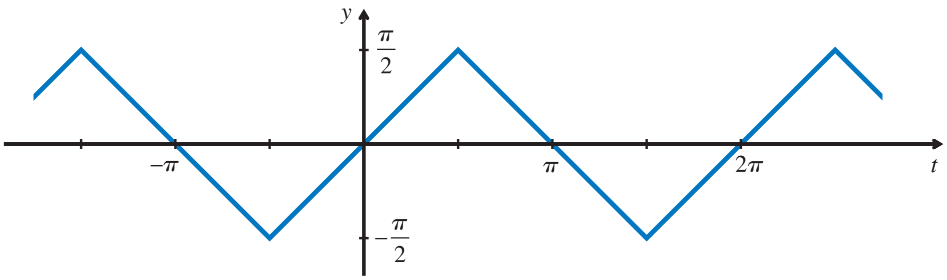


Figure 7.5.1 A continuous periodic function.

In the early nineteenth century, while studying the problem of the conduction of heat, Fourier had the brilliant idea of trying to represent an arbitrary function in $CP_{2\pi}$ as a linear combination of the set of functions

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots\}$$

This idea developed into Fourier analysis, which is now one of the essential tools in quantum physics, communication engineering, and many other areas.

We formulate the questions and ideas as follows. (The proofs of the statements are discussed below.)

(i) For any n, the set of functions $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx\}$ is an orthogonal set with respect to the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$$

The set is therefore an orthogonal basis for the subspace of $CP_{2\pi}$ that it spans. This subspace will be denoted $CP_{2\pi,n}$.

(ii) Given an arbitrary function f in $CP_{2\pi}$, how well can it be approximated by a function in $CP_{2\pi,n}$? We expect from our experience with distance and subspaces that the closest approximation to f in $CP_{2\pi,n}$ is $\text{proj}_{CP_{2\pi,n}}(f)$. The coefficients for Fourier's representation of f by a linear combination of $\{1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots\}$, called **Fourier coefficients**, are found by considering this projection.

(iii) We hope that the approximation improves as n gets larger. Since the distance from f to the n-th approximation $\text{proj}_{CP_{2\pi,n}}(f)$ is $\|\text{perp}_{CP_{2\pi,n}} f\|$, to test if the approximation improves, we must examine whether $\|\text{perp}_{CP_{2\pi,n}}(f)\| \to 0$ as $n \to \infty$.

Let us consider these statements in more detail.

(i) **The orthogonality of constants, sines, and cosines with respect to the inner product** $\langle f, g\rangle = \int\limits_{-\pi}^{\pi} f(x)g(x)\,dx$

These results follow by standard trigonometric integrals and trigonometric identities:

$$\int_{-\pi}^{\pi} \sin nx\,dx = -\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \cos nx\,dx = \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = \int_{-\pi}^{\pi} \frac{1}{2}(\sin(m+n)x - \sin(m-n)x)\,dx = 0$$

and for $m \neq n$,

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos(m+n)x + \cos(m-n)x)\,dx = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos(m-n)x - \cos(m+n)x)\,dx = 0$$

Hence, the set $\{1, \cos x, \sin x, \dots, \cos nx, \sin nx\}$ is orthogonal. To use this as a basis for projection arguments, it is necessary to calculate $\|1\|^2$, $\|\cos mx\|^2$, and $\|\sin mx\|^2$:

$$\|1\|^2 = \int_{-\pi}^{\pi} 1\,dx = 2\pi$$

$$\|\cos mx\|^2 = \int_{-\pi}^{\pi} \cos^2 mx\,dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos 2mx)\,dx = \pi$$

$$\|\sin mx\|^2 = \int_{-\pi}^{\pi} \sin^2 mx\,dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2mx)\,dx = \pi$$

(ii) **The Fourier coefficients of f as coordinates of a projection with respect to the orthogonal basis for $CP_{2\pi,n}$**

The procedure for finding the closest approximation $\mathrm{proj}_{CP_{2\pi,n}}(f)$ in $CP_{2\pi,n}$ to an arbitrary function f in $CP_{2\pi}$ is parallel to the procedure in Sections 7.2 and 7.4. That is, we use the projection formula, given an orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for a subspace $\mathbb{S}$:

$$\mathrm{proj}_{\mathbb{S}}(\vec{x}) = \frac{\langle \vec{v}_1, \vec{x}\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\langle \vec{v}_n, \vec{x}\rangle}{\|\vec{v}_n\|^2}\vec{v}_n$$

There is a standard way to label the coefficients of this linear combination:

$$\begin{aligned}\mathrm{proj}_{CP_{2\pi,n}}(f) = {}& \frac{a_0}{2}1 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx \\ & + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx\end{aligned}$$

The factor $\frac{1}{2}$ in the coefficient of 1 appears here because $\|1\|^2$ is equal to 2π, while the other basis vectors have length squared equal to π. Thus, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_m = \frac{\langle \cos mx, f\rangle}{\|\cos mx\|^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx, \quad 1 \le m \le n$$

$$b_m = \frac{\langle \sin mx, f\rangle}{\|\sin mx\|^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx, \quad 1 \le m \le n$$

(iii) **Is** $\text{proj}_{CP_{2\pi,n}}(f)$ **equal to** f **in the limit as** $n \to \infty$**?**

As $n \to \infty$, the sum becomes an infinite series called the **Fourier series** for f. The question being asked is a question about the convergence of series—and in fact, about series of functions. Such questions are raised in calculus (or analysis) and are beyond the scope of this book. (The short answer is "yes, the series converges to f provided that f is continuous." The problem becomes more complicated if f is allowed to be piecewise continuous.) Questions about convergence are important in physical and engineering applications.

EXAMPLE 7.5.1

Determine $\text{proj}_{CP_{2\pi,3}}(f)$ for the function $f(x)$ defined by $f(x) = |x|$ if $-\pi \le x \le \pi$ and $f(x + 2\pi) = f(x)$ for all x.

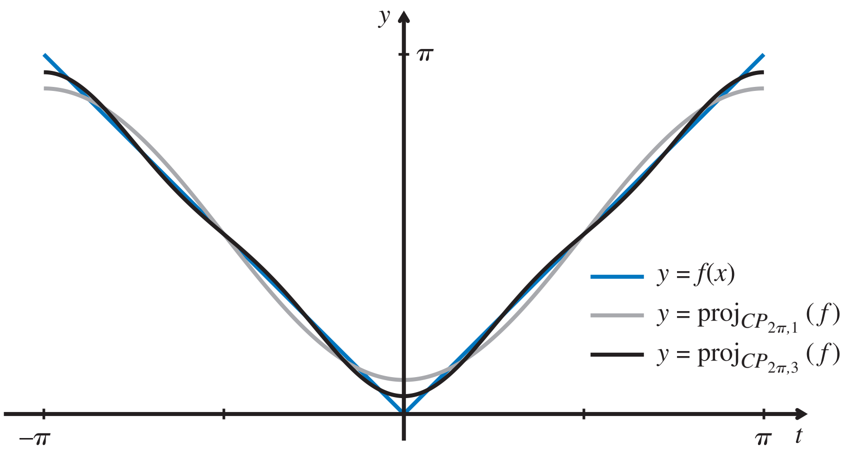
Solution: We have

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \pi \\
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos x\,dx = -\frac{4}{\pi} \\
a_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos 2x\,dx = 0 \\
a_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos 3x\,dx = -\frac{4}{9\pi} \\
b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin x\,dx = 0 \\
b_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin 2x\,dx = 0 \\
b_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin 3x\,dx = 0
\end{aligned}
$$

Hence,

$$\text{proj}_{CP_{2\pi,3}}(f) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$$

The results are shown below.



EXAMPLE 7.5.2

Determine $\text{proj}_{CP_{2\pi,3}}(f)$ for the function $f(x)$ defined by

$$f(x) = \begin{cases} -\pi - x & \text{if } -\pi \le x \le -\pi/2 \\ x & \text{if } -\pi/2 < x \le \pi/2 \\ \pi - x & \text{if } \pi/2 < x \le \pi \end{cases}$$

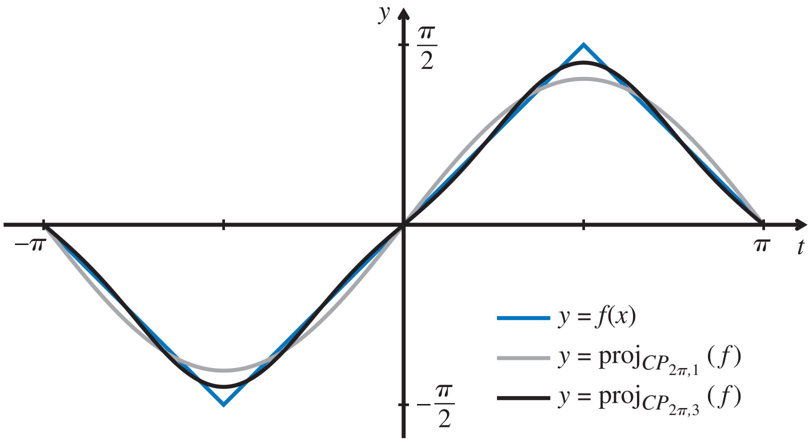
Solution: We have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\,dx = 0 \\
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos x\,dx = 0 \\
a_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos 2x\,dx = 0 \\
a_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos 3x\,dx = 0 \\
b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\sin x\,dx = \frac{4}{\pi} \\
b_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\sin 2x\,dx = 0 \\
b_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\sin 3x\,dx = -\frac{4}{9\pi}
\end{aligned}$$

Hence,

$$\text{proj}_{CP_{2\pi,3}}(f) = \frac{4}{\pi}\sin x - \frac{4}{9\pi}\sin 3x$$

The results are shown below.



PROBLEMS 7.5

Practice Problems

For Problems A1–A3, use a computer to calculate $\text{proj}_{CP_{2\pi,n}}(f)$ for $n = 3, 7,$ and, 11. Graph the function f and each of the projections on the same plot.

A1 $f(x) = x^2, -\pi \le x \le \pi$

A2 $f(x) = e^x, -\pi \le x \le \pi$

A3 $f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$

For Problems A4–A10, use a computer to calculate $\text{proj}_{CP_{2\pi,3}}(f)$.

A4 $f(x) = \sin^2 x, -\pi \le x \le \pi$

A5 $f(x) = 1 - x + x^3, -\pi \le x \le \pi$

A6 $f(x) = 1 + x^2, -\pi \le x \le \pi$

A7 $f(x) = \pi - x, -\pi \le x \le \pi$

A8 $f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0 \\ x & \text{if } 0 < x \le \pi \end{cases}$

A9 $f(x) = \begin{cases} -1 & \text{if } -\pi \le x \le 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$

A10 $f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le -1 \\ 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{if } 1 < x \le \pi \end{cases}$

Homework Problems

For Problems B1–B4, calculate $\text{proj}_{CP_{2\pi,2}}(f)$.

B1 $f(x) = \cos^2 x, -\pi \le x \le \pi$

B2 $f(x) = -x, -\pi \le x \le \pi$

B3 $f(x) = \sin^3 x, -\pi \le x \le \pi$

B4 $f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0 \\ x & \text{if } 0 < x \le \pi \end{cases}$

CHAPTER REVIEW

Suggestions for Student Review

1. What is meant by *an orthogonal set of vectors* in $\mathbb{R}^n$? What is the difference between an orthogonal basis and an orthonormal basis? (Section 7.1)
2. Illustrate with a sketch why it is easy to write a vector $\vec{x} \in \mathbb{R}^2$ as a linear combination of an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$. (Section 7.1)
3. Define an orthogonal matrix and list the properties of an orthogonal matrix. (Section 7.1)
4. What are the essential properties of a projection onto a subspace of $\mathbb{R}^n$? How do you calculate a projection onto a subspace? What is the relationship between the formula for a projection and writing a vector as a linear combination of an orthogonal basis? (Section 7.2)
5. State the Approximation Theorem. Illustrate with a sketch in $\mathbb{R}^2$. (Section 7.2)
6. Does every subspace of $\mathbb{R}^n$ have an orthonormal basis? What about the zero subspace? How do you find an orthonormal basis? Describe the Gram-Schmidt Procedure. (Section 7.2)
7. Explain how to use the method of least squares and its relationship to the Fundamental Theorem of Linear Algebra. (Section 7.3)
8. What is an overdetermined system? How do you best approximate a solution of an inconsistent overdetermined system? What answer would our algorithm give if the overdetermined system was consistent? (Section 7.3)
9. What are the essential properties of an inner product? Give an example of an inner product on $P_2(\mathbb{R})$. Create a few different inner products for $M_{2\times 2}(\mathbb{R})$. (Section 7.4)
10. Let $\mathbb{V}$ be a vector space. If $\vec{v}, \vec{w} \in \mathbb{V}$ are orthogonal with respect to one inner product on $\mathbb{V}$, will $\vec{v}$ and $\vec{w}$ be orthogonal with respect to all inner products on $\mathbb{V}$? Give some examples. (Section 7.4)

Chapter Quiz

For Problems E1–E3, determine whether the set is orthogonal or orthonormal. Show how you decide.

E1 $\left\{\frac{1}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}0\\1\\1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right\}$

E2 $\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}0\\0\\1\\-2\end{bmatrix}\right\}$

E3 $\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\right\}$

E4 Find an orthogonal basis for the orthogonal complement of the subspace $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$ of $\mathbb{R}^3$.

E5 Let $\vec{x} = \begin{bmatrix}1\\3\\-1\\2\end{bmatrix}$. Given that $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}$ is an orthogonal basis for the subspace $\mathbb{S} = \operatorname{Span}\mathcal{B}$ of $\mathbb{R}^4$, use Theorem 7.1.2 to write $\vec{x}$ as a linear combination of the vectors in $\mathcal{B}$.

E6 Let $\vec{x} = \begin{bmatrix}1\\3\\-1\end{bmatrix}$ and let $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\2\\-3\end{bmatrix}, \begin{bmatrix}3\\3\\-4\end{bmatrix}, \begin{bmatrix}3\\3\\5\end{bmatrix}\right\}$. Calculate $\operatorname{proj}_{\mathbb{S}}(\vec{x})$ and $\operatorname{perp}_{\mathbb{S}}(\vec{x})$.

For Problems E7 and E8, determine whether the function $\langle\ ,\ \rangle$ defines an inner product on $M_{2\times 2}(\mathbb{R})$. Explain how you decide in each case.

E7 $\langle A, B\rangle = \det(AB)$

E8 $\langle A, B\rangle = a_{11}b_{11} + 2a_{12}b_{12} + 2a_{21}b_{21} + a_{22}b_{22}$

E9 Let $\mathbb{S}$ be the subspace of $\mathbb{R}^4$ defined by

$$\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\3\\1\end{bmatrix}\right\}$$

(a) Apply the Gram-Schmidt Procedure to the given spanning set to produce an orthogonal basis for $\mathbb{S}$.

(b) Determine the point in $\mathbb{S}$ closest to $\vec{x} = \begin{bmatrix}1\\-2\\-1\\1\end{bmatrix}$.

E10 Create an orthogonal basis for $\mathbb{R}^3$ that contains the vector $\vec{w}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$.

For Problems E11 and E12, find the best-fitting equation of the given form for the given set of data points.

E11 $y = a + bt$: $(-1, 2), (0, 3), (1, 5), (1, 6)$

E12 $y = a + ct^2$: $(-1, 4), (0, 1), (1, 5)$

For Problems E13 and E14, determine the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$.

E13
$$\begin{aligned} x_1 + 2x_2 &= 3\\ x_1 + x_2 &= 1\\ x_2 &= -1 \end{aligned}$$

E14
$$\begin{aligned} x_1 - 2x_2 &= 1\\ 2x_1 + x_2 &= 2\\ 3x_1 + 2x_2 &= 2 \end{aligned}$$

For Problems E15–E20, determine whether the statement is true or false. Justify your answer.

E15 If $P, R \in M_{n\times n}(\mathbb{R})$ are orthogonal, then so is PR.

E16 If $P \in M_{m\times n}(\mathbb{R})$ has orthonormal columns, then P is orthogonal.

E17 If $P \in M_{m\times n}(\mathbb{R})$ has orthonormal columns, then $P^T P = I$.

E18 If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, then for any $\vec{x} \in \mathbb{R}^n$ we have

$$\operatorname{proj}_{\mathbb{S}}(\vec{x}) = \frac{\langle \vec{x}, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\langle \vec{x}, \vec{v}_k\rangle}{\|\vec{v}_k\|^2}\vec{v}_k$$

E19 If $\mathbb{S}$ is a subspace for $\mathbb{R}^n$ and $\vec{x} \in \mathbb{S}$, then $\operatorname{perp}_{\mathbb{S}}(\vec{x}) = \vec{0}$.

E20 Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$ and let $\vec{x} \in \mathbb{R}^n$. Then, there is a unique vector $\vec{y} \in \mathbb{S}$ that has the property

$$\|\vec{x} - \vec{y}\| < \|\vec{x} - \vec{v}\|$$

for all $\vec{v} \in \mathbb{S}$, $\vec{v} \neq \vec{y}$.

E21 On $P_2(\mathbb{R})$ define the inner product $\langle \mathbf{p}, \mathbf{q}\rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$ and let $\mathbb{S} = \operatorname{Span}\{1, x - x^2\}$.

(a) Find an orthogonal basis for $\mathbb{S}$.

(b) Determine $\operatorname{proj}_{\mathbb{S}}(1 + x + x^2)$ and $\operatorname{perp}_{\mathbb{S}}(1 + x + x^2)$.

Further Problems

These exercises are intended to be challenging.

F1 Let $\mathbb{S}$ be a finite-dimensional subspace of an inner product space $\mathbb{V}$. Prove that $\text{proj}_{\mathbb{S}}$ is independent of the orthogonal basis chosen for $\mathbb{S}$. That is, for any $\mathbf{v} \in \mathbb{V}$, $\text{proj}_{\mathbb{S}}(\mathbf{v})$ will always be the same vector even if we change which orthogonal basis is being used for $\mathbb{S}$.

F2 (**Isometries of $\mathbb{R}^3$**)

(a) A linear mapping is an **isometry** of $\mathbb{R}^3$ if

$$\|L(\vec{x})\| = \|\vec{x}\|$$

for every $\vec{x} \in \mathbb{R}^3$. Prove that an isometry preserves dot products.

(b) Show that a linear mapping L is an isometry if and only if the standard matrix of L is orthogonal. (Hint: see Chapter 3 Problem F7 and Section 7.1 Problem C3.)

(c) Explain why an isometry of $\mathbb{R}^3$ must have 1 or 3 real eigenvalues, counting multiplicity. Based on Section 7.1 Problem C3 (b), the eigenvalues must be ± 1.

(d) Let A be the standard matrix of an isometry L. Suppose that 1 is an eigenvalue of A with eigenvector $\vec{u}$. Let $\vec{v}$ and $\vec{w}$ be vectors such that $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthonormal basis for $\mathbb{R}^3$ and let $P = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$. Show that

$$P^T AP = \begin{bmatrix} 1 & 0_{12} \\ 0_{21} & A^* \end{bmatrix}$$

where the right-hand side is a partitioned matrix, with 0_{ij} being the $i \times j$ zero matrix, and with A^* being a 2×2 orthogonal matrix. Moreover, show that the eigenvalues of A are 1 and the eigenvalues of A^*.

Note that an analogous form can be obtained for $\text{P}^\text{T}\text{AP}$ *in the case where one eigenvalue is -1.*

(e) Use Chapter 3 Problem F7 to analyze the A^* of part (d) and explain why every isometry of $\mathbb{R}^3$ is the identity mapping, a reflection, a composition of reflections, a rotation, or a composition of a reflection and a rotation.

F3 A linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is called an **involution** if $L \circ L = \text{Id}$. In terms of its standard matrix, this means that $A^2 = I$. Prove that any two of the following imply the third.

(a) A is the matrix of an involution.

(b) A is symmetric.

(c) A is an isometry.

F4 The sum $\mathbb{S} + \mathbb{T}$ of subspaces of a finite dimensional vector space $\mathbb{V}$ is defined in the Chapter 4 Further Problems. Prove that $(\mathbb{S} + \mathbb{T})^\perp = \mathbb{S}^\perp \cap \mathbb{T}^\perp$.

F5 Finding a sequence of approximations to some vector (or function) $\mathbf{v}$ in a possibly infinite-dimensional inner product space $\mathbb{V}$ can often be described by requiring the i-th approximation to be the closest vector $\mathbf{v}$ in some finite-dimensional subspace $\mathbb{S}_i$ of $\mathbb{V}$, where the subspaces are required to satisfy

$$\mathbb{S}_1 \subset \mathbb{S}_2 \subset \cdots \subset \mathbb{S}_i \subset \cdots \subset \mathbb{V}$$

The i-th approximation is then $\text{proj}_{\mathbb{S}_i}(\mathbf{v})$. Prove that the approximations improve as i increases in the sense that

$$\|\mathbf{v} - \text{proj}_{\mathbb{S}_{i+1}}(\mathbf{v})\| \leq \|\mathbf{v} - \text{proj}_{\mathbb{S}_i}(\mathbf{v})\|$$

F6 ***QR*-factorization.** Suppose that A is an invertible $n \times n$ matrix. Prove that A can be written as the product of an orthogonal matrix Q and an upper triangular matrix R: $A = QR$.

(Hint: apply the Gram-Schmidt Procedure to the columns of A, starting with the first column.)

Note that this QR*-factorization is important in a numerical procedure for determining eigenvalues of symmetric matrices.*

MyLab Math

Go to MyLab Math to practice many of this chapter's exercises as often as you want. The guided solutions help you find an answer step by step. You'll find a personalized study plan available to you, too!

CHAPTER 8

Symmetric Matrices and Quadratic Forms

CHAPTER OUTLINE

Symmetric matrices and quadratic forms arise naturally in many physical applications. For example, the strain matrix describing the deformation of a solid and the inertia tensor of a rotating body are symmetric (Section 8.4). We have also seen that the matrix of a projection is symmetric since a real inner product is symmetric. We now use our work with diagonalization and inner products to explore the theory of symmetric matrices and quadratic forms.

In this chapter we will make frequent use of the formula $\vec{x}^T\vec{y} = \vec{x}\cdot\vec{y}$ (see the bottom of page 160).

8.1 Diagonalization of Symmetric Matrices

Definition
Symmetric Matrix

A matrix $A \in M_{n\times n}(\mathbb{R})$ is said to be **symmetric** if $A^T = A$ or, equivalently, if $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$.

EXERCISE 8.1.1

Determine whether $A = \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix}$ and/or $B = \begin{bmatrix} 2 & -3 \\ 3 & 0 \end{bmatrix}$ is symmetric.

Symmetric matrices have the following useful property.

Theorem 8.1.1

A matrix $A \in M_{n\times n}(\mathbb{R})$ is symmetric if and only if $\vec{x}\cdot(A\vec{y}) = (A\vec{x})\cdot\vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

You are asked to prove Theorem 8.1.1 in Problem C2.

Our goal in this section is to look at the diagonalization of symmetric matrices. We begin with an example.

EXAMPLE 8.1.1

Diagonalize the symmetric matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$.

Solution: We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 1$$

Using the quadratic formula, we find that the roots of the characteristic polynomial are $\lambda_1 = -1 + \sqrt{2}$ and $\lambda_2 = -1 - \sqrt{2}$. Thus, the resulting diagonal matrix is

$$D = \begin{bmatrix} -1+\sqrt{2} & 0 \\ 0 & -1-\sqrt{2} \end{bmatrix}$$

For $\lambda_1 = -1 + \sqrt{2}$, we have

$$A - \lambda_1 I = \begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -1-\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \right\}$.

Similarly, for $\lambda_2 = -1 - \sqrt{2}$, we have

$$A - \lambda_2 I = \begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -1+\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} \right\}$.

Hence, A is diagonalized by $P = \begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} -1+\sqrt{2} & 0 \\ 0 & -1-\sqrt{2} \end{bmatrix}$.

Observe that the eigenvectors of the matrix A in Example 8.1.1 are orthogonal:

$$\begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} = (1+\sqrt{2})(1-\sqrt{2}) + 1(1) = 1 - 2 + 1 = 0$$

Using Theorem 8.1.1, we can prove that eigenvectors of a symmetric matrix corresponding to different eigenvalues are always orthogonal.

Theorem 8.1.2

If $\vec{v}_1, \vec{v}_2$ are eigenvectors of a symmetric matrix $A \in M_{n\times n}(\mathbb{R})$ corresponding to distinct eigenvalues λ_1, λ_2, then $\vec{v}_1$ is orthogonal to $\vec{v}_2$.

Proof: Assume that $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$, $\lambda_1 \neq \lambda_2$. Theorem 8.1.1 gives

$$\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = (\lambda_1\vec{v}_1) \cdot \vec{v}_2 = (A\vec{v}_1) \cdot \vec{v}_2 = \vec{v}_1 \cdot (A\vec{v}_2) = \vec{v}_1 \cdot (\lambda_2\vec{v}_2) = \lambda_2(\vec{v}_1 \cdot \vec{v}_2)$$

Hence, $(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$. But, $\lambda_1 \neq \lambda_2$, so $\vec{v}_1 \cdot \vec{v}_2 = 0$ as required. ■

We know that any non-zero scalar multiple of an eigenvector $\vec{v}$ of a matrix A corresponding to λ is also an eigenvector of A corresponding to λ. Consequently, by Theorem 8.1.2, if a symmetric matrix with all distinct eigenvalues is diagonalizable, then it can be diagonalized by an orthogonal matrix P (by normalizing the columns). To extend what we did in Chapter 6, we make the following definitions.

Definition
Orthogonally Similar

Two matrices $A, B \in M_{n\times n}(\mathbb{R})$ are said to be **orthogonally similar** if there exists an orthogonal matrix P such that

$$P^T AP = B$$

Remark

Since P is orthogonal, we have that $P^T = P^{-1}$ and hence if A and B are orthogonally similar, then they are similar. Therefore, all the properties of similar matrices still hold. In particular, if A and B are orthogonally similar, then $\operatorname{rank} A = \operatorname{rank} B$, $\operatorname{tr} A = \operatorname{tr} B$, $\det A = \det B$, and A and B have the same eigenvalues.

Definition
Orthogonally Diagonalizable

A matrix $A \in M_{n\times n}(\mathbb{R})$ is said to be **orthogonally diagonalizable** if there exists an orthogonal matrix P and diagonal matrix D such that

$$P^T AP = D$$

that is, if A is orthogonally similar to a diagonal matrix.

EXAMPLE 8.1.2

Find an orthogonal matrix that diagonalizes $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

Solution: We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 1)$$

So, the eigenvalues are $\lambda_1 = 3$, and $\lambda_2 = -1$.
For $\lambda_1 = 3$, we get

$$A - \lambda_1 I = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.
For $\lambda_2 = -1$, we get

$$A - \lambda_2 I = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

As foretold by Theorem 8.1.2, the vectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form an orthogonal set. Hence, if we normalize them, we find that A is diagonalized by the orthogonal matrix

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

EXAMPLE 8.1.3

Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$.

Solution: We have

$$C(\lambda) = \begin{vmatrix} 5-\lambda & -4 & -2 \\ -4 & 5-\lambda & -2 \\ -2 & -2 & 8-\lambda \end{vmatrix} = -\lambda(\lambda - 9)^2$$

So, the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 0$.
For $\lambda_1 = 9$, we get

$$A - \lambda_1 I = \begin{bmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ -2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$. However, observe that $\vec{w}_1$ and $\vec{w}_2$ are not orthogonal. Since we want an orthonormal basis of eigenvectors of A, we need to find an orthonormal basis for the eigenspace of λ_1. We can do this by applying the Gram-Schmidt Procedure to $\{\vec{w}_1, \vec{w}_2\}$.
Pick $\vec{v}_1 = \vec{w}_1$ and let $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$. We find that

$$\vec{v}_2 = \operatorname{perp}_{\mathbb{S}_1}(\vec{w}_2) = \vec{w}_2 - \frac{\vec{v}_1 \cdot \vec{w}_2}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

Then, $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for the eigenspace of λ_1.
For $\lambda_2 = 0$, we get

$$A - \lambda_2 I = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\{\vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$.
Normalizing $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, we find that A is diagonalized by the orthogonal matrix

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix} \quad \text{to} \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EXERCISE 8.1.2

Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

The Principal Axis Theorem

To prove that every symmetric matrix is orthogonally diagonalizable, we use the following two results.

Theorem 8.1.3 If $A \in M_{n\times n}(\mathbb{R})$ is a symmetric matrix, then all eigenvalues of A are real.

The proof of this theorem requires properties of complex numbers and hence is postponed until Chapter 9. See Theorem 9.5.1.

Theorem 8.1.4 Triangularization Theorem
If $A \in M_{n\times n}(\mathbb{R})$ has all real eigenvalues, then A is orthogonally similar to an upper triangular matrix T.

The proof is not helpful for our purposes and so is omitted.

Theorem 8.1.5 Principal Axis Theorem
A matrix $A \in M_{n\times n}(\mathbb{R})$ is symmetric if and only if it is orthogonally diagonalizable.

Proof: We will prove if A is symmetric, then A is orthogonally diagonalizable and leave the proof of the converse as Problem C3.

Assume A is symmetric. By Theorem 8.1.3 all eigenvalues of A are real. Therefore, we can apply the Triangularization Theorem to get that there exists an orthogonal matrix P such that $P^TAP = T$ is upper triangular. Since A is symmetric, we have that $A^T = A$ and hence

$$T^T = (P^TAP)^T = P^TA^T(P^T)^T = P^TAP = T$$

Therefore, T is also a symmetric matrix. But, if T is upper triangular, then T^T is lower triangular, and so T is both upper and lower triangular. Consequently, T is diagonal. Hence, we have that $P^TAP = T$ is diagonal, so A is orthogonally similar to a diagonal matrix. ■

Remarks

1. Note that Theorem 8.1.2 applies only to eigenvectors that correspond to different eigenvalues. As we saw in Example 8.1.3, eigenvectors that correspond to the same eigenvalue do not need to be orthogonal. Thus, as in Example 8.1.3, if an eigenvalue of a symmetric matrix has algebraic multiplicity greater than 1, it may be necessary to apply the Gram-Schmidt Procedure to find an orthogonal basis for its eigenspace.

2. The eigenvectors in an orthogonal matrix that diagonalizes a symmetric matrix A are called the **principal axes** for A. We will see why this definition makes sense in Section 8.3.

PROBLEMS 8.1

Practice Problems

For Problems A1–A4, decide whether the matrix is symmetric.

A1 $A = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$

A2 $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A3 $C = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & -2 & 1 \end{bmatrix}$

A4 $D = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

For Problems A5–A16, orthogonally diagonalize the matrix.

A5 $A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$

A6 $A = \begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$

A7 $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$

A8 $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

A9 $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

A10 $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ -2 & -2 & 0 \end{bmatrix}$

A11 $A = \begin{bmatrix} 1 & 8 & 4 \\ 8 & 1 & -4 \\ 4 & -4 & 7 \end{bmatrix}$

A12 $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

A13 $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

A14 $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

A15 $A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & -2 & -2 \\ -4 & -2 & 1 \end{bmatrix}$

A16 $A = \begin{bmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{bmatrix}$

Homework Problems

For Problems B1–B4, decide whether the matrix is symmetric.

B1 $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

B2 $B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

B3 $C = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

B4 $D = \begin{bmatrix} -1 & 4 & 1 \\ 4 & -2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

For Problems B5–B16, orthogonally diagonalize the matrix.

B5 $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

B6 $A = \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$

B7 $A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$

B8 $A = \begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix}$

B9 $A = \begin{bmatrix} 2 & -2 & -5 \\ -2 & -5 & -2 \\ -5 & -2 & 2 \end{bmatrix}$

B10 $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix}$

B11 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

B12 $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

B13 $A = \begin{bmatrix} 2 & -4 & -4 \\ -4 & 2 & -4 \\ -4 & -4 & 2 \end{bmatrix}$

B14 $A = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

B15 $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$

B16 $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 7 \end{bmatrix}$

Conceptual Problems

C1 Let $A, B \in M_{n\times n}(\mathbb{R})$ be symmetric. Determine which of the following is also symmetric.

(a) $A + B$ (b) $A^T A$ (c) AB (d) A^2

C2 Prove Theorem 8.1.1.

C3 Show that if A is orthogonally diagonalizable, then A is symmetric.

C4 Prove that if A is an invertible symmetric matrix, then A^{-1} is orthogonally diagonalizable.

For Problems C5 and C6, find a 2×2 symmetric matrix with the given eigenvalues and corresponding eigenvectors.

C5 $\lambda_1 = 4, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 = 6, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

C6 $\lambda_1 = -1, \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \lambda_2 = 3, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

For Problems C7–C11, determine whether the statement is true or false. Justify your answer.

C7 If the matrix P diagonalizes a symmetric matrix A, then P is orthogonal.

C8 Every orthogonal matrix is orthogonally diagonalizable.

C9 If A and B are orthogonally diagonalizable, then AB is orthogonally diagonalizable.

C10 If A is orthogonally similar to a symmetric matrix B, then A is orthogonally diagonalizable.

C11 Every eigenvalue of a symmetric matrix has its geometric multiplicity equal to its algebraic multiplicity.

8.2 Quadratic Forms

In Chapter 3, we saw the relationship between matrix mappings and linear mappings. We now explore the relationship between symmetric matrices and an important class of functions called **quadratic forms***, which are not linear. Quadratic forms appear in geometry, statistics, calculus, topology, and many other disciplines. We shall see in the next section how quadratic forms and our special theory of diagonalization of symmetric matrices can be used to graph conic sections and quadric surfaces.*

Quadratic Forms

Consider the symmetric matrix $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

$$\begin{aligned} \vec{x}^T A\vec{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2/2 \\ bx_1/2 + cx_2 \end{bmatrix} \\ &= ax_1^2 + bx_1x_2 + cx_2^2 \end{aligned}$$

We call the expression $ax_1^2 + bx_1x_2 + cx_2^2$ a quadratic form on $\mathbb{R}^2$ (or in the variables x_1 and x_2). Thus, corresponding to every symmetric matrix A, there is a quadratic form

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = ax_1^2 + bx_1x_2 + cx_2^2$$

On the other hand, given a quadratic form $Q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$, we can reconstruct the symmetric matrix $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ by choosing $(A)_{11}$ to be the coefficient of x_1^2, $(A)_{12} = (A)_{21}$ to be half of the coefficient of x_1x_2, and $(A)_{22}$ to be the coefficient of x_2^2. We deal with the coefficient of x_1x_2 in this way to ensure that A is symmetric.

EXAMPLE 8.2.1

Determine the symmetric matrix corresponding to the quadratic form

$$Q(\vec{x}) = 2x_1^2 - 4x_1x_2 - x_2^2$$

Solution: The corresponding symmetric matrix A is

$$A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$$

Notice that we could have written $ax_1^2 + bx_1x_2 + cx_2^2$ in terms of other *asymmetric* matrices. For example,

$$ax_1^2 + bx_1x_2 + cx_2^2 = \vec{x}^T \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \vec{x} = \vec{x}^T \begin{bmatrix} a & 2b \\ -b & c \end{bmatrix} \vec{x}$$

Many choices are possible. However, we agree always to choose the symmetric matrix for two reasons. First, it gives us a unique (symmetric) matrix corresponding to a given quadratic form. Second, the choice of the symmetric matrix A allows us to apply the special theory available for symmetric matrices. We now use this to extend the definition of quadratic form to n variables.

Definition
Quadratic Form

A **quadratic form** on $\mathbb{R}^n$, with corresponding $n \times n$ symmetric matrix A, is a function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q(\vec{x}) = \vec{x}^T A\vec{x}, \qquad \text{for all } \vec{x} \in \mathbb{R}^n$$

EXAMPLE 8.2.2

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -1 \end{bmatrix}$. Find the quadratic form $Q(\vec{x})$ corresponding to A.

Solution: We have

$$\begin{aligned} Q(x_1, x_2, x_3) &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1x_1 + 2x_2 - 3x_3 \\ 2x_1 - 4x_2 + 0x_3 \\ -3x_1 + 0x_2 - 1x_3 \end{bmatrix} \\ &= x_1(1x_1 + 2x_2 - 3x_3) + x_2(2x_1 - 4x_2 + 0x_3) + x_3(-3x_1 + 0x_2 - 1x_3) \\ &= x_1^2 + 4x_1x_2 - 6x_1x_3 - 4x_2^2 - x_3^2 \end{aligned}$$

Observe from the multiplication of $\vec{x}^T A\vec{x}$ in Example 8.2.2 that multiplying by $\vec{x}$ on the right of A makes the first column of A correspond to x_1, the second column of A to x_2, etc. Similarly, multiplying on the left of A by $\vec{x}^T$ makes the first row of A correspond to x_1, the second row of A to x_2, etc.

Thus, for any symmetric matrix A, the coefficient b_{ij} of $x_i x_j$ in the quadratic form $Q(\vec{x}) = \vec{x}^T A\vec{x}$ is given by

$$b_{ij} = \begin{cases} a_{ii} & \text{if } i = j \\ 2a_{ij} & \text{if } i < j \end{cases}$$

On the other hand, given a quadratic form $Q(\vec{x}) = b_{11}x_1^2 + b_{12}x_1x_2 + \cdots + b_{nn}x_n^2$ on $\mathbb{R}^n$, we can construct the corresponding symmetric matrix A by taking

$$(A)_{ij} = \begin{cases} b_{ii} & \text{if } i = j \\ \frac{1}{2}b_{ij} & \text{if } i \neq j \end{cases}$$

EXAMPLE 8.2.3

Find the corresponding symmetric matrix for each of the following quadratic forms.

(a) $Q(\vec{x}) = 3x_1^2 + 5x_1x_2 + 2x_2^2$

Solution: The corresponding symmetric matrix $A = \begin{bmatrix} 3 & 5/2 \\ 5/2 & 2 \end{bmatrix}$.

(b) $Q(\vec{x}) = x_1^2 + 4x_1x_2 + x_1x_3 + 4x_2^2 + 2x_2x_3 + 2x_3^2$

Solution: The corresponding symmetric matrix $A = \begin{bmatrix} 1 & 2 & 1/2 \\ 2 & 4 & 1 \\ 1/2 & 1 & 2 \end{bmatrix}$.

EXERCISE 8.2.1

Find the quadratic form corresponding to each of the following symmetric matrices.

(a) $\begin{bmatrix} 4 & 1/2 \\ 1/2 & \sqrt{2} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix}$

EXERCISE 8.2.2

Find the corresponding symmetric matrix for each of the following quadratic forms.

(a) $Q(\vec{x}) = x_1^2 - 2x_1x_2 - 3x_2^2$

(b) $Q(\vec{x}) = 2x_1^2 + 3x_1x_2 - x_1x_3 + 4x_2^2 + x_3^2$

(c) $Q(\vec{x}) = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2$

Observe that the symmetric matrix corresponding to $Q(\vec{x}) = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2$ is in fact diagonal. This motivates the following definition.

Definition
Diagonal Form

A quadratic form $Q(\vec{x}) = b_{11}x_1^2 + b_{12}x_1x_2 + \cdots + b_{nn}x_n^2$ is in **diagonal form** if all the coefficients b_{jk} with $j \neq k$ are equal to 0. Equivalently, $Q(\vec{x})$ is in diagonal form if its corresponding symmetric matrix is diagonal.

EXAMPLE 8.2.4

The quadratic form $Q(\vec{x}) = 3x_1^2 - 2x_2^2 + 4x_3^2$ is in diagonal form.

The quadratic form $Q(\vec{x}) = 2x_1^2 - 4x_1x_2 + 3x_2^2$ is not in diagonal form.

Since each quadratic form is defined by a symmetric matrix, we should expect that diagonalizing the symmetric matrix should also diagonalize the quadratic form. We first demonstrate this with an example and then prove the result in general.

EXAMPLE 8.2.5

Consider the quadratic form

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = 17x_1^2 + 12x_1x_2 + 8x_2^2$$

The corresponding symmetric matrix $A = \begin{bmatrix} 17 & 6 \\ 6 & 8 \end{bmatrix}$ is orthogonally diagonalized by $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$ to $D = \begin{bmatrix} 20 & 0 \\ 0 & 5 \end{bmatrix}$. Use the change of variables $\vec{x} = P\vec{y}$ to express $Q(\vec{x})$ in terms of $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Solution: We have

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = (P\vec{y})^T A(P\vec{y}) = \vec{y}^T(P^T AP)\vec{y} = \vec{y}^T \begin{bmatrix} 20 & 0 \\ 0 & 5 \end{bmatrix}\vec{y} = 20y_1^2 + 5y_2^2$$

Recall that $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ is a change of coordinates matrix from coordinates with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ to standard coordinates. So, in Example 8.2.5, we put $Q(\vec{x})$ into diagonal form by writing it with respect to the orthonormal basis $\mathcal{B} = \left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \right\}$. The vector $\vec{y}$ is just the $\mathcal{B}$-coordinates with respect to $\vec{x}$. See Example 1.2.12 on page 27 to view this geometrically. We now prove this in general.

Theorem 8.2.1 Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix and let $Q(\vec{x}) = \vec{x}^T A\vec{x}$. If P is an orthogonal matrix that diagonalizes A, then the change of variables $\vec{x} = P\vec{y}$ brings $Q(\vec{x})$ into diagonal form. In particular, we get

$$Q(\vec{x}) = \lambda_1 y_1^2 + \cdots \lambda_n y_n^2$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A corresponding to the columns of P.

Proof: Since P orthogonal diagonalizes A, we have that $P^T AP = D$ is diagonal where the diagonal entries of D are the eigenvalues $\lambda_1, \dots, \lambda_n$ of A. Hence,

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = (P\vec{y})^T A(P\vec{y}) = \vec{y}^T (P^T AP)\vec{y} = \vec{y}^T D\vec{y} = \lambda_1 y_1^2 + \cdots \lambda_n y_n^2$$

■

EXAMPLE 8.2.6 Let $Q(\vec{x}) = x_1^2 + 4x_1x_2 + x_2^2$. Find a diagonal form of $Q(\vec{x})$ and an orthogonal matrix P that brings it into this form.

Solution: The corresponding symmetric matrix is $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 1)$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.
For $\lambda_1 = 3$, we get

$$A - \lambda_1 I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector for λ_1 is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\vec{v}_1\}$.
For $\lambda_2 = -1$, we get

$$A - \lambda_2 I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector for λ_2 is $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\vec{v}_2\}$.

Therefore, we see that A is orthogonally diagonalized by $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. Thus, by Theorem 8.2.1 we get that the change of variables $\vec{x} = P\vec{y}$ brings $Q(\vec{x})$ into the form

$$Q(\vec{x}) = 3y_1^2 - y_2^2$$

EXERCISE 8.2.3

Let $Q(\vec{x}) = 4x_1x_2 - 3x_2^2$. Find a diagonal form of $Q(\vec{x})$ and an orthogonal matrix P that brings it into this form.

Classifications of Quadratic Forms

Definition
Positive Definite
Negative Definite
Indefinite
Positive Semidefinite
Negative Semidefinite

A quadratic form $Q(\vec{x})$ on $\mathbb{R}^n$ is

(1) **positive definite** if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$.
(2) **negative definite** if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$.
(3) **indefinite** if $Q(\vec{x}) > 0$ for some $\vec{x}$ and $Q(\vec{x}) < 0$ for some $\vec{x}$.
(4) **positive semidefinite** if $Q(\vec{x}) \geq 0$ for all $\vec{x}$.
(5) **negative semidefinite** if $Q(\vec{x}) \leq 0$ for all $\vec{x}$.

These concepts are useful in applications. For example, we shall see in Section 8.3 that the graph of $Q(\vec{x}) = 1$ in $\mathbb{R}^2$ is an ellipse if and only if $Q(\vec{x})$ is positive definite.

EXAMPLE 8.2.7

Classify the quadratic forms $Q_1(\vec{x}) = 3x_1^2 + 4x_2^2$, $Q_2(\vec{x}) = x_1^2 - x_2^2$, and $Q_3(\vec{x}) = -2x_1^2 - x_2^2$.

Solution: $Q_1(\vec{x})$ is positive definite since $Q_1(\vec{x}) = 3x_1^2 + 4x_2^2 > 0$ for all $\vec{x} \neq \vec{0}$.
$Q_2(\vec{x})$ is indefinite since $Q_2(1,0) = 1 > 0$ and $Q_2(0,1) = -1 < 0$.
$Q_3(\vec{x})$ is negative definite since $Q_3(\vec{x}) = -2x_1^2 - x_2^2 < 0$ for all $\vec{x} \neq \vec{0}$.

The quadratic forms in Example 8.2.7 were easy to classify since they were in diagonal form. The following theorem gives us an easy way to classify general quadratic forms.

Theorem 8.2.2

If $Q(\vec{x}) = \vec{x}^T A\vec{x}$ where $A \in M_{n\times n}(\mathbb{R})$ is a symmetric matrix, then

(1) $Q(\vec{x})$ is positive definite if and only if all eigenvalues of A are positive.
(2) $Q(\vec{x})$ is negative definite if and only if all eigenvalues of A are negative.
(3) $Q(\vec{x})$ is indefinite if and only if some of the eigenvalues of A are positive and some are negative.

Proof: We prove (1) and leave (2) and (3) as Problems C1 and C2.

By Theorem 8.2.1, there exists an orthogonal matrix P such that

$$Q(\vec{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\vec{x} = P\vec{y}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Clearly, $Q(\vec{x}) > 0$ for all $\vec{y} \neq \vec{0}$ if and only if the eigenvalues are all positive. Moreover, since P is orthogonal, it is invertible. Hence, $\vec{x} = \vec{0}$ if and only if $\vec{y} = \vec{0}$ since $\vec{x} = P\vec{y}$. Thus we have shown that $Q(\vec{x})$ is positive definite if and only if all eigenvalues of A are positive. ■

EXAMPLE 8.2.8

Classify the following quadratic forms.

$$Q_1(\vec{x}) = 4x_1^2 + 8x_1x_2 + 3x_2^2$$
$$Q_2(\vec{x}) = -2x_1^2 - 2x_1x_2 + 2x_1x_3 - 2x_2^2 + 2x_2x_3 - 2x_3^2$$

Solution: The symmetric matrix corresponding to $Q_1(\vec{x})$ is $A = \begin{bmatrix} 4 & 4 \\ 4 & 3 \end{bmatrix}$. The characteristic polynomial of A is $C(\lambda) = \lambda^2 - 7\lambda - 4$. Using the quadratic formula, we find that the eigenvalues of A are $\lambda_1 = \dfrac{7+\sqrt{65}}{2}$ and $\lambda_2 = \dfrac{7-\sqrt{65}}{2}$. Clearly $\lambda_1 > 0$. Observe that $\sqrt{65} > 7$, so $\lambda_2 < 0$. Hence, $Q_1(\vec{x})$ is indefinite.

The symmetric matrix corresponding to $Q_2(\vec{x})$ is $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$. The characteristic polynomial of A is $C(\lambda) = -(\lambda+1)^2(\lambda+4)$. Thus, the eigenvalues of A are -1, -1, and -4. Therefore, $Q_2(\vec{x})$ is negative definite.

EXERCISE 8.2.4

Classify the following quadratic forms.

(a) $Q_1(\vec{x}) = 5x_1^2 + 4x_1x_2 + 2x_2^2$

(b) $Q_2(\vec{x}) = 2x_1^2 - 6x_1x_2 - 6x_1x_3 + 3x_2^2 + 4x_2x_3 + 3x_3^2$

Since every symmetric matrix corresponds uniquely to a quadratic form, it makes sense to classify a symmetric matrix by classifying its corresponding quadratic form. That is, for example, we will say a symmetric matrix A is positive definite if and only if the quadratic form $Q(\vec{x}) = \vec{x}^T A\vec{x}$ is positive definite. Observe that this implies that we can use Theorem 8.2.2 to classify symmetric matrices as well.

EXAMPLE 8.2.9

Classify the following symmetric matrices.

(a) $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

Solution: We have $C(\lambda) = \lambda^2 - 6\lambda + 5$. Thus, the eigenvalues of A are 5 and 1, so A is positive definite.

(b) $A = \begin{bmatrix} -2 & 2 & -4 \\ 2 & -4 & -2 \\ -4 & -2 & 2 \end{bmatrix}$

Solution: We have $C(\lambda) = -(\lambda+4)(\lambda^2 - 28)$. Thus, the eigenvalues of A are -4, $2\sqrt{7}$, and $-2\sqrt{7}$, so A is indefinite.

PROBLEMS 8.2

Practice Problems

For Problems A1–A4, determine the quadratic form corresponding to the given symmetric matrix.

A1 $\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$

A2 $\begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}$

A3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 3 & -1 \end{bmatrix}$

A4 $\begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

For Problems A5–A10, classify the symmetric matrix.

A5 $\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$

A6 $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

A7 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 6 \\ 0 & 6 & 7 \end{bmatrix}$

A8 $\begin{bmatrix} -3 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$

A9 $\begin{bmatrix} 7 & 2 & -1 \\ 2 & 10 & -2 \\ -1 & -2 & 7 \end{bmatrix}$

A10 $\begin{bmatrix} -4 & -5 & 5 \\ -5 & 2 & 1 \\ 5 & 1 & 2 \end{bmatrix}$

For Problems A11–A19:

(a) Determine the symmetric matrix corresponding to $Q(\vec{x})$.

(b) Express $Q(\vec{x})$ in diagonal form and give the orthogonal matrix that brings it into this form.

(c) Classify $Q(\vec{x})$.

A11 $Q(\vec{x}) = x_1^2 - 3x_1x_2 + x_2^2$

A12 $Q(\vec{x}) = 5x_1^2 - 4x_1x_2 + 2x_2^2$

A13 $Q(\vec{x}) = -7x_1^2 + 4x_1x_2 - 4x_2^2$

A14 $Q(\vec{x}) = -2x_1^2 - 6x_1x_2 - 2x_2^2$

A15 $Q(\vec{x}) = -2x_1^2 + 12x_1x_2 + 7x_2^2$

A16 $Q(\vec{x}) = x_1^2 - 2x_1x_2 + 6x_1x_3 + x_2^2 + 6x_2x_3 - 3x_3^2$

A17 $Q(\vec{x}) = -4x_1^2 + 2x_1x_2 - 5x_2^2 - 2x_2x_3 - 4x_3^2$

A18 $Q(\vec{x}) = 3x_1^2 - 2x_1x_2 - 2x_1x_3 + 5x_2^2 + 2x_2x_3 + 3x_3^2$

A19 $Q(\vec{x}) = 3x_1^2 - 4x_1x_2 + 8x_1x_3 + 6x_2^2 + 4x_2x_3 + 3x_3^2$

Homework Problems

For Problems B1–B6, determine the quadratic form corresponding to the given symmetric matrix.

B1 $\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$

B2 $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

B3 $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

B4 $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

B5 $\begin{bmatrix} 3 & 6 & 5 \\ 6 & 0 & 2 \\ 5 & 2 & 1 \end{bmatrix}$

B6 $\begin{bmatrix} 2 & 1 & 2 \\ 1 & -3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

For Problems B7–B12, classify the symmetric matrix.

B7 $\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$

B8 $\begin{bmatrix} -6 & 6 \\ 6 & -11 \end{bmatrix}$

B9 $\begin{bmatrix} 3 & 0 & 6 \\ 0 & 2 & 0 \\ 6 & 0 & -2 \end{bmatrix}$

B10 $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

B11 $\begin{bmatrix} 1 & 4 & 4 \\ 4 & 3 & 0 \\ 4 & 0 & -1 \end{bmatrix}$

B12 $\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

For Problems B13–B24:

(a) Determine the symmetric matrix corresponding to $Q(\vec{x})$.

(b) Express $Q(\vec{x})$ in diagonal form and give the orthogonal matrix that brings it into this form.

(c) Classify $Q(\vec{x})$.

B13 $Q(\vec{x}) = x_1^2 + 8x_1x_2 + x_2^2$

B14 $Q(\vec{x}) = -2x_1^2 + 12x_1x_2 - 7x_2^2$

B15 $Q(\vec{x}) = 2x_1^2 + 4x_1x_2 + 5x_2^2$

B16 $Q(\vec{x}) = -5x_1^2 + 12x_1x_2 - 10x_2^2$

B17 $Q(\vec{x}) = -3x_1^2 + 12x_1x_2 - 8x_2^2$

B18 $Q(\vec{x}) = -2x_1^2 - 6x_1x_2 + 6x_2^2$

B19 $Q(\vec{x}) = -x_1^2 + 2x_1x_2 + 4x_1x_3 - x_2^2 + 4x_2x_3 + 2x_3^2$

B20 $Q(\vec{x}) = 4x_1^2 - 2x_1x_2 + 2x_1x_3 + 4x_2^2 - 2x_2x_3 + 4x_3^2$

B21 $Q(\vec{x}) = -6x_1^2 + 4x_1x_3 - 6x_2^2 + 8x_2x_3 - 5x_3^2$

B22 $Q(\vec{x}) = -2x_1^2 - 3x_2^2 + 4x_2x_3 - 3x_3^2$

B23 $Q(\vec{x}) = -x_1^2 + 4x_1x_2 - 2x_1x_3 + 2x_2^2 - 4x_2x_3 - x_3^2$

B24 $Q(\vec{x}) = 4x_1^2 + 4x_1x_2 - 4x_1x_3 + 3x_2^2 + 5x_3^2$

Conceptual Problems

C1 Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix. Prove that $Q(\vec{x})$ is negative definite if and only if all eigenvalues of A are negative.

C2 Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix. Prove that $Q(\vec{x})$ is indefinite if and only if some of the eigenvalues of A are positive and some are negative.

C3 Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix. Prove that $Q(\vec{x})$ is positive semidefinite if and only if all of the eigenvalues of A are non-negative.

C4 Let $A \in M_{m\times n}(\mathbb{R})$. Prove that $A^T A$ is positive semidefinite.

For Problems C5–C8, assume that A is a positive definite symmetric matrix.

C5 Prove that the diagonal entries of A are all positive.

C6 Prove that A is invertible.

C7 Prove that A^{-1} is positive definite.

C8 Prove that $P^T AP$ is positive definite for any orthogonal matrix P.

C9 A matrix B is called **skew-symmetric** if $B^T = -B$. Given a square matrix A, define the **symmetric part** of A to be

$$A^+ = \frac{1}{2}(A + A^T)$$

and the **skew-symmetric** part of A to be

$$A^- = \frac{1}{2}(A - A^T)$$

(a) Verify that A^+ is symmetric, A^- is skew-symmetric, and $A = A^+ + A^-$.
(b) Prove that the diagonal entries of A^- are 0.
(c) Determine expressions for typical entries $(A^+)_{ij}$ and $(A^-)_{ij}$ in terms of the entries of A.
(d) Prove that for every $\vec{x} \in \mathbb{R}^n$,

$$\vec{x}^T A\vec{x} = \vec{x}^T A^+\vec{x}$$

(Hint: use the fact that $A = A^+ + A^-$ and prove that $\vec{x}^T A^-\vec{x} = \vec{0}$.)

C10 In this problem, we show that general inner products on $\mathbb{R}^n$ are not different in interesting ways from the standard inner product.
Let $\langle\, ,\rangle$ be an inner product on $\mathbb{R}^n$ and let $\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$ be the standard basis.
(a) Verify that for any $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\langle \vec{x}, \vec{y}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \langle \vec{e}_i, \vec{e}_j\rangle$$

(b) Let G be the $n \times n$ matrix defined by $g_{ij} = \langle \vec{e}_i, \vec{e}_j\rangle$. Verify that

$$\langle \vec{x}, \vec{y}\rangle = \vec{x}^T G\vec{y}$$

(c) Use the properties of an inner product to verify that G is symmetric and positive definite.
(d) By adapting the proof of Theorem 8.2.1, show that there is a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ such that in $\mathcal{B}$-coordinates,

$$\langle \vec{x}, \vec{y}\rangle = \lambda_1 \tilde{x}_1 \tilde{y}_1 + \cdots + \lambda_n \tilde{x}_n \tilde{y}_n$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of G. In particular,

$$\langle \vec{x}, \vec{y}\rangle = \|\vec{x}\|^2 = \sum_{i=1}^{n} \lambda_i \tilde{x}_1{}^2$$

(e) Introduce a new basis $C = \{\vec{w}_1, \dots, \vec{w}_n\}$ by defining $\vec{w}_i = \vec{v}_i/\sqrt{\lambda_i}$. Use an asterisk to denote C-coordinates, so that $\vec{x} = x_1^*\vec{w}_1 + \cdots + x_n^*\vec{w}_n$. Verify that

$$\langle \vec{w}_i, \vec{w}_j\rangle = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

and that

$$\langle \vec{x}, \vec{y}\rangle = x_1^* y_1^* + \cdots + x_n^* y_n^*$$

Thus, with respect to the inner product $\langle\, ,\rangle$, C is an orthonormal basis, and in C-coordinates, the inner product of two vectors looks just like the standard dot product.

8.3 Graphs of Quadratic Forms

In $\mathbb{R}^2$, it is often of interest to know the graph of an equation of the form $Q(\vec{x}) = k$, where $Q(\vec{x})$ is a quadratic form on $\mathbb{R}^2$ and k is a constant. If we were interested in only one or two particular graphs, it might be sensible to simply use a computer. However, by applying diagonalization to the problem of determining these graphs, we see a very clear interpretation of eigenvectors. We also consider a concrete useful application of a change of coordinates. Moreover, this approach to these graphs leads to a classification of the various possibilities; all of the graphs of the form $Q(\vec{x}) = k$ in $\mathbb{R}^2$ can be divided into a few standard cases. Classification is a useful process because it allows us to say "I really need to understand only these few standard cases." A classification of these graphs is given later in this section.

In general it is difficult to identify the shape of the graph of

$$ax_1^2 + bx_1x_2 + cx_2^2 = k$$

It is even more difficult to try to sketch the graph. However, it is relatively easy to sketch the graph of

$$ax_1^2 + cx_2^2 = k$$

Thus, our strategy to sketch the graph of a quadratic form $Q(\vec{x}) = k$ is to first bring it into diagonal form. Of course, we first need to determine how diagonalizing the quadratic form will affect the graph.

Theorem 8.3.1 Let $Q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$ where $a, b,$ and c are not all zero and let P be an orthogonal matrix which diagonalizes $Q(\vec{x})$. If $\det P = 1$, then P corresponds to a rotation in $\mathbb{R}^2$.

Proof: Since A is symmetric, by the Principal Axis Theorem, there exists an orthonormal basis $\{\vec{v}, \vec{w}\}$ of $\mathbb{R}^2$ of eigenvectors of A. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. Since $\vec{v}$ is a unit vector, we must have

$$1 = \|\vec{v}\|^2 = v_1^2 + v_2^2$$

Hence, the entries v_1 and v_2 lie on the unit circle. Therefore, there exists an angle θ such that $v_1 = \cos\theta$ and $v_2 = \sin\theta$. Moreover, since $\vec{w}$ is a unit vector orthogonal to $\vec{v}$, we must have $\vec{w} = \pm\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. We choose $\vec{w} = +\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ so that $\det P = 1$. Hence we have

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This corresponds to a rotation by θ. Finally, from our work in Section 8.2, we know that this change of coordinates matrix brings Q into diagonal form. ■

Remark

If we picked $\vec{w} = -\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, we would find that $\det P = -1$ and that P corresponds to a rotation and a reflection.

In practice, we do not need to calculate the angle of rotation. When we orthogonally diagonalize $Q(\vec{x})$ with $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$, the change of coordinates $\vec{x} = P\vec{y}$ causes a rotation of the y_1- and y_2-axes. In particular, since the y_1-axis is spanned by the first standard basis vector $\vec{e}_1$, we get that the image of the y_1-axis in the x_1x_2-plane is spanned by

$$\vec{x} = P\vec{e}_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_1$$

Similarly, since the y_2-axis is spanned by the second standard basis vector $\vec{e}_2$, we get that the image of the y_2-axis in the x_1x_2-plane is spanned by

$$\vec{x} = P\vec{e}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{v}_2$$

We demonstrate this with two examples.

EXAMPLE 8.3.1

Sketch the graph of the equation $3x_1^2 + 4x_1x_2 = 16$.

Solution: The quadratic form $Q(\vec{x}) = 3x_1^2 + 4x_1x_2$ corresponds to the symmetric matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$. Its characteristic polynomial is

$$C(\lambda) = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

Thus, the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -1$. Thus, by an orthogonal change of coordinates, the equation can be brought into the diagonal form:

$$4y_1^2 - y_2^2 = 16$$

This is an equation of a hyperbola in the y_1y_2-plane. We observe that the y_1-intercepts are $(2, 0)$ and $(-2, 0)$, and there are no intercepts on the y_2-axis. The asymptotes of the hyperbola are determined by the equation $4y_1^2 - y_2^2 = 0$. Solving for y_2, we determine that the asymptotes are lines with equations $y_2 = 2y_1$ and $y_2 = -2y_1$. With this information, we obtain the graph in Figure 8.3.1.

Next, we need to find a basis for each eigenspace of A.

For $\lambda_1 = 4$,

$$A - \lambda_1 I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\{\vec{v}_1\}$, where $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -1$,

$$A - \lambda_2 I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\{\vec{v}_2\}$, where $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Hence, A is orthogonally diagonalized by $P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

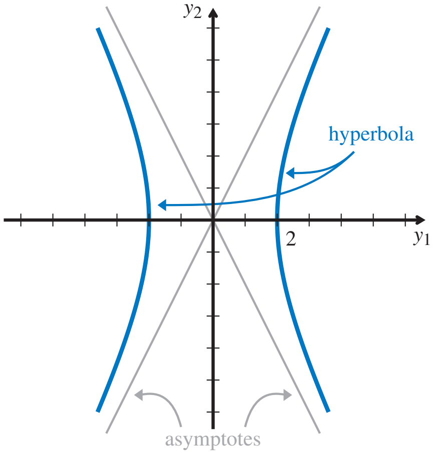


Figure 8.3.1 The graph of $4y_1^2 - y_2^2 = 16$.

EXAMPLE 8.3.1
(continued)

Now we sketch the graph of $3x_1^2 + 4x_1x_2 = 16$. In the x_1x_2-plane, we draw the y_1-axis in the direction of $\vec{v}_1$. (For clarity, in Figure 8.3.2 we have shown the vector $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ instead of $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.) We also draw the y_2-axis in the direction of $\vec{v}_2$. Then, relative to these new axes, we sketch the graph of the hyperbola $4y_1^2 - y_2^2 = 16$. The graph in Figure 8.3.2 is also the graph of the original equation $3x_1^2 + 4x_1x_2 = 16$.

In order to include the asymptotes in the sketch, we solve the change of variables $\vec{x} = P\vec{y}$ for $\vec{y}$ to get

$$\vec{y} = P^T\vec{x}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This gives

$$y_1 = \frac{1}{\sqrt{5}}(2x_1 + x_2)$$

$$y_2 = \frac{1}{\sqrt{5}}(-x_1 + 2x_2)$$

Then one asymptote is

$$y_2 = 2y_1$$

$$\frac{1}{\sqrt{5}}(-x_1 + 2x_2) = \frac{2}{\sqrt{5}}(2x_1 + x_2)$$

$$0 = x_1$$

The other asymptote is

$$y_2 = -2y_1$$

$$\frac{1}{\sqrt{5}}(-x_1 + 2x_2) = -\frac{2}{\sqrt{5}}(2x_1 + x_2)$$

$$x_2 = -\frac{3}{4}x_1$$

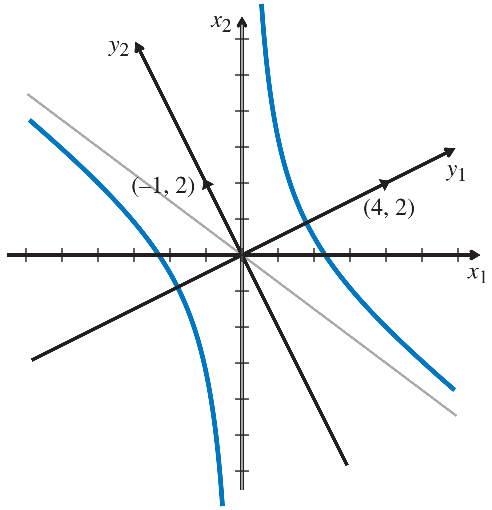


Figure 8.3.2 The graph of $3x_1^2 + 4x_1x_2 = 16$.

Remark

In Example 8.3.1, we chose $\vec{v}_2$ so that $\det P = 1$. It is a valuable exercise to see what would happen if we had chosen $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ instead.

EXAMPLE 8.3.2

Sketch the graph of the equation $6x_1^2 + 4x_1x_2 + 3x_2^2 = 14$.

Solution: The corresponding symmetric matrix is $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$. We get

$$C(\lambda) = \lambda^2 - 9\lambda + 14 = (\lambda - 2)(\lambda - 7)$$

Thus, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 7$. Hence, the equation can be brought into the diagonal form

$$2y_1^2 + 7y_2^2 = 14$$

This is the equation of an ellipse with y_1-intercepts $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$ and y_2-intercepts $(0, \sqrt{2})$ and $(0, -\sqrt{2})$. We get the ellipse in Figure 8.3.3.

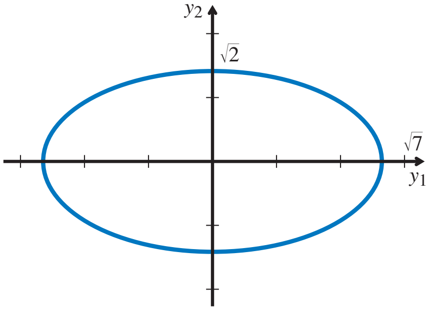


Figure 8.3.3 The graph of $2y_1^2 + 7y_2^2 = 14$.

For $\lambda_1 = 2$,

$$A - \lambda_1 I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\{\vec{v}_1\}$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

For $\lambda_2 = 7$,

$$A - \lambda_2 I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\{\vec{v}_2\}$, where $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

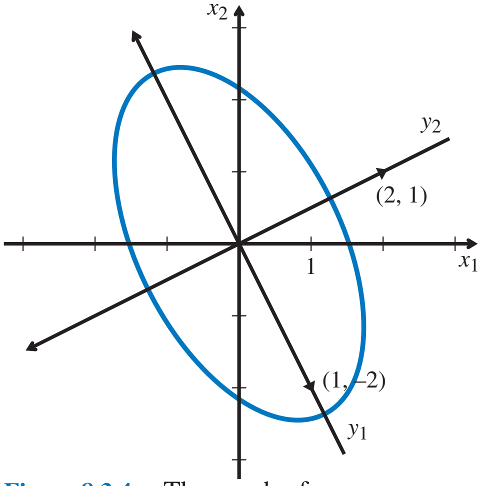


Figure 8.3.4 The graph of $6x_1^2 + 4x_1x_2 + 3x_2^2 = 14$.

In the x_1x_2-plane, we draw the y_1-axis in the direction of $\vec{v}_1$ and the y_2-axis in the direction of $\vec{v}_2$. Then, relative to these new axes, we sketch the graph of the ellipse $2y_1^2 + 7y_2^2 = 14$. This gives us the graph of $6x_1^2 + 4x_1x_2 + 3x_2^2 = 14$ in Figure 8.3.4.

Since diagonalizing a quadratic form corresponds to a rotation, to classify all the graphs of equations of the form $Q(\vec{x}) = k$, we diagonalize and rewrite the equation in the form $\lambda_1 y_1^2 + \lambda_2 y_2^2 = k$. Here, λ_1 and λ_2 are the eigenvalues of the corresponding symmetric matrix. The distinct possibilities are displayed in Table 8.3.1.

Table 8.3.1 Graphs of $\lambda_1 x_1^2 + \lambda_2 x_2^2 = k$

	$k > 0$	$k = 0$	$k < 0$
$\lambda_1 > 0, \lambda_2 > 0$	ellipse	point (0,0)	empty set
$\lambda_1 > 0, \lambda_2 = 0$	parallel lines	line $x_1 = 0$	empty set
$\lambda_1 > 0, \lambda_2 < 0$	hyperbola	intersecting lines	hyperbola
$\lambda_1 = 0, \lambda_2 < 0$	empty set	line $x_2 = 0$	parallel lines
$\lambda_1 < 0, \lambda_2 < 0$	empty set	point (0,0)	ellipse

The cases where $k = 0$ or one eigenvalue is zero may be regarded as **degenerate cases** (not general cases). The **nondegenerate cases** are the ellipses and hyperbolas, which are **conic sections**. (A conic section is a curve obtained in $\mathbb{R}^3$ as the intersection of a cone and a plane.) Notice that the cases of a single point, a single line, and intersecting lines can also be obtained as the intersection of a cone and a plane passing through the vertex of the cone. However, the cases of parallel lines (in Table 8.3.1) are not obtained as the intersection of a cone and a plane.

It is also important to realize that one class of conic sections, parabolas, does not appear in Table 8.3.1. In $\mathbb{R}^2$, the equation of a parabola is a quadratic equation, but it contains first-degree terms. Since a quadratic form contains only second-degree terms, an equation of the form $Q(\vec{x}) = k$ cannot be a parabola.

The classification provided by Table 8.3.1 suggests that it might be interesting to consider how degenerate cases arise as limiting cases of nondegenerate cases. For example, Figure 8.3.5 shows that the case of parallel lines ($y = \pm$ constant) arises from the family of ellipses $\lambda x_1^2 + x_2^2 = 1$ as λ tends to 0.

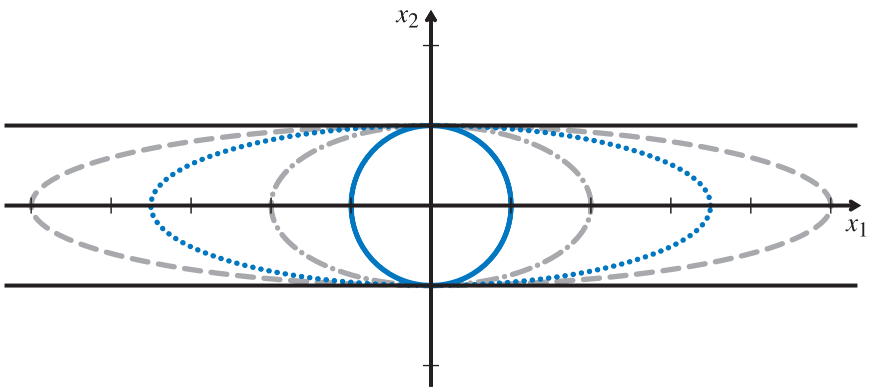


Figure 8.3.5 A family of ellipses $\lambda x_1^2 + x_2^2 = 1$. The circle occurs for $\lambda = 1$; as λ decreases, the ellipses get "fatter"; for $\lambda = 0$, the graph is a pair of lines.

Figure 8.3.6 shows that the case of intersecting lines ($k = 0$) separates the case of hyperbolas with intercepts on the x_1-axis ($x_1^2 - 2x_2^2 = k$, $k > 0$) from the case of hyperbolas with intercepts on the x_2-axis ($x_1^2 - 2x_2^2 = k$, $k < 0$).

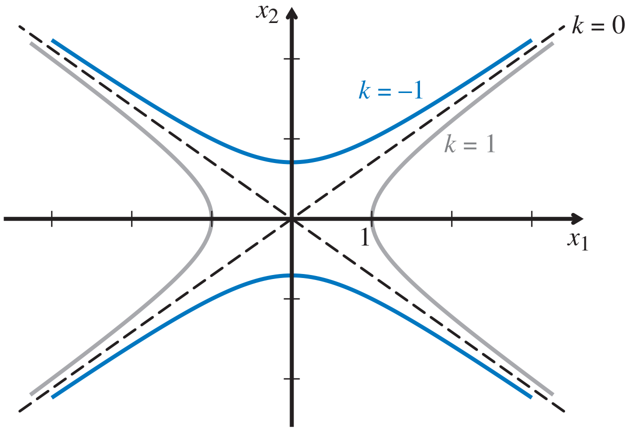


Figure 8.3.6 Graphs of $x_1^2 - 2x_2^2 = k$ for $k \in \{-1, 0, 1\}$.

EXERCISE 8.3.1

Diagonalize the quadratic form and sketch the graph of the equation $x_1^2+2x_1x_2+x_2^2 = 2$. Show both the original axes and the new axes.

Graphs of $Q(\vec{x}) = k$ in $\mathbb{R}^3$

For a quadratic equation of the form $Q(\vec{x}) = k$ in $\mathbb{R}^3$, there are similar results to what we did above. However, because there are three variables instead of two, there are more possibilities. The nondegenerate cases give ellipsoids, hyperboloids of one sheet, and hyperboloids of two sheets. These graphs are called **quadric surfaces**.

The usual standard form for the equation of an ellipsoid is $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$. This is the case obtained by diagonalizing $Q(\vec{x}) = k$ where the eigenvalues and k are all non-zero and have the same sign. In particular, it is obtained by taking

$$a^2 = k/\lambda_1, \qquad b^2 = k/\lambda_2, \qquad c^2 = k/\lambda_3$$

An ellipsoid is shown in Figure 8.3.7. The positive intercepts on the coordinate axes are $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

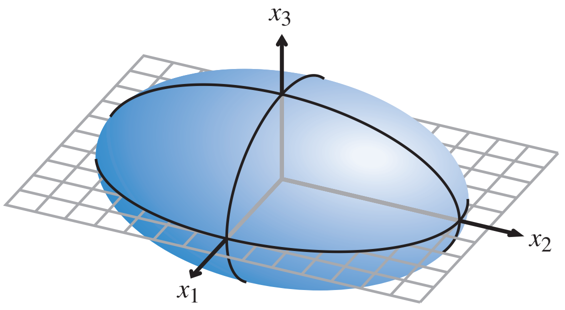


Figure 8.3.7 An ellipsoid in standard position.

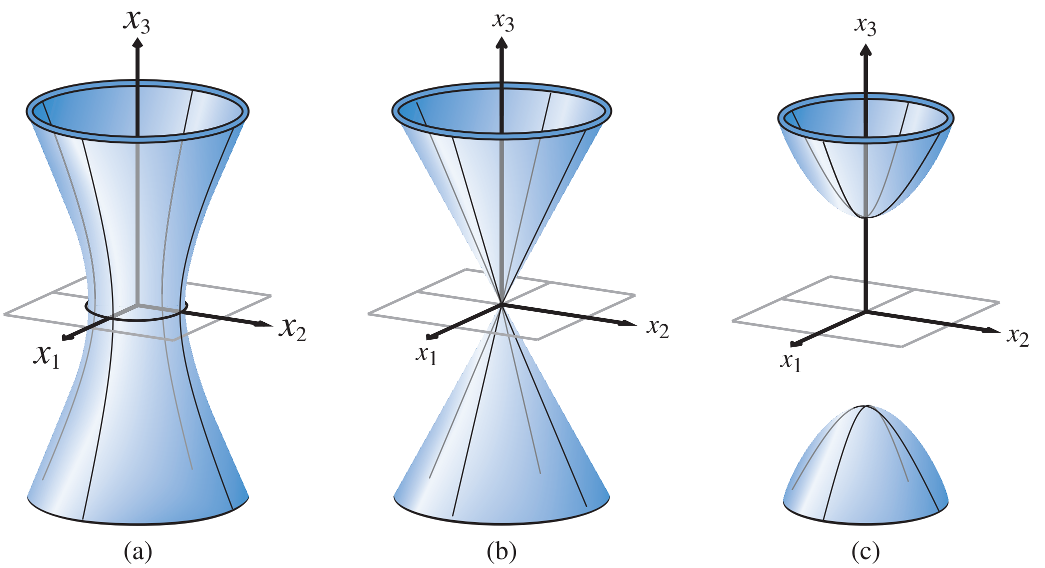


Figure 8.3.8 Graphs of $4x_1^2 + 4x_2^2 - x_3^2 = k$. (a) $k = 1$; a hyperboloid of one sheet. (b) $k = 0$; a cone. (c) $k = -1$; a hyperboloid of two sheets.

The standard form of the equation for a hyperboloid of one sheet is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1$$

This form is obtained when k and two eigenvalues of the matrix of Q are positive and the third eigenvalue is negative. It is also obtained when k and two eigenvalues are negative and the other eigenvalue is positive. If the equation is rewritten as

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 + \frac{x_3^2}{c^2}$$

then it is clear that for every x_3 there are values of x_1 and x_2 that satisfy the equation, so that the surface is all one piece (or one sheet). A hyperboloid of one sheet is shown in Figure 8.3.8 (a).

The standard form of the equation for a hyperboloid of two sheets is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = -1$$

This form is obtained when k and one eigenvalue is negative and the other eigenvalues are positive, or when k and one eigenvalue are positive and the other eigenvalues are negative. Notice that if this is rewritten as

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = -1 + \frac{x_3^2}{c^2}$$

it is clear that for every $|x_3| < c$, there are no values of x_1 and x_2 that satisfy the equation. Therefore, the graph consists of two pieces (or two sheets), one with $x_3 \geq c$ and the other with $x_3 \leq -c$. A hyperboloid of two sheets is shown in Figure 8.3.8 (c).

It is interesting to consider the family of surfaces obtained by varying k in the equation

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = k$$

as in Figure 8.3.8. When $k = 1$, the surface is a hyperboloid of one sheet; as k decreases towards 0, the "waist" of the hyperboloid shrinks until at $k = 0$ it has "pinched in" to a single point and the hyperboloid of one sheet becomes a cone. As k decreases towards -1, the waist has disappeared, and the graph is now a hyperboloid of two sheets.

Table 8.3.2 Graphs of $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = k$

	$k > 0$	$k = 0$
$\lambda_1, \lambda_2, \lambda_3 > 0$	ellipsoid	point $(0, 0, 0)$
$\lambda_1, \lambda_2 > 0, \lambda_3 = 0$	elliptic cylinder	x_3-axis
$\lambda_1, \lambda_2 > 0, \lambda_3 < 0$	hyperboloid of one sheet	cone
$\lambda_1 > 0, \lambda_2 = 0, \lambda_3 = 0$	parallel planes	x_2x_3-plane
$\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$	hyperbolic cylinder	intersecting planes
$\lambda_1 > 0, \lambda_2, \lambda_3 < 0$	hyperboloid of two sheets	cone
$\lambda_1 = 0, \lambda_2, \lambda_3 < 0$	empty set	x_1-axis
$\lambda_1 < 0, \lambda_2 = 0, \lambda_3 = 0$	empty set	x_2x_3-plane
$\lambda_1, \lambda_2, \lambda_3 < 0$	empty set	point $(0, 0, 0)$

Table 8.3.2 displays the possible cases for $Q(\vec{x}) = k$ in $\mathbb{R}^3$. The nondegenerate cases are the ellipsoids and hyperboloids. Note that the hyperboloid of two sheets appears in the form $\dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = k, k > 0$.

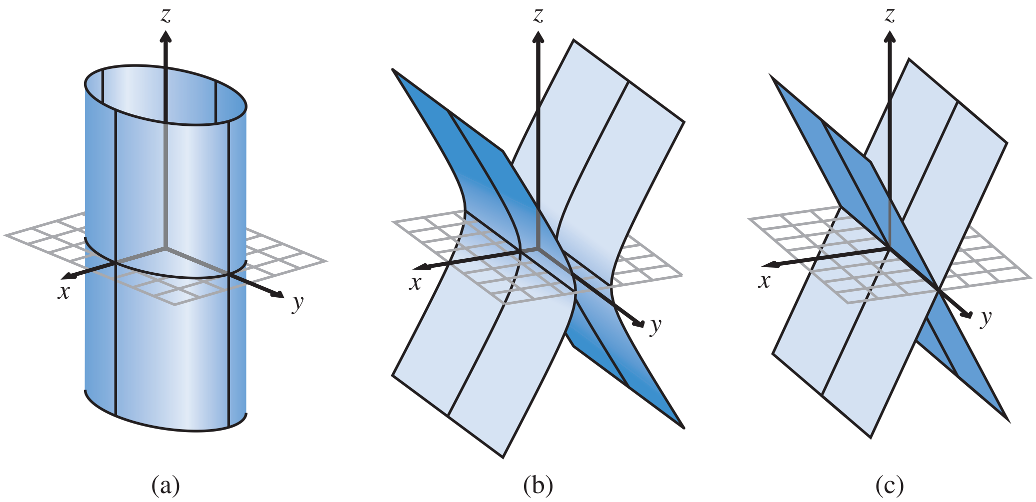


Figure 8.3.9 Some degenerate quadric surfaces. (a) an elliptic cylinder $x_1^2 + \lambda x_2^2 = 1$, parallel to the x_3-axis. (b) a hyperbolic cylinder $\lambda x_1^2 - x_3^2 = 1$, parallel to the x_2-axis. (c) intersecting planes $\lambda x_1^2 - x_3^2 = 0$.

Figure 8.3.9 shows some degenerate quadric surfaces. Note that paraboloidal surfaces do not appear as graphs of the form $Q(\vec{x}) = k$ in $\mathbb{R}^3$ for the same reason that parabolas do not appear in Table 8.3.1 for $\mathbb{R}^2$: their equations contain first-degree terms.

PROBLEMS 8.3

Practice Problems

A1 Sketch the graph of $2x_1^2 + 4x_1x_2 - x_2^2 = 6$. Show both the original axes and the new axes.

A2 Sketch the graph of $2x_1^2 + 6x_1x_2 + 10x_2^2 = 11$. Show both the original axes and the new axes.

A3 Sketch the graph of $4x_1^2 - 6x_1x_2 + 4x_2^2 = 12$. Show both the original axes and the new axes.

A4 Sketch the graph of $5x_1^2 + 6x_1x_2 - 3x_2^2 = 15$. Show both the original axes and the new axes.

A5 Sketch the graph of $x_1^2 - 4x_1x_2 + x_2^2 = 8$. Show both the original axes and the new axes.

A6 Sketch the graph of $x_1^2 + 4x_1x_2 + x_2^2 = 8$. Show both the original axes and the new axes.

A7 Sketch the graph of $3x_1^2 - 4x_1x_2 + 3x_2^2 = 32$. Show both the original axes and the new axes.

For Problems A8–A13, identify the shape of the graph of $\vec{x}^T A\vec{x} = 1$ and the shape of the graph of $\vec{x}^T A\vec{x} = -1$.

A8 $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

A9 $A = \begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$

A10 $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

A11 $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ -2 & -2 & 0 \end{bmatrix}$

A12 $A = \begin{bmatrix} 1 & 8 & 4 \\ 8 & 1 & -4 \\ 4 & -4 & 7 \end{bmatrix}$

A13 $A = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$

For Problems A14–A17, express $Q(\vec{x})$ in diagonal form and use the diagonal form to determine the shape of the surface $Q(\vec{x}) = k$ for $k = 1, 0, -1$.

A14 $Q(\vec{x}) = -x_1^2 + 12x_1x_2 + 4x_2^2$

A15 $Q(\vec{x}) = x_1^2 + 2x_1x_2 - 4x_1x_3 + x_2^2 - 4x_2x_3 + 4x_3^2$

A16 $Q(\vec{x}) = 2x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_2x_3 + 4x_3^2$

A17 $Q(\vec{x}) = 4x_1^2 + 2x_1x_2 + 5x_2^2 - 2x_2x_3 + 4x_3^2$

Homework Problems

B1 Sketch the graph of $4x_1^2 - 24x_1x_2 + 11x_2^2 = 20$. Show both the original axes and the new axes.

B2 Sketch the graph of $2x_1^2 + 12x_1x_2 + 7x_2^2 = 11$. Show both the original axes and the new axes.

B3 Sketch the graph of $3x_1^2 - 4x_1x_2 + 6x_2^2 = 14$. Show both the original axes and the new axes.

B4 Sketch the graph of $12x_1^2 + 10x_1x_2 - 12x_2^2 = 13$. Show both the original axes and the new axes.

B5 Sketch the graph of $9x_1^2 + 4x_1x_2 + 6x_2^2 = 90$. Show both the original axes and the new axes.

B6 Sketch the graph of $x_1^2 + 6x_1x_2 - 7x_2^2 = 32$. Show both the original axes and the new axes.

For Problems B7–B14, identify the shape of the graph of $\vec{x}^T A\vec{x} = 1$ and the shape of the graph of $\vec{x}^T A\vec{x} = -1$.

B7 $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

B8 $A = \begin{bmatrix} -2 & -3 \\ -3 & -10 \end{bmatrix}$

B9 $A = \begin{bmatrix} 26 & 10 \\ 10 & 5 \end{bmatrix}$

B10 $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

B11 $A = \begin{bmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & -5 \end{bmatrix}$

B12 $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & 2 \\ -1 & 2 & 3 \end{bmatrix}$

B13 $A = \begin{bmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

B14 $A = \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix}$

For Problems B15–B21, express $Q(\vec{x})$ in diagonal form and use the diagonal form to determine the shape of the surface $Q(\vec{x}) = k$ for $k = 1, 0, -1$.

B15 $Q(\vec{x}) = 5x_1^2 + 4x_1x_2 + 2x_2^2$

B16 $Q(\vec{x}) = x_1^2 - 4x_1x_2 + x_2^2$

B17 $Q(\vec{x}) = -3x_1^2 + 8x_1x_2 - 9x_2^2$

B18 $Q(\vec{x}) = x_1^2 + 2x_1x_2 + x_2^2$

B19 $Q(\vec{x}) = x_1^2 + 6x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_3 + 5x_3^2$

B20 $Q(\vec{x}) = x_1^2 + 4x_1x_2 + 4x_1x_3 + 5x_2^2 + 6x_2x_3 + 5x_3^2$

B21 $Q(\vec{x}) = -x_1^2 + 2x_1x_2 - 6x_1x_3 + x_2^2 - 2x_2x_3 - x_3^2$

8.4 Applications of Quadratic Forms

Some may think of mathematics as only a set of rules for doing calculations. However, a theorem such as the Principal Axis Theorem is often important because it provides a simple way of thinking about complicated situations. The Principal Axis Theorem plays an important role in the two applications described here.

Small Deformations

A small deformation of a solid body may be understood as the composition of three stretches along the principal axes of a symmetric matrix together with a rigid rotation of the body.

Consider a body of material that can be deformed when it is subjected to some external forces. This might be, for example, a piece of steel under some load. Fix an origin of coordinates $\vec{0}$ in the body; to simplify the story, suppose that this origin is left unchanged by the deformation. Suppose that a material point in the body, which is at $\vec{x}$ before the forces are applied, is moved by the forces to the point

$$f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))$$

where we have assumed that $f(\vec{0}) = \vec{0}$. The problem is to understand this deformation f so that it can be related to the properties of the body.

For many materials under reasonable forces, the deformation is small; this means that the point $f(\vec{x})$ is not far from $\vec{x}$. It is convenient to introduce a parameter β to describe how small the deformation is. To do this, we define a function $h(\vec{x})$ by

$$f(\vec{x}) = \vec{x} + \beta h(\vec{x})$$

For many materials, an arbitrary small deformation is well approximated by its "best linear approximation," the derivative. In this case, the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is approximated near the origin by the linear transformation with matrix $\left[\frac{\partial f_j}{\partial x_k}(\vec{0})\right]$, so that in this approximation, a point originally at $\vec{v}$ is moved (approximately) to $\left[\frac{\partial f_j}{\partial x_k}(\vec{0})\right]\vec{v}$. (This is a standard calculus approximation.)

In terms of the parameter β and the function h, this matrix can be written as

$$\left[\frac{\partial f_j}{\partial x_k}(\vec{0})\right] = I + \beta G$$

where $G = \left[\frac{\partial h_j}{\partial x_k}(\vec{0})\right]$. In this situation, it is useful to write G as $G = E + W$, where

$$E = \frac{1}{2}(G + G^T)$$

is its symmetric part, and

$$W = \frac{1}{2}(G - G^T)$$

is its skew-symmetric part, as in Section 8.2 Problem C9.

The next step is to observe that we can write

$$I + \beta G = I + \beta(E + W) = (I + \beta E)(I + \beta W) - \beta^2 EW$$

Since β is assumed to be small, β^2 is very small and may be ignored. (Such treatment of terms like β^2 can be justified by careful discussion of the limit at $\beta \to 0$.)

The small deformation we started with is now described as the composition of two linear transformations, one with matrix $I+\beta E$ and the other with matrix $I+\beta W$. It can be shown that $I+\beta W$ describes a small rigid rotation of the body; a rigid rotation does not alter the distance between any two points in the body. (The matrix βW is called an *infinitesimal rotation*.)

Finally, we have the linear transformation with matrix $I+\beta E$. This matrix is symmetric, so there exist principal axes such that the symmetric matrix is diagonalized to $\begin{bmatrix} 1+\epsilon_1 & 0 & 0 \\ 0 & 1+\epsilon_2 & 0 \\ 0 & 0 & 1+\epsilon_3 \end{bmatrix}$. (It is equivalent to diagonalize βE and add the result to I, because I is transformed to itself under any orthonormal change of coordinates.) Since β is small, it follows that the numbers ϵ_j are small in magnitude, and therefore $1+\epsilon_j > 0$. This diagonalized matrix can be written as the product of the three matrices:

$$\begin{bmatrix} 1+\epsilon_1 & 0 & 0 \\ 0 & 1+\epsilon_2 & 0 \\ 0 & 0 & 1+\epsilon_3 \end{bmatrix} = \begin{bmatrix} 1+\epsilon_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\epsilon_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\epsilon_3 \end{bmatrix}$$

It is now apparent that, excluding rotation, the small deformation can be represented as the composition of three stretches along the principal axes of the matrix βE. The quantities $\epsilon_1, \epsilon_2,$ and ϵ_3 are related to the external and internal forces in the material by elastic properties of the material. (βE is called the infinitesimal strain; this notation is not quite the standard notation. This will be important if you read further about this topic in a book on continuum mechanics.)

The Inertia Tensor

For the purpose of discussing the rotation motion of a rigid body, information about the mass distribution within the body is summarized in a symmetric matrix N called the **inertia tensor**. The tensor is easiest to understand if principal axes are used so that the matrix is diagonal; in this case, the diagonal entries are simply the moments of inertia about the principal axes, and the moment of inertia about any other axis can be calculated in terms of these **principal moments of inertia**. In general, the **angular momentum** vector $\vec{J}$ of the rotating body is equal to $N\vec{\omega}$, where $\vec{\omega}$ is the **instantaneous angular velocity** vector. The vector $\vec{J}$ is a scalar multiple of $\vec{\omega}$ if and only if $\vec{\omega}$ is an eigenvector of N—that is, if and only if the axis of rotation is one of the principal axes of the body. This is a beginning to an explanation of how the body wobbles during rotation ($\vec{\omega}$ need not be constant) even though $\vec{J}$ is a conserved quantity. That is, $\vec{J}$ is constant if no external force is applied.

Suppose that a rigid body is rotating about some point in the body that remains fixed in space throughout the rotation. Make this fixed point the origin $(0,0,0)$. Suppose that there are coordinate axes fixed in space and also three reference axes that are fixed in the body (so that they rotate with the body). At any time t, these body axes make certain angles with respect to the space axes; at a later time $t+\Delta t$, the body axes have moved to a new position. Since $(0,0,0)$ is fixed and the body is rigid, the body axes have moved only by a rotation, and it is a fact that any rotation in $\mathbb{R}^3$ is determined by its axis and an angle. Call the unit vector along this axis $\vec{u}(t+\Delta t)$ and denote the angle by $\Delta\theta$. Now let $\Delta t \to 0$; the unit vector $\vec{u}(t+\Delta t)$ must tend to a limit $\vec{u}(t)$, and this determines the **instantaneous axis of rotation at time** t. Also, as $\Delta t \to 0$, $\frac{\Delta\theta}{\Delta t} \to \frac{d\theta}{dt}$, the **instantaneous rate of rotation about the axis**. The **instantaneous angular velocity** is defined to be the vector $\vec{\omega} = \left(\frac{d\theta}{dt}\right)\vec{u}(t)$.

(It is a standard exercise to show that the instantaneous linear velocity $\vec{v}(t)$ at some point in the body whose space coordinates are given by $\vec{x}(t)$ is determined by $\vec{v} = \vec{\omega} \times \vec{x}$.)

To use concepts such as energy and momentum in the discussion of rotating motion, it is necessary to introduce moments of inertia.

For a single mass m at the point (x_1, x_2, x_3) the **moment of inertia about the x_3-axis** is defined to be $m(x_1^2 + x_2^2)$; this will be denoted by n_{33}. The factor $(x_1^2 + x_2^2)$ is simply the square of the distance of the mass from the x_3-axis. There are similar definitions of the moments of inertia about the x_1-axis (denoted by n_{11}) and about the x_2-axis (denoted by n_{22}).

For a general axis ℓ through the origin with unit direction vector $\vec{u}$, the moment of inertia of the mass about ℓ is defined to be m multiplied by the square of the distance of m from ℓ. Thus, if we let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the moment of inertia in this case is

$$m\|\operatorname{perp}_{\vec{u}} \vec{x}\|^2 = m[\vec{x} - (\vec{x} \cdot \vec{u})\vec{u}]^T[\vec{x} - (\vec{x} \cdot \vec{u})\vec{u}] = m(\|\vec{x}\|^2 - (\vec{x} \cdot \vec{u})^2)$$

With some manipulation, using $\vec{u}^T\vec{u} = 1$ and $\vec{x} \cdot \vec{u} = \vec{x}^T\vec{u}$, we can verify that this is equal to the expression

$$\vec{u}^T m(\|\vec{x}\|^2 I - \vec{x}\vec{x}^T)\vec{u}$$

Because of this, for the single point mass m at $\vec{x}$, we define the **inertia tensor** N to be the 3×3 matrix

$$N = m(\|\vec{x}\|^2 I - \vec{x}\vec{x}^T)$$

(Vectors and matrices are special kinds of "tensors"; for our present purposes, we simply treat N as a matrix.) With this definition, the moment of inertia about an axis with unit direction $\vec{u}$ is

$$\vec{u}^T N \vec{u}$$

It is easy to check that N is the matrix with components n_{11}, n_{22}, and n_{33} as given above, and for $i \neq j$, $n_{ij} = -mx_ix_j$. It is clear that this matrix N is symmetric because $\vec{x}\vec{x}^T$ is a symmetric 3×3 matrix. (The term mx_ix_j is called a *product of inertia*. This name has no special meaning; the term is simply a product that appears as an entry in the inertia tensor.)

It is easy to extend the definition of moments of inertia and the inertia tensor to bodies that are more complicated than a single point mass. Consider a rigid body that can be thought of as k masses joined to each other by weightless rigid rods. The moment of inertia of the body about the x_3-axis is determined by taking the moment of inertia about the x_3-axis of each mass and simply adding these moments; the moments about the x_1- and x_2-axes, and the products of the inertia are defined similarly. The inertia tensor of this body is just the sum of the inertia tensors of the k masses; since it is the sum of symmetric matrices, it is also symmetric. If the mass is distributed continuously, the various moments and products of inertia are determined by definite integrals. In any case, the inertia tensor N is still defined, and is still a symmetric matrix.

Since N is a symmetric matrix, it can be brought into diagonal form by the Principal Axis Theorem. The diagonal entries are then the moments of inertia with respect to the principal axes, and these are called the **principal moments of inertia**. Denote these by N_1, N_2, and N_3. Let $\mathcal{P}$ denote the orthonormal basis consisting of eigenvectors of N (which means these vectors are unit vectors along the principal axes).

Suppose an arbitrary axis ℓ is determined by the unit vector $\vec{u}$ such that $[\vec{u}]_{\mathcal{P}} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$.

Then, from the discussion of quadratic forms in Section 8.2, the moment of inertia about this axis ℓ is simply

$$\begin{aligned} \vec{u}^T N \vec{u} &= \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \\ &= p_1^2 N_1 + p_2^2 N_2 + p_3^2 N_3 \end{aligned}$$

This formula is greatly simplified because of the use of the principal axes.

It is important to get equations for rotating motion that corresponds to Newton's equation:

The rate of change of momentum equals the applied force.

The appropriate equation is

The rate of change of angular momentum equals the applied torque.

It turns out that the right way to define the angular momentum vector $\vec{J}$ for a general body is

$$\vec{J} = N(t)\vec{\omega}(t)$$

Note that in general N is a function of time since it depends on the positions at time t of each of the masses making up the solid body. Understanding the possible motions of a rotating body depends on determining $\vec{\omega}(t)$, or at least saying something about it. In general, this is a very difficult problem, but there will often be important simplifications if N is diagonalized by the Principal Axis Theorem. Note that $\vec{J}(t)$ is parallel to $\vec{\omega}(t)$ if and only if $\vec{\omega}(t)$ is an eigenvector of $N(t)$.

PROBLEM 8.4

Conceptual Problem

C1 Show that if P is an orthogonal matrix that diagonalizes the symmetric matrix βE to a matrix with diagonal entries ϵ_1, ϵ_2, and ϵ_3, then P also diagonalizes $(I + \beta E)$ to a matrix with diagonal entries $1 + \epsilon_1$, $1 + \epsilon_2$, and $1 + \epsilon_3$.

8.5 Singular Value Decomposition

The theory of orthogonal diagonalization of symmetric matrices is extremely useful and powerful, but it has one drawback; it applies only to symmetric matrices. In many real world situations, matrices are not only not symmetric nor diagonalizable, they are not even square. So, the natural question to ask is if we can mimic orthogonal diagonalization for non-square matrices.

Let $A \in M_{m\times n}(\mathbb{R})$. To truly mimic orthogonal diagonalization we would need to find an orthogonal matrix P such that

$$P^T AP = D$$

is an $m \times n$ **rectangular diagonal matrix**; that is, a matrix where all entries are 0 except possibly entries along the main diagonal.

However, we know that this is not possible since the sizes do not work out if $n \neq m$. In particular, we require that P^T is an $m \times m$ matrix for the matrix-matrix multiplication $P^T A$ to be defined, and we need P to be an $n \times n$ matrix for AP to be defined. Thus, rather than looking for a single orthogonal matrix P, we will try to find an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$U^T AV = \Sigma$$

is an $m \times n$ rectangular diagonal matrix. Rather than emphasizing the matrix Σ, we typically solve this equation for A to get a matrix decomposition

$$A = U\Sigma V^T$$

The difficult question is how to find such a matrix decomposition. If we are going to mimic orthogonal diagonalization, it would be helpful to relate A to a symmetric matrix. The key is to recognize that both $A^T A$ and AA^T are symmetric matrices. Indeed

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

and

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

Thus, if such a matrix decomposition exists, we find that

$$\begin{aligned} A^T A &= (U\Sigma V^T)^T (U\Sigma V^T) \\ &= V\Sigma^T U^T U\Sigma V^T && \text{by properties of transposes} \\ &= V\Sigma^T I\Sigma V^T && \text{since } U \text{ is orthogonal} \\ A^T A &= V\Sigma^T \Sigma V^T \end{aligned} \tag{8.1}$$

We observe that $\Sigma^T \Sigma$ will be a diagonal matrix, and so we can rewrite equation (8.1) as

$$V^T (A^T A)V = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$$

where $\sigma_i = (\Sigma)_{ii}$.

Observe that this is an orthogonal diagonalization of $A^T A$. In particular, it shows us that we want the diagonal entries of Σ to be the square roots of the eigenvalues of $A^T A$.

Definition
Singular Values

Let $A \in M_{m\times n}(\mathbb{R})$. The **singular values** $\sigma_1, \ldots, \sigma_n$ of A are the square roots of the eigenvalues of A^TA arranged so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Remark

We use the convention that the singular values are always ordered from greatest to least, as it is helpful in a variety of applications to know the position of the largest singular values.

EXAMPLE 8.5.1

Find the singular values of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$.

Solution: We have $A^TA = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. The characteristic polynomial of A^TA is $C(\lambda) = -\lambda^2(\lambda - 15)$. Thus, the eigenvalues (ordered from greatest to least) are $\lambda_1 = 15$, $\lambda_2 = 0$, and $\lambda_3 = 0$. So, the singular values of A are $\sigma_1 = \sqrt{15}$, $\sigma_2 = 0$, and $\sigma_3 = 0$.

EXAMPLE 8.5.2

Find the singular values of $B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution: We have $B^TB = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$. The characteristic polynomial of B^TB is $C(\lambda) = (\lambda - 2)(\lambda - 7)$. Hence, the eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = 2$. Therefore, the singular values of B are $\sigma_1 = \sqrt{7}$ and $\sigma_2 = \sqrt{2}$.

EXAMPLE 8.5.3

Find the singular values of $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution: We have $A^TA = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$. Hence, the eigenvalues of A^TA are $\lambda_1 = 6$ and $\lambda_2 = 5$. Thus, the singular values of A are $\sigma_1 = \sqrt{6}$ and $\sigma_2 = \sqrt{5}$.

We have $B^TB = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$. The characteristic polynomial of B^TB is

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ 0 & 1 & 5-\lambda \end{vmatrix} = -\lambda(\lambda - 5)(\lambda - 6)$$

Hence, the eigenvalues of B^TB are $\lambda_1 = 6$, $\lambda_2 = 5$, and $\lambda_3 = 0$. Thus, the singular values of B are $\sigma_1 = \sqrt{6}$, $\sigma_2 = \sqrt{5}$, and $\sigma_3 = 0$.

You are asked to prove that a matrix and its transpose have the same non-zero singular values in Problem C5.

EXERCISE 8.5.1

Find the singular values of $A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$.

Theorem 8.5.1

If $A \in M_{m\times n}(\mathbb{R})$ and $\text{rank}(A) = r$, then A has r non-zero singular values.

You are guided through the proof of Theorem 8.5.1 in Problem C1.

We can now define our desired decomposition.

Definition

Singular Value Decomposition (SVD)

Left Singular Vectors

Right Singular Vectors

Let $A \in M_{m\times n}(\mathbb{R})$ with $\text{rank}(A) = r$ and non-zero singular values $\sigma_1, \dots, \sigma_r$. If U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and Σ is an $m \times n$ rectangular diagonal matrix with $(\Sigma)_{ii} = \sigma_i$ for $1 \leq i \leq r$ and all other entries 0, then

$$A = U\Sigma V^T$$

is called a **singular value decomposition (SVD)** of A. The orthonormal columns of U are called **left singular vectors** of A. The orthonormal columns of V are called **right singular vectors** of A.

Equation (8.1) not only indicates how to define the singular values of a matrix A, but also shows us that the desired orthogonal matrix V for a singular value decomposition of A is a matrix which orthogonally diagonalizes A^TA (ensuring that we order the columns of V so that the corresponding eigenvalues are ordered from greatest to least).

We now need to define the columns of U so that we will indeed have $A = U\Sigma V^T$. Let $V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ and $U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$. Rewriting $A = U\Sigma V^T$ as $AV = U\Sigma$ we find that

$$\begin{aligned} A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} &= \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}\begin{bmatrix} \sigma_1\vec{e}_1 & \cdots & \sigma_r\vec{e}_r & \vec{0} & \cdots & \vec{0} \end{bmatrix} \\ \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} &= \begin{bmatrix} \sigma_1\vec{u}_1 & \cdots & \sigma_r\vec{u}_r & \vec{0} & \cdots & \vec{0} \end{bmatrix} \end{aligned}$$

Hence, we require that

$$A\vec{v}_i = \sigma_i\vec{u}_i \Rightarrow \vec{u}_i = \frac{1}{\sigma_i}A\vec{v}_i, \quad 1 \leq i \leq r$$

We now prove that the vectors $\{\vec{u}_1, \dots, \vec{u}_r\}$ indeed form an orthonormal basis.

Theorem 8.5.2

Let $A \in M_{m\times n}(\mathbb{R})$ with $\text{rank}(A) = r$ and non-zero singular values $\sigma_1, \dots, \sigma_r$. Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of the eigenvectors of A^TA arranged so that the corresponding eigenvalues are ordered from greatest to least. If we define

$$\vec{u}_i = \frac{1}{\sigma_i}A\vec{v}_i, \quad 1 \leq i \leq r$$

then $\{\vec{u}_1, \dots, \vec{u}_r\}$ is an orthonormal basis for Col A.

Proof: Observe that for $1 \leq i, j \leq r$, we have

$$\begin{aligned}\vec{u}_i \cdot \vec{u}_j &= \frac{1}{\sigma_i}(A\vec{v}_i) \cdot \frac{1}{\sigma_j}\left(A\vec{v}_j\right) \\ &= \frac{1}{\sigma_i\sigma_j}(A\vec{v}_i)^T(A\vec{v}_j) \\ &= \frac{1}{\sigma_i\sigma_j}\vec{v}_i^T A^T A\vec{v}_j\end{aligned}$$

We have assumed that $\vec{v}_j$ is an eigenvector of A^TA corresponding to the eigenvalue $\lambda_j = \sigma_j^2$. Thus, we have

$$\begin{aligned}\vec{u}_i \cdot \vec{u}_j &= \frac{1}{\sigma_i\sigma_j}\vec{v}_i^T(\sigma_j^2\vec{v}_j) \\ &= \frac{\sigma_j}{\sigma_i}\vec{v}_i^T(\vec{v}_j) \\ &= \frac{\sigma_j}{\sigma_i}\vec{v}_i \cdot \vec{v}_j = \begin{cases}0 & \text{if } i \neq j \\ 1 & \text{if } i = j\end{cases}\end{aligned}$$

because $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal set. By definition, $\vec{u}_i \in \operatorname{Col}(A)$ and, since $\dim \operatorname{Col}(A) = r$, Theorem 4.3.7 implies that this is an orthonormal basis for $\operatorname{Col} A$. ■

We now have an algorithm for finding the singular value decomposition of a matrix.

Algorithm 8.5.1

To find a singular value decomposition of $A \in M_{m\times n}(\mathbb{R})$ with $\operatorname{rank}(A) = r$:

(1) Find the eigenvalues $\lambda_1, \dots, \lambda_n$ of A^TA arranged from greatest to least and a corresponding orthonormal set of eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. Let $V = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}$.

(2) Define $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq r$, and let Σ be the $m \times n$ matrix such that $(\Sigma)_{ii} = \sigma_i$ for $1 \leq i \leq r$ and all other entries of Σ are 0.

(3) Compute $\vec{u}_i = \dfrac{1}{\sigma_i}A\vec{v}_i$ for $1 \leq i \leq r$, and then extend the set $\{\vec{u}_1, \dots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_m\}$ for $\mathbb{R}^m$. Take $U = \begin{bmatrix}\vec{u}_1 & \cdots & \vec{u}_m\end{bmatrix}$.

Then, $A = U\Sigma V^T$ is a singular value decomposition of A.

Remark

Since $\{\vec{u}_1, \dots, \vec{u}_r\}$ is an orthonormal basis for $\operatorname{Col}(A)$, by the Fundamental Theorem of Linear Algebra, one way to extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to an orthonormal basis for $\mathbb{R}^m$ is by adding an orthonormal basis for $\operatorname{Null}(A^T)$.

EXAMPLE 8.5.4

Find a singular value decomposition of $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: Our first step is to orthogonally diagonalize $A^T A$. We find that

$$A^T A = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

has eigenvalues (ordered from greatest to least) of $\lambda_1 = 16$, $\lambda_2 = 6$, and $\lambda_3 = 0$ and corresponding orthonormal eigenvectors are

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Hence, we define $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$.

The non-zero singular values of A are $\sigma_1 = \sqrt{16} = 4$, $\sigma_2 = \sqrt{6}$. Hence, we take

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}$$

Now, we compute

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{4}\begin{bmatrix} 4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -3/\sqrt{3} \\ 3/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Since this forms an orthonormal basis for $\mathbb{R}^2$, we take

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Then, $A = U\Sigma V^T$ is a singular value decomposition of A.

EXERCISE 8.5.2

Find a singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$.

EXAMPLE 8.5.5

Find a singular value decomposition of $B = \begin{bmatrix} 2 & -4 \\ 2 & 2 \\ -4 & 0 \\ 1 & 4 \end{bmatrix}$.

Solution: Our first step is to orthogonally diagonalize $B^T B$. We have

$$B^T B = \begin{bmatrix} 25 & 0 \\ 0 & 36 \end{bmatrix}$$

The eigenvalues of $B^T B$ are $\lambda_1 = 36$ and $\lambda_2 = 25$ with corresponding orthonormal eigenvectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Consequently, we take

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The singular values of B are $\sigma_1 = \sqrt{36} = 6$ and $\sigma_2 = \sqrt{25} = 5$, so we have

$$\Sigma = \begin{bmatrix} 6 & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Next, we compute

$$\vec{u}_1 = \frac{1}{\sigma_1} B\vec{v}_1 = \frac{1}{6}\begin{bmatrix} -4 \\ 2 \\ 0 \\ 4 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} B\vec{v}_2 = \frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ -4 \\ 1 \end{bmatrix}$$

But, we have only two vectors in $\mathbb{R}^4$, so we need to extend $\{\vec{u}_1, \vec{u}_2\}$ to an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ for $\mathbb{R}^4$. To do this we will find an orthonormal basis for Null(B^T). Row reducing B^T gives

$$B^T \sim \begin{bmatrix} 1 & 0 & -2/3 & -1/2 \\ 0 & 1 & -4/3 & 1 \end{bmatrix}$$

Hence, a basis for Null(B^T) is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix} \right\}$. Applying the Gram-Schmidt Procedure to this set and then normalizing gives

$$\vec{u}_3 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{u}_4 = \frac{1}{15}\begin{bmatrix} 8 \\ 8 \\ 9 \\ 4 \end{bmatrix}$$

Thus, we take $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$ and we get $B = U\Sigma V^T$.

Applications of the Singular Value Decomposition

Pseudoinverse In Problem C4 of Section 7.3 we defined the pseudoinverse (**Moore-Penrose inverse**) of $A \in M_{m\times n}(\mathbb{R})$ as a matrix A^+ such that

$$A^+ = (A^TA)^{-1}A^T$$

However, this formula works only if A^TA is invertible. To overcome this drawback we can use the singular value decomposition.

Definition
Pseudoinverse

Let $A \in M_{m\times n}(\mathbb{R})$ with singular value decomposition $A = U\Sigma V^T$ and non-zero singular values $\sigma_1, \dots, \sigma_r$. Define Σ^+ to be the $n \times m$ matrix such that $(\Sigma^+)_{ii} = \frac{1}{\sigma i}$ for $1 \leq i \leq r$ and all other entries of Σ^+ are 0. We define the pseudoinverse A^+ of A by

$$A^+ = V\Sigma^+U^T$$

EXAMPLE 8.5.6

Find a pseudoinverse A^+ for the matrix $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ of Example 8.5.4. Verify that $AA^+A = A$ and $A^+AA^+ = A^+$.

Solution: We found in Example 8.5.4 that a singular value decomposition for A is $A = U\Sigma V^T$ where

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

Thus, we take

$$\Sigma^+ = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/\sqrt{6} \\ 0 & 0 \end{bmatrix}$$

Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} -1/24 & 7/24 \\ -1/6 & 1/6 \\ 7/24 & -1/24 \end{bmatrix}$$

Multiplying, we find that $AA^+ = I$, so we have $AA^+A = A$.

Computing A^+AA^+ by first multiplying A^+A gives

$$\begin{aligned} A^+AA^+ &= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix} \begin{bmatrix} -1/24 & 7/24 \\ -1/6 & 1/6 \\ 7/24 & -1/24 \end{bmatrix} \\ &= \begin{bmatrix} -1/24 & 7/24 \\ -1/6 & 1/6 \\ 7/24 & -1/24 \end{bmatrix} \\ &= A^+ \end{aligned}$$

EXAMPLE 8.5.7

Find a pseudoinverse B^+ for the matrix $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

Solution: We have that

$$B^T B = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

The eigenvalues of $B^T B$ are $\lambda_1 = 10$ and $\lambda_2 = 0$ with corresponding orthonormal eigenvectors $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Consequently, we take

$$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

The non-zero singular value of B is $\sigma_1 = \sqrt{10}$. So, we take

$$\Sigma^+ = \begin{bmatrix} 1/\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$$

Next, we compute

$$\vec{u}_1 = \frac{1}{\sigma_1} B\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since we only have one vector in $\mathbb{R}^2$, we need to extend $\{\vec{u}_1\}$ to an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{R}^2$. We see that we can take $\vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Hence,

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Then

$$B^+ = V\Sigma^+ U^T = \begin{bmatrix} 1/10 & 1/10 \\ 1/5 & 1/5 \end{bmatrix}$$

One useful property of the pseudoinverse is that it gives us a formula for the solution of the normal system $A^T A\vec{x} = A^T\vec{b}$ in the method of least squares with the smallest length.

Theorem 8.5.3

Let $A \in M_{m\times n}(\mathbb{R})$. The solution of $A^T A\vec{x} = A^T\vec{b}$ with the smallest length is

$$\vec{x} = A^+\vec{b}$$

EXERCISE 8.5.3

Use Theorem 8.5.3 to determine the vector $\vec{x}$ that minimizes $\|B\vec{x} - \vec{b}\|$ for the system

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

Effective Rank In real world applications we often have extremely large matrices and hence the computations are all performed by a computer. One important area of concern is the numerical stability of the calculations. That is, we must be careful of changes due to round-off error. This is best demonstrated with an example.

EXAMPLE 8.5.8

Consider the matrix $A = \begin{bmatrix} 1/3 & 1/2 & 1 \\ 1/7 & 1/4 & 1 \\ 10/21 & 3/4 & 2 \end{bmatrix}$. It is easy to check that the third row is the sum of the first two rows and hence $\text{rank}(A) = 2$.

However, if we replace the fractions with decimals accurate to two decimals places, we would get $A' = \begin{bmatrix} 0.33 & 0.50 & 1 \\ 0.14 & 0.25 & 1 \\ 0.48 & 0.75 & 2 \end{bmatrix}$. The sum of the first two rows no longer equals the third rows, so $\text{rank}(A') = 3$.

To counteract this problem, we can use Theorem 8.5.1, which says that $\text{rank}(A)$ is equal to the number of non-zero singular values. In particular, we can define the **effective rank** of a matrix to be the number of singular values which are not less than some specified tolerance.

For example, the singular values for A' to two decimal places (our number of significant digits) are $\sigma_1 = 2.68$, $\sigma_2 = 0.20$, and $\sigma_3 = 0.00$ (the actual value is approximately $\sigma_3 = 0.0046$). Hence, the effective rank of A' is 2.

Image Compression Consider a digital image that is $1080x1920$ pixels in size (standard size for a high definition television). Each pixel is a value from 0 to 255 for its grayscale. Hence, a digital image is essentially an $m = 1080$ by $n = 1920$ matrix where each entry is an integer value from 0 to 255. Since it takes 8 bits to store a number from 0 to 255, such an image takes 16 588 800 bits. Transmitting large amounts of data like this is expensive, so we want to find a way to compress the information with minimal loss of quality.

It turns out that a singular value decomposition of the image matrix still perfectly represents the image. This immediately indicates that we can throw away any information corresponding to 0 singular values. Since many images contain redundant information, this amounts to substantial savings. However, we can do even better. The information corresponding to the larger singular values is the most important. That is, if k is the effective rank of the matrix, then retaining only the information corresponding to the first k singular values (since the singular values are ordered from greatest to least) will still give a very accurate representation of the original picture.

PROBLEM 8.5

Practice Problems

For Problems A1–A3, find the non-zero singular values of the matrix.

A1 $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ **A2** $\begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix}$ **A3** $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

For Problems A4–A13, find a singular value decomposition of the matrix.

A4 $\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$ **A5** $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ **A6** $\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$

A7 $\begin{bmatrix} 2 & -1 \\ 2 & -1 \\ 2 & -1 \end{bmatrix}$ **A8** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$ **A9** $\begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$

A10 $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **A11** $\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

A12 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ **A13** $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -2 & -1 \\ 1 & 3 \end{bmatrix}$

For Problems A14–A16, find the pseudoinverse of the matrix.

A14 $\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$ **A15** $\begin{bmatrix} 2 & -1 \\ 2 & -1 \\ 2 & -1 \end{bmatrix}$ **A16** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

Homework Problems

For Problems B1–B3, find the non-zero singular values of the matrix.

B1 $\begin{bmatrix} -\sqrt{2} & 0 \\ 0 & 3 \end{bmatrix}$ **B2** $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ **B3** $\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

For Problems B4–B18, find a singular value decomposition of the matrix.

B4 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ **B5** $\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$ **B6** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

B7 $\begin{bmatrix} -1 & 2 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$ **B8** $\begin{bmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$ **B9** $\begin{bmatrix} 2 & 1 \\ -2 & -2 \\ 1 & 2 \end{bmatrix}$

B10 $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}$ **B11** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$ **B12** $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}$

B13 $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **B14** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

B15 $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ **B16** $\begin{bmatrix} 1 & 3 & -1 \\ 1 & 3 & -1 \end{bmatrix}$

B17 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ **B18** $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$

For Problems B19–B23, find the pseudoinverse of the matrix.

B19 $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ **B20** $\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ **B21** $\begin{bmatrix} 2 & 1 \end{bmatrix}$

B22 $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$ **B23** $\begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ **B24** $\begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Conceptual Problems

C1 Let $A \in M_{m\times n}(\mathbb{R})$ with $\operatorname{rank}(A) = r$. Prove Theorem 8.5.1 using the following steps.
(a) Prove that the number of non-zero eigenvalues of a symmetric matrix B equals $\operatorname{rank}(B)$.
(b) Prove that $\operatorname{Null}(A^TA) = \operatorname{Null}(A)$.
(c) Use (b) to prove that $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$.
(d) Conclude that A has r non-zero singular values.

C2 Let $A \in M_{m\times n}(\mathbb{R})$ and let $P \in M_{m\times m}(\mathbb{R})$ be an orthogonal matrix. Prove that PA has the same singular values as A.

C3 Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix. Prove that the singular values of A are the absolute values of the eigenvalues of A.

C4 Let $U\Sigma V^T$ be a singular value decomposition for an $m \times n$ matrix A with rank r. Find, with proof, an orthonormal basis for Row(A), Col(A), Null(A), and Null(A^T) from the columns of U and V.

C5 Let $A \in M_{m\times n}(\mathbb{R})$. Prove that A and A^T have the same non-zero singular values.

C6 Let $A \in M_{m\times n}(\mathbb{R})$. Prove that the left singular vectors of A are all eigenvectors of AA^T.

C7 Let $A \in M_{m\times n}(\mathbb{R})$ with rank(A) $= r$. Show that a singular value decomposition $A = U\Sigma V^T$ allows us to write A as

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

For Problems C8 and C9, let $A \in M_{n\times n}(\mathbb{R})$ with singular value decomposition $U\Sigma V^T$.

C8 Prove that if $A^T A$ is invertible, then

$$(A^T A)^{-1} A^T = V\Sigma^+ U^T$$

C9 Prove that A^+ has the following properties.

(a) $AA^+A = A$
(b) $A^+AA^+ = A^+$
(c) $(AA^+)^T = AA^+$
(d) $(A^+A)^T = A^+A$

CHAPTER REVIEW

Suggestions for Student Review

1 What does it mean for two matrices to be orthogonally similar? How does this relate to the concept of two matrices being similar? (Section 8.1)

2 List the properties of a symmetric matrix. How does the theory of diagonalization for symmetric matrices differ from the theory of diagonalization for general square matrices? (Section 8.1)

3 Explain the algorithm for orthogonally diagonalizing a symmetric matrix. When do you need to use the Gram-Schmidt procedure? (Section 8.1)

4 Explain the connection between quadratic forms and symmetric matrices. How do you find the symmetric matrix corresponding to a quadratic form and vice versa? Why did we choose to relate a quadratic form with a symmetric matrix as opposed to an asymmetric matrix? (Section 8.2)

5 How does diagonalization of the symmetric matrix enable us to diagonalize the quadratic form? What is the geometric interpretation of this? (Sections 8.2, 8.3)

6 List the classifications of a quadratic form. How does diagonalizing the corresponding symmetric matrix help us classify a quadratic form? (Section 8.2)

7 What role do eigenvalues play in helping us understand the graphs of equations $Q(\vec{x}) = k$, where $Q(\vec{x})$ is a quadratic form? How are the classifications of a quadratic form related to its graph? (Section 8.3)

8 Define the principal axes of a symmetric matrix A. How do the principal axes of A relate to the graph of $Q(\vec{x}) = \vec{x}^T A\vec{x} = k$? (Section 8.3)

9 When diagonalizing a symmetric matrix A, we know that we can choose the eigenvalues in any order. How would changing the order in which we pick the eigenvalues change the graph of $Q(\vec{x}) = \vec{x}^T A\vec{x} = k$? Explain. (Section 8.3)

10 Try to find applications of symmetric matrices and/or quadratic forms related to your chosen field of study. (Sections 8.1, 8.2, 8.3, 8.4)

11 Explain the similarities and differences between orthogonal diagonalization and the singular value decomposition. (Section 8.5)

12 Write the algorithm for finding a singular value decomposition of a matrix. Is the singular value decomposition of a matrix unique? (Section 8.5)

13 Research applications of the singular value decomposition. Why is the singular value decomposition particularly useful? (Section 8.5)

Chapter Quiz

E1 Let $A = \begin{bmatrix} 2 & -3 & 2 \\ -3 & 3 & 3 \\ 2 & 3 & 2 \end{bmatrix}$. Find an orthogonal matrix P such that $P^TAP = D$ is diagonal.

For Problems E2 and E3, write the quadratic form corresponding to the given symmetric matrix.

E2 $\begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$

E3 $\begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 4 \\ -2 & 4 & 1 \end{bmatrix}$

For Problems E4 and E5:

(a) Determine the symmetric matrix corresponding to $Q(\vec{x})$.
(b) Express $Q(\vec{x})$ in diagonal form and give the orthogonal matrix that brings it into this form.
(c) Classify $Q(\vec{x})$.
(d) Describe the shape of $Q(\vec{x}) = 1$ and $Q(\vec{x}) = 0$.

E4 $Q(\vec{x}) = 5x_1^2 + 4x_1x_2 + 5x_2^2$

E5 $Q(\vec{x}) = 2x_1^2 - 6x_1x_2 - 6x_1x_3 - 3x_2^2 + 4x_2x_3 - 3x_3^2$

E6 By diagonalizing the quadratic form, make a sketch of the graph of

$$5x_1^2 - 2x_1x_2 + 5x_2^2 = 12$$

in the x_1x_2-plane. Show the new and old coordinate axes.

E7 Sketch the graph of the hyperbola

$$x_1^2 + 4x_1x_2 + x_2^2 = 8$$

in the x_1x_2-plane, including the asymptotes in the sketch.

For Problems E8 and E9, find a singular value decomposition of the matrix.

E8 $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

E9 $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$

E10 Find the pseudoinverse of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

E11 Prove that if A is a positive definite symmetric matrix, then $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$ is an inner product on $\mathbb{R}^n$.

E12 Prove that if $A \in M_{4\times4}(\mathbb{R})$ is symmetric with characteristic polynomial $C(\lambda) = (\lambda - 3)^4$, then $A = 3I$.

E13 Find a 2×2 symmetric matrix such that a basis for the eigenspace of $\lambda_1 = 2$ is $\left\{\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right\}$, and a basis for the eigenspace of $\lambda_2 = -1$ is $\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$.

E14 Prove that a square matrix A is invertible if and only if 0 is not a singular value of A.

For Problems E15–E22, determine whether the statement is true or false. Justify your answer.

E15 If A and B are orthogonally similar, then A^2 and B^2 are orthogonally similar.

E16 If A and B are orthogonally similar and A is symmetric, then B is symmetric.

E17 If $\vec{v}_1$ and $\vec{v}_2$ are both eigenvectors of a symmetric matrix A, then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

E18 If A is orthogonally diagonalizable, then A is symmetric.

E19 If $Q(\vec{x})$ is positive definite, then $Q(\vec{x}) > 0$ for all $\vec{x} \in \mathbb{R}^n$.

E20 If A is an indefinite symmetric matrix, then A is invertible.

E21 A symmetric matrix with all negative entries is negative definite.

E22 If A is a square matrix, then $|\det A|$ is equal to the product of the singular values of A.

Further Problems

These exercises are intended to be challenging.

F1 Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ be the standard matrix of a linear mapping L. (Observe that A is not diagonalizable.)

(a) Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ of right singular vectors of A.

(b) Find a basis C for $\mathbb{R}^3$ of left singular vectors of A.

(c) Determine ${}_C[L]_{\mathcal{B}}$.

F2 In Chapter 7 Problem F6, we saw the QR-factorization: an invertible $n \times n$ matrix A can be expressed in the form $A = QR$, where Q is orthogonal and R is upper triangular. Let $A_1 = RQ$, and prove that A_1 is orthogonally similar to A and hence has the same eigenvalues as A. (By repeating this process, $A = Q_1R_1$, $A_1 = R_1Q_1$, $A_1 = Q_2R_2$, $A_2 = R_2Q_2$, ..., one obtains an effective numerical procedure for determining eigenvalues of a symmetric matrix.)

F3 Suppose that $A \in M_{n\times n}(\mathbb{R})$ is a positive semidefinite symmetric matrix. Prove that A has a square root. That is, show that there is a positive semidefinite symmetric matrix B such that $B^2 = A$. (Hint: suppose that Q diagonalizes A to D so that $Q^TAQ = D$. Define C to be a positive square root for D and let $B = QCQ^T$.)

F4 (a) If $A \in M_{n\times n}(\mathbb{R})$, prove that A^TA is symmetric and positive semidefinite. (Hint: consider $A\vec{x} \cdot A\vec{x}$.)

(b) If A is invertible, prove that A^TA is positive definite.

F5 (a) Suppose that $A \in M_{n\times n}(\mathbb{R})$ is invertible. Prove that A can be expressed as a product of an orthogonal matrix Q and a positive definite symmetric matrix U, $A = QU$. This is known as a **polar decomposition** of A. (Hint: use Problems F3 and F4; let U be the square root of A^TA and let $Q = AU^{-1}$.)

(b) Let $V = QUQ^T$. Show that V is symmetric and that $A = VQ$. Moreover, show that $V^2 = AA^T$, so that V is a positive definite symmetric square root of AA^T.

For part (c), we will use the following definition. A linear mapping $L(\vec{x}) = A\vec{x}$ is said to be **orientation-preserving** if $\det A > 0$.

(c) Suppose that $A \in M_{3\times 3}(\mathbb{R})$ is the matrix of an orientation-preserving linear mapping L. Show that L is the composition of a rotation following three stretches along mutually orthogonal axes. (This result follows from part (a), facts about isometries of $\mathbb{R}^3$, and ideas in Section 8.4. In fact, this is a finite version of the result for infinitesimal strain in Section 8.4.)

CHAPTER 9

Complex Vector Spaces

CHAPTER OUTLINE

When first encountering imaginary numbers, many students wonder at the point of looking at numbers which are not real. In 1572, Rafael Bombelli was the first to show that numbers involving square roots of negative numbers could be used to solve real world problems. Currently, complex numbers are used to solve problems in a wide variety of areas. Some examples are electronics, control theory, quantum mechanics, and fluid dynamics. Our goal in this chapter is to extend everything we did in Chapters 1–8 to allow the use of complex numbers instead of just real numbers.

9.1 Complex Numbers

Introduction

The first numbers we encounter as children are the natural numbers $1, 2, 3, \dots$ and so on. In school, we soon found out that, in order to perform certain subtractions, we had to extend our concept of number to the integers, which include the natural numbers. Then, so that division by a non-zero number could always be carried out, the concept of number was extended to the rational numbers, which include the integers. Next, the concept of number was extended to the real numbers, which include all the rational numbers.

Now, to solve the equation $x^2 + 1 = 0$ we have to extend our concept of number one more time. We define the number i to be a number such that $i^2 = -1$. The system of numbers of the form $x + yi$ where $x, y \in \mathbb{R}$ is called the **complex numbers**. Note that the real numbers are included as those complex numbers with $b = 0$. As in the case with all the previous extensions of our understanding of number, some people are initially uncertain about the "meaning" of the "new" numbers. However, the complex numbers have a consistent set of rules of arithmetic, and the extension to complex numbers is justified by the fact that they allow us to solve important mathematical and physical problems that we could not solve using only real numbers.

The Geometry of Complex Numbers

Definition
Complex Number

A **complex number** is a number of the form $z = x + yi$, where $x, y \in \mathbb{R}$ and i is an element such that $i^2 = -1$. The set of all complex numbers is denoted by $\mathbb{C}$.

EXAMPLE 9.1.1

Some examples of complex numbers are $2 - 5i$, $0 + 3i$ (usually written as $3i$), $4 + 0i$ (usually written as 4).

Definition
Real Part
Imaginary Part

If $z = x + yi \in \mathbb{C}$, then we say that the **real part** of z is x and write $\mathrm{Re}(z) = x$. We say that the **imaginary part** of z is y (*not* yi), and we write $\mathrm{Im}(z) = y$.

EXAMPLE 9.1.2

For $z = 2 - 5i$ we have $\mathrm{Re}(z) = 2$ and $\mathrm{Im}(z) = -5$.

Remarks

1. The form $z = x + yi$ is called the **standard form** of a complex number.
2. If $y \neq 0$, then $z = yi$ is said to be "purely imaginary."
3. In physics and engineering, j is sometimes used in place of i since the letter i is often used to denote electric current.
4. It is sometimes convenient to write $x + iy$ instead of $x + yi$. This is particularly common with the polar form for complex numbers, which is discussed below.

At this point it is useful to observe that we can think of any complex number $z = x + yi$ as a point (x, y) in $\mathbb{R}^2$. When doing this, we refer to the x-axis as the **real axis**, denoted Re, and the y-axis as the **imaginary axis**, denoted Im, and call the xy-plane the **complex plane**. See Figure 9.1.1. A picture of this kind is sometimes called an **Argand diagram**.

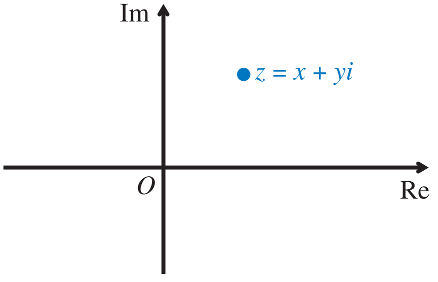


Figure 9.1.1 The complex plane.

One advantage of looking at complex numbers as points in the complex plane is that it gives us geometric motivations for properties and operations on complex numbers.

For example, we know that two points (x, y) and (u, v) in $\mathbb{R}^2$ are equal if $x = u$ and $y = v$, so we make the corresponding definition for complex numbers.

Definition
Equality

Two complex numbers $z = x + yi$ and $w = u + vi$ are **equal** if $x = u$ and $y = v$.

Similarly, since the absolute value function $|x|$ measures the distance that x is from the origin, we define the absolute value (called the modulus) of a complex number z to be the distance z is from the origin.

Definition
Modulus

For a complex number $z = x + yi$, the real number

$$|z| = \sqrt{x^2 + y^2}$$

is called the **modulus** of z.

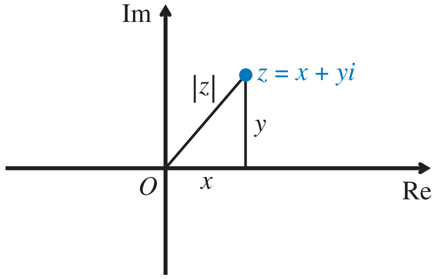


Figure 9.1.2 The Pythagorean Theorem gives the distance from z to O: $|z| = \sqrt{x^2 + y^2}$.

EXAMPLE 9.1.3

Determine the modulus (absolute value) of $z_1 = 2 - 2i$ and $z_2 = -1 + \sqrt{3}i$.

Solution: We have

$$|z_1| = |2 - 2i| = \sqrt{(2)^2 + (-2)^2} = \sqrt{8}$$

$$|z_2| = |-1 + \sqrt{3}i| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

EXERCISE 9.1.1

Determine the modulus of $z_1 = -3$ and $z_2 = -1 - i$ and illustrate with an Argand diagram.

The Arithmetic of Complex Numbers

We also use the geometry of the complex plane to define **addition** and **real scalar multiplication** of complex numbers.

Definition
Addition
Real Scalar Multiplication

For complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ we define

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

For any real number c we define

$$cz_1 = cx_1 + cy_1 i$$

Note that when we write $z_1 - z_2$ we mean $z_1 + (-1)z_2$.

EXAMPLE 9.1.4

Perform the following operations.

(a) $(2 + 3i) + (5 - 4i)$

Solution: $(2 + 3i) + (5 - 4i) = (2 + 5) + (3 - 4)i = 7 - i$

(b) $(2 + 3i) - (5 - 4i)$

Solution: $(2 + 3i) - (5 - 4i) = (2 - 5) + (3 - (-4))i = -3 + 7i$

(c) $4(2 - 3i)$

Solution: $4(2 - 3i) = 8 - 12i$

CONNECTION

(Section 4.7 required.) It is not difficult to show that under the operations of addition and real scalar multiplication defined above, $\mathbb{C}$ is a real vector space with basis $\{1, i\}$. Therefore, by Theorem 4.7.3, $\mathbb{C}$, as a real vector space, is in fact isomorphic to $\mathbb{R}^2$.

So far, we have only defined multiplication of a complex number by a real scalar. We, of course, also want to be able to multiply two complex numbers together. To define this, we write $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, and require that the product satisfy the distributive property. We get

$$\begin{aligned} z_1 z_2 &= (x_1 + y_1 i)(x_2 + y_2 i) \\ &= x_1x_2 + x_1y_2 i + x_2y_1 i + y_1y_2 i^2 \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i \end{aligned}$$

Definition
Multiplication

If $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i \in \mathbb{C}$, then we define

$$z_1 z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i$$

EXAMPLE 9.1.5

Calculate $(3 - 2i)(-2 + 5i)$.

Solution: $(3 - 2i)(-2 + 5i) = [3(-2) - (-2)(5)] + [3(5) + (-2)(-2)]i = 4 + 19i$

EXERCISE 9.1.2

Calculate the following:

(a) $(1 - 4i) + (2 + 5i)$
(b) $(2 + 3i) - (1 - 3i)$
(c) $(2 + 2\mathrm{i})\mathrm{i}$
(d) $(1 - 3\mathrm{i})(2 + \mathrm{i})$
(e) $(3 - 2\mathrm{i})(3 + 2\mathrm{i})$

In order to give a systematic method for expressing the quotient of two complex numbers as a complex number in standard form, it is useful to introduce the **complex conjugate**.

Definition
Complex Conjugate

The **complex conjugate** $\bar{z}$ of the complex number $z = x + yi$ is defined by

$$\bar{z} = x - yi$$

EXAMPLE 9.1.6

Find the complex conjugate of $2 + 5i$, $-3 - 2i$, and 4.

Solution: We have

$$\overline{2 + 5i} = 2 - 5i$$
$$\overline{-3 - 2i} = -3 + 2i$$
$$\overline{4} = 4$$

Geometrically, the complex conjugate is a reflection of $z = x + yi$ over the real axis.

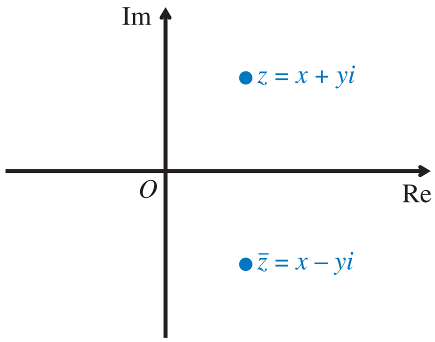


Figure 9.1.3 The geometry of $\bar{z}$.

Theorem 9.1.1 **Properties of the Complex Conjugate**

For complex numbers $z_1 = x + yi$ and z_2 with $x, y \in \mathbb{R}$, we have

(1) $\overline{\overline{z_1}} = z_1$
(2) z_1 is real if and only if $\overline{z_1} = z_1$
(3) z_1 is purely imaginary if and only if $\overline{z_1} = -z_1$, $z_1 \neq 0$
(4) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
(5) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
(6) $\overline{z_1^n} = \overline{z_1}^n$
(7) $z_1 + \overline{z_1} = 2\operatorname{Re}(z_1) = 2x$
(8) $z_1 - \overline{z_1} = i2\operatorname{Im}(z_1) = i2y$
(9) $z_1\overline{z_1} = x^2 + y^2 = |z_1|^2$

EXERCISE 9.1.3 Prove properties (1), (2), and (4) in Theorem 9.1.1.

The proofs of the remaining properties are left as Problem C1.

The **quotient** of two complex numbers can now be displayed as a complex number in standard form by multiplying both the numerator and the denominator by the complex conjugate of the denominator and simplifying. In particular, if $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i \neq 0$, then

$$\begin{aligned}\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} &= \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)}\\ &= \frac{(x_1x_2 + y_1y_2) + (y_1x_2 - x_1y_2)i}{x_2^2 + y_2^2}\\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}i\end{aligned}$$

Notice that the quotient is defined for every pair of complex numbers z_1, z_2, provided that the denominator is not zero.

EXAMPLE 9.1.7 Write $\dfrac{2 + 5i}{3 - 4i}$ in standard form.

Solution: We have

$$\frac{2 + 5i}{3 - 4i} = \frac{(2 + 5i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{(6 - 20) + (8 + 15)i}{9 + 16} = -\frac{14}{25} + \frac{23}{25}i$$

EXERCISE 9.1.4 Calculate the following quotients.

(a) $\dfrac{1 + i}{1 - i}$ (b) $\dfrac{2i}{1 + i}$ (c) $\dfrac{4 - i}{1 + 5i}$

The operations on complex numbers have the following important properties.

Theorem 9.1.2 If $z_1, z_2, z_3 \in \mathbb{C}$, then

(1) $z_1 + z_2 \in \mathbb{C}$
(2) $z_1 + z_2 = z_2 + z_1$
(3) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
(4) $z_1 + 0 = z_1$
(5) For each $z \in \mathbb{C}$ there exists $(-z) \in \mathbb{C}$ such that $z + (-z) = 0$.
(6) $z_1 z_2 \in \mathbb{C}$
(7) $z_1 z_2 = z_2 z_1$
(8) $z_1(z_2 z_3) = (z_1 z_2) z_3$
(9) $1 z_1 = z_1$
(10) For each $z \in \mathbb{C}$ with $z \neq 0$, there exists $(1/z) \in \mathbb{C}$ such that $z(1/z) = 1$.
(11) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

CONNECTION

Any set with two operations of addition and multiplication that satisfies properties (1) to (11) is called a **field**. Both the set of rational numbers $\mathbb{Q}$, and the set of real numbers $\mathbb{R}$ are also fields. There are other fields which play important roles in many real world applications of linear algebra.

Another important property is the familiar triangle inequality.

Theorem 9.1.3 If $z_1, z_2 \in \mathbb{C}$, then

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Polar Form

One of the reasons complex numbers are so helpful for solving problems is that they can be represented in many different ways. So far, we have seen the algebraic form $z = x + yi$ and the vector form (x, y). We now look at another way of representing complex numbers.

Rather than locating a complex number $z = x + yi$ in the complex plane using its Cartesian coordinates, we can locate it by using its modulus $r = |z|$ (distance from the origin) and any angle θ between the positive real axis and a line segment between the origin and z. See Figure 9.1.4.

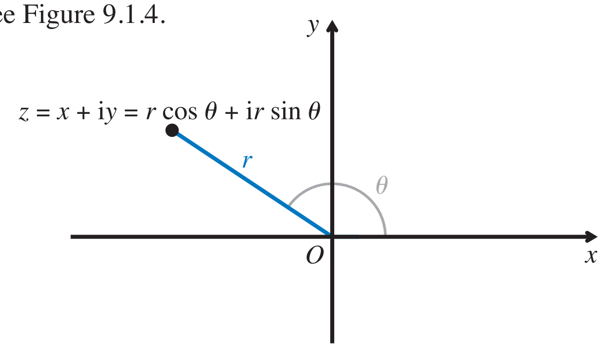


Figure 9.1.4 Polar representation of z.

We begin by making the following definition.

Definition
Argument

Let $z = x + yi \in \mathbb{C}$ with $z \neq 0$. If θ is any angle such that

$$x = |z| \cos\theta \quad \text{and} \quad y = |z| \sin\theta$$

then the angle θ is called an **argument** of z and we write

$$\theta = \arg z$$

It is important to observe that every complex number has infinitely many arguments. In particular, if θ is an argument of a complex number z, then all the values

$$\theta + 2\pi k, \quad k \in \mathbb{Z}$$

are also arguments of z.

For many applications, we prefer to choose the unique argument θ that satisfies $-\pi < \theta \leq \pi$ called the **principal argument** of z. It is denoted Arg z.

EXAMPLE 9.1.8

Find all arguments of $z_1 = 1 - i$.

Solution: To calculate any argument of z_1, we first need to calculate the modulus.

$$|z_1| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Hence, any argument θ of z_1 satisfies

$$1 = \sqrt{2}\cos\theta, \quad \text{and} \quad -1 = \sqrt{2}\sin\theta$$

So, $\cos\theta = \dfrac{1}{\sqrt{2}}$ and $\sin\theta = -\dfrac{1}{\sqrt{2}}$. This gives

$$\theta = -\frac{\pi}{4} + 2\pi k, \quad k \in \mathbb{Z}$$

EXAMPLE 9.1.9

Determine the principal argument of $z_2 = \sqrt{3} + i$.

Solution: We have

$$|z_2| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

and so, any argument θ of z_2 satisfies

$$\sqrt{3} = 2\cos\theta, \quad \text{and} \quad 1 = 2\sin\theta$$

So, $\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = \dfrac{1}{2}$. This gives

$$\theta = \frac{\pi}{6} + 2\pi k, \quad k \in \mathbb{Z}$$

Therefore, the principal value is Arg $z_2 = \dfrac{\pi}{6}$.

Remarks

1. The angles may be measured in radians or degrees. We will always use radians.
2. It is tempting but incorrect to write $\theta = \arctan(y/x)$. Remember that you need two trigonometric functions to locate the correct quadrant for z. Also note that y/x is not defined if $x = 0$.

Definition
Polar Form

The **polar form** of a complex number z is

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and θ is an argument of z.

EXAMPLE 9.1.10 Determine the modulus, an argument, and a polar form of $z_1 = 2-2i$ and $z_2 = -1+\sqrt{3}i$.

Solution: We have

$$|z_1| = |2-2i| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

Any argument θ of z_1 satisfies

$$2 = 2\sqrt{2}\cos\theta \quad \text{and} \quad -2 = 2\sqrt{2}\sin\theta$$

So, $\cos\theta = \dfrac{1}{\sqrt{2}}$ and $\sin\theta = -\dfrac{1}{\sqrt{2}}$, which gives $\theta = -\dfrac{\pi}{4} + 2\pi k$, $k \in \mathbb{Z}$. Hence, a polar form of z_1 is

$$z_1 = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

For z_2, we have

$$|z_2| = \left|-1 + \sqrt{3}i\right| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$$

Since $-1 = 2\cos\theta$ and $\sqrt{3} = 2\sin\theta$, we get $\theta = \dfrac{2\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$. Thus, a polar form of z_2 is

$$z_2 = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

EXERCISE 9.1.5 Determine the modulus, an argument, and a polar form of $z_1 = \sqrt{3} + i$ and $z_2 = -1 - i$.

EXERCISE 9.1.6 Let $z = r(\cos\theta + i\sin\theta)$. Prove that the modulus of $\bar{z}$ equals the modulus of z and an argument of $\bar{z}$ is $-\theta$.

The polar form is particularly convenient for multiplication and division because of the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

Theorem 9.1.4 For any complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, we have

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$$

In words, the modulus of a product is the product of the moduli of the factors, while an argument of a product is the sum of the arguments.

Theorem 9.1.5 For any complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, with $z_2 \neq 0$, we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

The proofs of Theorems 9.1.4 and Theorem 9.1.5 are left for you to complete in Problem C4.

Theorem 9.1.6 If $z = r(\cos\theta + i\sin\theta)$ with $r \neq 0$, then $\dfrac{1}{z} = \dfrac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right)$.

EXERCISE 9.1.7 Describe Theorem 9.1.5 in words and use it to prove Theorem 9.1.6.

EXAMPLE 9.1.11 Calculate $(1 - i)(-\sqrt{3} + i)$ and $\dfrac{2 + 2i}{1 + \sqrt{3}i}$ using polar form.

Solution: We have

$$\begin{aligned}
(1 - i)(-\sqrt{3} + i) &= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)\\
&= 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4} + \frac{5\pi}{6}\right) + i\sin\left(-\frac{\pi}{4} + \frac{5\pi}{6}\right)\right)\\
&= 2\sqrt{2}\left(\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right)\\
\frac{2 + 2i}{1 + \sqrt{3}i} &= \frac{2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}{2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)}\\
&= \sqrt{2}\left(\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)\\
&= \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)
\end{aligned}$$

EXERCISE 9.1.8

Calculate $(2-2i)(-1+\sqrt{3}i)$ and $\dfrac{2-2i}{-1+\sqrt{3}i}$ using polar form.

Powers and the Complex Exponential

From the rule for products, we find that

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

Then

$$\begin{aligned} z^3 = z^2 z &= r^2 r\big(\cos(2\theta+\theta) + i\sin\theta(2\theta+\theta)\big) \\ &= r^3(\cos 3\theta + i\sin 3\theta) \end{aligned}$$

Theorem 9.1.7 de Moivre's Formula

If $z = r(\cos\theta + i\sin\theta)$ with $r \neq 0$, then for any integer n we have

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Proof: For $n = 0$, we have $z^0 = 1 = r^0(\cos 0 + i\sin 0)$. To prove that the theorem holds for positive integers, we proceed by induction. Assume that the result is true for some integer $k \geq 0$. Then

$$\begin{aligned} z^{k+1} = z^k z &= r^k r[\cos(k\theta+\theta) + i\sin(k\theta+\theta)] \\ &= r^{k+1}[\cos((k+1)\theta) + i\sin((k+1)\theta)] \end{aligned}$$

Therefore, the result is true for all non-negative integers n. Then, by Theorem 9.1.4, for any positive integer m, we have

$$\begin{aligned} z^{-m} = (z^m)^{-1} &= (r^m(\cos m\theta + i\sin m\theta))^{-1} \\ &= r^{-m}\big(\cos(-m\theta) + i\sin(-m\theta)\big) \end{aligned}$$

Hence, the result also holds for all negative integers $n = -m$. ■

EXAMPLE 9.1.12

Use de Moivre's Formula to calculate $(2+2i)^3$.

Solution:

$$\begin{aligned}(2+2i)^3 &= \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)\right)\right]^3\\ &= (2\sqrt{2})^3\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)\\ &= 16\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\\ &= -16+16i\end{aligned}$$

In the case where $r = 1$, de Moivre's Formula reduces to

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

This is formally just like one of the exponential laws, $(e^{\theta})^n = e^{n\theta}$. We use this idea to define e^z for any $z \in \mathbb{C}$, where e is the usual natural base for exponentials ($e \approx 2.71828$). We begin with a useful formula of Euler.

Definition
Euler's Formula

For any $\theta \in \mathbb{R}$ we have

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Observe that Euler's Formula allows us to write the polar form of a complex number z more compactly. In particular, we can now write

$$z = re^{i\theta}$$

where $r = |z|$ and θ is any argument of z. One advantage of this form is that de Moivre's Formula can be written as

$$z^n = r^n e^{in\theta}$$

Remarks

1. One interesting consequence of Euler's Formula is that $e^{i\pi} + 1 = 0$. In one formula, we have five of the most important numbers in mathematics: $0, 1, e, i$, and π.

2. One area where Euler's Formula has important applications is ordinary differential equations. There, one often uses the fact that

$$e^{(a+bi)t} = e^{at}e^{ibt} = e^{at}(\cos bt + i\sin bt)$$

EXAMPLE 9.1.13 Calculate the following using the polar form.

(a) $(2+2i)^3$ (b) $(2i)^3$ (c) $(\sqrt{3}+i)^5$

Solution:

$$(2+2i)^3 = \left(2\sqrt{2}e^{i\pi/4}\right)^3 = (2\sqrt{2})^3 e^{i(3\pi/4)} = 16\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4)) = -16+16i$$

$$(2i)^3 = \left(2e^{i\pi/2}\right)^3 = 2^3 e^{i(3\pi/2)} = 8(\cos(3\pi/2) + i\sin(3\pi/2)) = -8i$$

$$(\sqrt{3}+i)^5 = \left(2e^{i\pi/6}\right)^5 = 2^5 e^{i5\pi/6} = 32(\cos(5\pi/6) + i\sin(5\pi/6)) = -16\sqrt{3}+16i$$

EXERCISE 9.1.9 Use polar form to calculate $(1-i)^5$ and $(-1-\sqrt{3}i)^5$.

n-th Roots

Using de Moivre's Formula for n-th powers is the key to finding n-th roots. Suppose that we need to find the n-th root of the non-zero complex number $z = re^{i\theta}$. That is, we need a number w such that $w^n = z$. Suppose that $w = Re^{i\phi}$. Then $w^n = z$ implies that

$$R^n e^{in\phi} = re^{i\theta}$$

Then R is the real n-th root of the positive real number r. However, because arguments of complex numbers are determined only up to the addition of $2\pi k$, all we can say about ϕ is that

$$n\phi = \theta + 2\pi k, \quad k \in \mathbb{Z}$$

or

$$\phi = \frac{\theta + 2\pi k}{n}, \quad k \in \mathbb{Z}$$

EXAMPLE 9.1.14 Find all the cube roots of 8.

Solution: We have $8 = 8e^{i(0+2\pi k)}$, $k \in \mathbb{Z}$. Thus, for any $k \in \mathbb{Z}$.

$$8^{1/3} = \left(8e^{i(0+2\pi k)}\right)^{1/3} = 8^{1/3}e^{i2k\pi/3}$$

If $k=0$, we have the root $w_0 = 2e^0 = 2$.
If $k=1$, we have the root $w_1 = 2e^{i2\pi/3} = -1+\sqrt{3}i$.
If $k=2$, we have the root $w_2 = 2e^{i4\pi/3} = -1-\sqrt{3}i$.
If $k=3$, we have the root $2e^{i2\pi} = 2 = w_0$.

By increasing k further, we simply repeat the roots we have already found. Similarly, consideration of negative k gives us no further roots. The number 8 has three third roots, ω_0, ω_1, and ω_2. In particular, these are the roots of equation $w^3 - 8 = 0$.

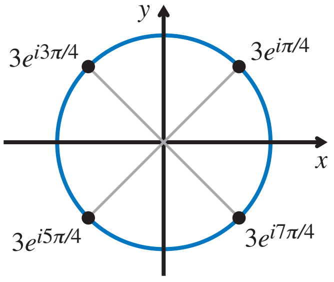
Theorem 9.1.8 If $z = re^{i\theta}$ is non-zero, then the n distinct n-th roots of z are

$$w_k = r^{1/n} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, \ldots, n-1$$

EXAMPLE 9.1.15 Find the fourth roots of -81 and illustrate in an Argand diagram.

Solution: We have $-81 = 81e^{i(\pi+2\pi k)}$. Thus, the fourth roots are

$$\begin{aligned} w_0 &= (81)^{1/4} e^{i(\pi+0)/4} = 3e^{i\pi/4} \\ w_1 &= (81)^{1/4} e^{i(\pi+2\pi)/4} = 3e^{i3\pi/4} \\ w_2 &= (81)^{1/4} e^{i(\pi+4\pi)/4} = 3e^{i5\pi/4} \\ w_3 &= (81)^{1/4} e^{i(\pi+6\pi)/4} = 3e^{i7\pi/4} \end{aligned}$$



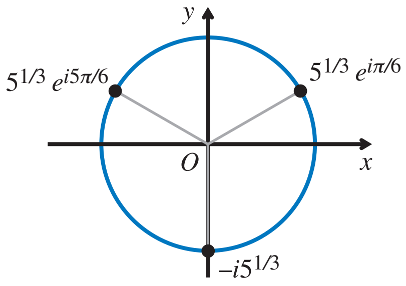
Plotting these roots shows that all four are points on the circle of radius 3 centred at the origin and that they are separated by equal angles of $\frac{\pi}{2}$.

In Examples 9.1.14 and 9.1.15, we took roots of numbers that were purely real: we were really solving $x^n - a = 0$, where $a \in \mathbb{R}$. As a contrast, let us consider roots of a number that is not real.

EXAMPLE 9.1.16 Find the third roots of $5i$ and illustrate in an Argand diagram.

Solution: $5i = 5e^{i\left(\frac{\pi}{2}+2k\pi\right)}$, so the cube roots are

$$\begin{aligned} w_0 &= 5^{1/3}\, e^{i\pi/6} &&= 5^{1/3}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\ w_1 &= 5^{1/3}\, e^{i5\pi/6} &&= 5^{1/3}\left(\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right) \\ w_2 &= 5^{1/3}\, e^{i3\pi/2} &&= 5^{1/3}(-i) \end{aligned}$$



Plotting these roots shows that all three are points on the circle of radius $5^{1/3}$ centred at the origin and that they are separated by equal angles of $\frac{2\pi}{3}$.

Examples 9.1.14, 9.1.15, and 9.1.16 all illustrate a general rule: the n-th roots of a complex number $z = re^{i\theta}$ all lie on the circle of radius $r^{1/n}$, and they are separated by equal angles of $2\pi/n$.

PROBLEMS 9.1

Practice Problems

For Problems A1–A6, draw an Argand diagram for z, and determine $\bar{z}$ and $|z|$.

A1 $z = 3 - 5i$ **A2** $z = 2 + 7i$ **A3** $z = -4i$
A4 $z = -1 - 2i$ **A5** $z = 2$ **A6** $z = -3 + 2i$

For Problems A7–A12, find a polar form and the principal argument of z.

A7 $z = -3 - 3i$ **A8** $z = \sqrt{3} - i$
A9 $z = -\sqrt{3} + i$ **A10** $z = -2 - 2\sqrt{3}i$
A11 $z = -3$ **A12** $z = -2 + 2i$

For Problems A13–A18, convert z into to standard form.

A13 $z = e^{-i\pi/3}$ **A14** $z = 2e^{-i\pi/3}$
A15 $z = 2e^{i\pi/3}$ **A16** $z = 3e^{i3\pi/4}$
A17 $z = 2e^{-i\pi/6}$ **A18** $z = e^{i5\pi/6}$

For Problems A19–A26, write the number in standard form.

A19 $(2 + 5i) + (3 + 2i)$ **A20** $(2 - 7i) + (-5 + 3i)$
A21 $(-3 + 5i) - (4 + 3i)$ **A22** $(-5 - 6i) - (9 - 11i)$
A23 $(1 + 3i)(3 - 2i)$ **A24** $(-2 - 4i)(3 - i)$
A25 $(1 - 6i)(-4 + i)$ **A26** $(-1 - i)(1 - i)$

For Problems A27–A30, determine Re(z) and Im(z).

A27 $z = 3 - 6i$ **A28** $z = (2 + 5i)(1 - 3i)$
A29 $z = \dfrac{4}{6 - i}$ **A30** $z = \dfrac{-1}{i}$

For Problems A31–A33, express the quotient in standard form.

A31 $\dfrac{1}{2 + 3i}$ **A32** $\dfrac{2 - 5i}{3 + 2i}$ **A33** $\dfrac{1 + 6i}{4 - i}$

For Problems A34–A37, use polar form to determine $z_1 z_2$ and $\dfrac{z_1}{z_2}$.

A34 $z_1 = 1 + i, z_2 = 1 + \sqrt{3}i$

A35 $z_1 = -\sqrt{3} - i, z_2 = 1 - i$

A36 $z_1 = -1 + \sqrt{3}i, z_2 = \sqrt{3} - i$

A37 $z_1 = -3 + 3i, z_2 = -\sqrt{3} + i$

For Problems A38–A41, use polar form to evaluate the expression.

A38 $(1 + i)^4$ **A39** $(3 - 3i)^3$
A40 $(-1 - \sqrt{3}i)^4$ **A41** $(-2\sqrt{3} + 2i)^5$

For Problems A42–A45, use polar form to determine all the indicated roots.

A42 $(-1)^{1/5}$ **A43** $(-16i)^{1/4}$
A44 $(-\sqrt{3} - i)^{1/3}$ **A45** $(1 + 4i)^{1/3}$

Homework Problems

For Problems B1–B6, draw an Argand diagram for z, and determine $\bar{z}$ and $|z|$.

B1 $z = 3 + 4i$ **B2** $z = -2 - 3i$ **B3** $z = 2 - \sqrt{3}i$
B4 $z = \frac{1}{2}i$ **B5** $z = 1 - 3i$ **B6** $z = -3 + i$

For Problems B7–B12, find a polar form and the principal argument of z.

B7 $z = 3 - 3i$ **B8** $z = -2i$
B9 $z = \sqrt{3}i$ **B10** $z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$
B11 $z = -3\sqrt{2} - 3\sqrt{2}i$ **B12** $z = -\frac{1}{4} - \frac{\sqrt{3}}{4}i$

For Problems B13–B20, convert z from polar form to standard form.

B13 $z = 3e^{i\pi/2}$ **B14** $z = 2e^{-i\pi}$
B15 $z = 4e^{i2\pi/3}$ **B16** $z = \sqrt{2}e^{i\pi}$
B17 $z = e^{-i\pi/4}$ **B18** $z = 6e^{i\pi/6}$
B19 $z = 3e^{-i2\pi/3}$ **B20** $z = e^{i\pi/3}$

For Problems B21–B34, evaluate the expression and write it in standard form.

B21 $(-1 + 2i) + (2 - 3i)$ **B22** $(3 - 4i) + (-3 - i)$
B23 $2i + (-1 - 2i)$ **B24** $(2 - i) - (3 - 3i)$
B25 $(-1 - 3i) - (-2 + i)$ **B26** $(4 + 2i) + (-2 - 3i)$
B27 $(-2 + i)(-1 - i)$ **B28** $(1 + i)(1 - i)$
B29 $(-1 - i)(1 + i)$ **B30** $(3 - i)(1 + 3i)$

B31 $(2+4i)(-1-3i)$

B32 $(-3+2i)(4-i)$

B33 $(1-2i)(-2+i)$

B34 $(1+\sqrt{3}i)(-\frac{3}{2}+\frac{3\sqrt{3}}{2}i)$

For Problems B35–B42, determine Re(z) and Im(z).

B35 $z=-2+3i$

B36 $z=\overline{3-4i}$

B37 $z=|-1+\sqrt{3}i|$

B38 $z=(1-2i)(2+2i)$

B39 $z=\dfrac{1}{2-3i}$

B40 $z=\dfrac{i}{1-i}$

B41 $z=5e^{-i\pi/2}$

B42 $z=2e^{i\pi/4}$

For Problems B43–B48, express the quotient in standard form.

B43 $\dfrac{1}{3-4i}$

B44 $\dfrac{2}{1+5i}$

B45 $\dfrac{5}{-1-2i}$

B46 $\dfrac{1-3i}{1+2i}$

B47 $\dfrac{3+5i}{-2+i}$

B48 $\dfrac{-2-3i}{4-3i}$

For Problems B49–B52, use polar form to determine z_1z_2 and $\dfrac{z_1}{z_2}$.

B49 $z_1=\sqrt{3}+i,\ z_2=-1-i$

B50 $z_1=2+2i,\ z_2=-2-2\sqrt{3}i$

B51 $z_1=2-2\sqrt{3}i,\ z_2=1+i$

B52 $z_1=-3\sqrt{2}+3\sqrt{2}i,\ z_2=1+\sqrt{3}i$

For Problems B53–B58, use polar form to convert z to standard form.

B53 $(1-i)^5$

B54 $(-2\sqrt{3}+2i)^3$

B55 $(\sqrt{(3)}+i)^4$

B56 $(2+2i)^4$

B57 $(-1+\sqrt{3}i)^4$

B58 $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^5$

For Problems B59–B64, use polar form to determine all the indicated roots.

B59 $(-1)^{1/5}$

B60 $(-16i)^{1/4}$

B61 $1^{1/5}$

B62 $(1+i)^{1/4}$

B63 $(-\sqrt{3}-i)^{1/3}$

B64 $(1+4i)^{1/3}$

Conceptual Problems

C1 Prove properties (3), (5), (6), (7), (8), and (9) of Theorem 9.1.1.

C2 If $z=r(\cos\theta+i\sin\theta)$, what is $|\overline{z}|$? What is an argument of $\overline{z}$?

C3 Use Euler's Formula to show that

(a) $\overline{e^{i\theta}}=e^{-i\theta}$

(b) $\cos\theta=\dfrac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$

(c) $\sin\theta=\dfrac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$

C4 Prove Theorem 9.1.4 and Theorem 9.1.5.

C5 Let $z_1,z_2\in\mathbb{C}$ such that z_1+z_2 and z_1z_2 are each negative real numbers. Prove that z_1 and z_2 must be real numbers.

C6 Prove that if $|z|=1$, $z\neq 1$, then $\operatorname{Re}\left(\frac{1}{1-z}\right)=\frac{1}{2}$.

C7 Use de Moivre's Formula to prove

$$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$$
$$\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$$

C8 Prove that for any $z\in\mathbb{C}$, $\operatorname{Re}(z)\leq|z|$.

C9 Derive the triangle inequality by following the steps below.

(a) Show that

$$|z_1+z_2|^2=(z_1+z_2)(\overline{z_1}+\overline{z_2})=z_1\overline{z_1}+(z_1\overline{z_2}+\overline{z_1\overline{z_2}})+z_2\overline{z_2}$$

(b) Use Theorem 9.1.1 (7) and C8 to show that

$$z_1\overline{z_2}+\overline{z_1\overline{z_2}}=2\operatorname{Re}(z_1\overline{z_2})\leq 2|z_1||z_2|$$

(c) Use the results above to obtain the inequality

$$|z_1+z_2|^2\leq(|z_1|+|z_2|)^2$$

9.2 Systems with Complex Numbers

In some applications, it is necessary to consider systems of linear equations with complex coefficients and complex right-hand sides. One physical application, discussed later in this section, is the problem of determining currents in electrical circuits with capacitors and inductive coils as well as resistance. We can solve systems with complex coefficients by using exactly the same elimination/row reduction procedures as for systems with real coefficients. Of course, our solutions will be complex, and any free variables will be allowed to take any complex value.

EXAMPLE 9.2.1

Solve the system of linear equations

$$\begin{aligned} z_1 + z_2 + z_3 &= 0 \\ (1-i)z_1 + z_2 &= i \\ (3-i)z_1 + 2z_2 + z_3 &= 1+2i \end{aligned}$$

Solution: The solution procedure is to write the augmented matrix for the system and row reduce the augmented matrix to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1-i & 1 & 0 & i \\ 3-i & 2 & 1 & 1+2i \end{array}\right] \begin{array}{l} \\ R_2-(1-i)R_1 \\ R_3-(3-i)R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & i & -1+i & i \\ 0 & -1+i & -2+i & 1+2i \end{array}\right] -iR_2 \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1+i & 1 \\ 0 & -1+i & -2+i & 1+2i \end{array}\right] \begin{array}{l} R_1-R_2 \\ \\ R_3+(1-i)R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -i & -1 \\ 0 & 1 & 1+i & 1 \\ 0 & 0 & i & 2+i \end{array}\right] \begin{array}{l} \\ \\ -iR_3 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -i & -1 \\ 0 & 1 & 1+i & 1 \\ 0 & 0 & 1 & 1-2i \end{array}\right] \begin{array}{l} R_1+iR_3 \\ R_2-(1+i)R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1+i \\ 0 & 1 & 0 & -2+i \\ 0 & 0 & 1 & 1-2i \end{array}\right]$$

Hence, the solution is $\vec{z} = \begin{bmatrix} 1+i \\ -2+i \\ 1-2i \end{bmatrix}$.

We can verify that

$$\begin{aligned} (1+i) + (-2+i) + (1-2i) &= 0 \\ (1-i)(1+i) + (-2+i) &= i \\ (3-i)(1+i) + 2(-2+i) + (1-2i) &= 1+2i \end{aligned}$$

Remark

Due to the sheer number of calculations involved in solving a system of linear equations with complex numbers, it is highly recommended that you check your answer whenever possible.

EXAMPLE 9.2.2

Solve the system

$$\begin{aligned}(1+i)z_1 + 2iz_2 &= 1\\ (1+i)z_2 + z_3 &= \frac{1}{2} - \frac{1}{2}i\\ z_1 - z_3 &= 0\end{aligned}$$

Solution: Row reducing the augmented matrix gives

$$\left[\begin{array}{ccc|c} 1+i & 2i & 0 & 1\\ 0 & 1+i & 1 & \frac{1-i}{2}\\ 1 & 0 & -1 & 0\end{array}\right]\begin{array}{l}\frac{1}{1+i}R_1\\ \frac{1}{1+i}R_2\\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1+i & 0 & \frac{1-i}{2}\\ 0 & 1 & \frac{1-i}{2} & -\frac{1}{2}i\\ 1 & 0 & -1 & 0\end{array}\right]\begin{array}{l}R_1-(1+i)R_2\\ \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & \frac{1-i}{2} & -\frac{1}{2}i\\ 1 & 0 & -1 & 0\end{array}\right]\begin{array}{l}\\ \\ R_3-R_1\end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & \frac{1-i}{2} & -\frac{1}{2}i\\ 0 & 0 & 0 & 0\end{array}\right]$$

Hence, z_3 is a free variable. We let $z_3 = t \in \mathbb{C}$. Then $z_1 = z_3 = t$, $z_2 = -\frac{1}{2}i - \frac{1-i}{2}t$, and the general solution is

$$\begin{bmatrix} z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix} 0\\ -i/2\\ 0\end{bmatrix} + t\begin{bmatrix} 1\\ -(1-i)/2\\ 1\end{bmatrix}, \quad t \in \mathbb{C}$$

We can verify that

$$\begin{aligned}(1+i)t + 2i\left(-\frac{1}{2}i - \frac{1-i}{2}t\right) &= 1\\ (1+i)\left(-\frac{1}{2}i - \frac{1-i}{2}t\right) + t &= \frac{1}{2} - \frac{1}{2}i\\ t - t &= 0\end{aligned}$$

EXERCISE 9.2.1

Solve the system

$$\begin{aligned}iz_1 + z_2 + 3z_3 &= -1-2i\\ iz_1 + iz_2 + (1+2i)z_3 &= 2+i\\ 2z_1 + (1+i)z_2 + 2z_3 &= 5-i\end{aligned}$$

Complex Numbers in Electrical Circuit Equations

For purposes of the following discussion only, we will denote by j the complex number such that $j^2 = -1$, so that we can use i to denote current.

In Section 2.4, we discussed electrical circuits with resistors. We now also consider capacitors and inductors, as well as alternating current. A simple capacitor can be thought of as two conducting plates separated by a vacuum or some dielectric. Charge can be stored on these plates, and it is found that the voltage across a capacitor at time t is proportional to the charge stored at that time:

$$V(t) = \frac{Q(t)}{C}$$

where Q is the charge and the constant C is called the *capacitance* of the capacitor.

The usual model of an inductor is a coil; because of magnetic effects, it is found that with time-varying current $i(t)$, the voltage across an inductor is proportional to the rate of change of current:

$$V(t) = L\frac{d\,i(t)}{dt}$$

where the constant of proportionality L is called the *inductance*.

As in the case of the resistor circuits, Kirchhoff's Laws applies: the sum of the voltage drops across the circuit elements must be equal to the applied electromotive force (voltage). Thus, for a simple loop with inductance L, capacitance C, resistance R, and applied electromotive force $E(t)$ (Figure 9.2.1), the circuit equation is

$$L\frac{d\,i(t)}{dt} + R\,i(t) + \frac{1}{C}Q(t) = E(t)$$

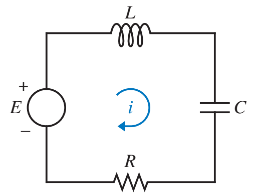


Figure 9.2.1 Kirchhoff's Voltage Law applied to an alternating current circuit.

For our purposes, it is easier to work with the derivative of this equation and use the fact that $\dfrac{dQ}{dt} = i$:

$$L\frac{d^2\,i(t)}{dt^2} + R\frac{d\,i(t)}{dt} + \frac{1}{C}i(t) = \frac{d\,E(t)}{dt}$$

In general, the solution to such an equation will involve the superposition (sum) of a steady-state solution and a transient solution. Here we will be looking only for the steady-state solution, in the special case where the applied electromotive force, and hence any current, is a single-frequency sinusoidal function. Thus, we can assume that

$$E(t) = Be^{j\omega t} \quad \text{and} \quad i(t) = Ae^{j\omega t}$$

where A and B are complex numbers that determine the amplitudes and phases of voltage and current, and ω is 2π multiplied by the frequency. Then

$$\frac{d\,i}{dt} = j\omega Ae^{j\omega t} = j\omega i$$
$$\frac{d^2 i}{dt^2} = (j\omega)^2 i = -\omega^2 i$$

and the circuit equation can be rewritten

$$-\omega^2 L\,i + j\omega R\,i + \frac{1}{C}i = \frac{dE}{dt}$$

Now consider a network of circuits with resistors, capacitors, inductors, electromotive force, and currents, as shown in Figure 9.2.2. As in Section 2.4, the currents are loops, so that the actual current across some circuit elements is the difference of two loop currents. (For example, across R_1, the actual current is $i_1 - i_2$.) From our assumption that we have only one single frequency source, we may conclude that the steady-state loop currents must be of the form

$$i_1(t) = A_1 e^{j\omega t}, \quad i_2(t) = A_2 e^{j\omega t}, \quad i_3(t) = A_3 e^{j\omega t}$$

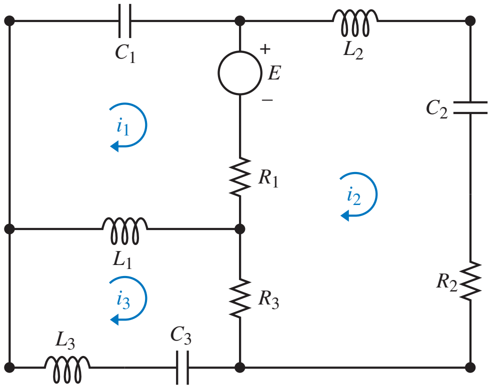


Figure 9.2.2 An alternating current network.

By applying Kirchhoff's Laws to the top-left loop, we find that

$$\left[-\omega^2 L_1(A_1 - A_3) + j\omega R_1(A_1 - A_2) + \frac{1}{C}A_1\right]e^{j\omega t} = -j\omega B e^{j\omega t}$$

If we write the corresponding equations for the other two loops, reorganize each equation, and divide out the non-zero common factor $e^{j\omega t}$, we obtain the following system of linear equations for the three variables A_1, A_2, and A_3:

$$\begin{aligned}
\left[-\omega^2 L_1 + \frac{1}{C_1} + j\omega R_1\right]A_1 - j\omega R_1 A_2 + \omega^2 L_1 A_3 &= -j\omega B\\
-j\omega R_1 A_1 + \left[-\omega^2 L_2 + \frac{1}{C_2} + j\omega(R_1 + R_2 + R_3)\right]A_2 - j\omega R_3 A_3 &= j\omega B\\
\omega^2 L_1 A_1 - j\omega R_3 A_2 + \left[-\omega^2(L_1 + L_3) + \frac{1}{C_3} + j\omega R_3\right]A_3 &= 0
\end{aligned}$$

Thus, we have a system of three linear equations with complex coefficients for the three variables A_1, A_2, and A_3. We can solve this system by standard elimination. We emphasize that this example is for illustrative purposes only: we have constructed a completely arbitrary network and provided the solution method for only part of the problem, in a special case. A much more extensive discussion is required before a reader will be ready to start examining realistic circuits to discover what they can do. But even this limited example illustrates the general point that to analyze some electrical networks, we need to solve systems of linear equations with complex coefficients.

PROBLEMS 9.2

Practice Problems

For Problems A1–A10, determine whether the system is consistent. If it is, find the general solution.

A1
$$\begin{aligned} z_1 - iz_2 &= 3 - i \\ 2z_1 + (1 - 3i)z_2 &= 8 - 2i \end{aligned}$$

A2
$$\begin{aligned} (1 - 2i)z_1 + iz_2 &= 1 + 2i \\ (-1 - 3i)z_1 + (1 + i)z_2 &= 3 - i \end{aligned}$$

A3
$$\begin{aligned} 2z_1 - 2iz_2 &= 8 - 8i \\ (1 - 2i)z_1 - 2z_2 &= -7 - 10i \end{aligned}$$

A4
$$\begin{aligned} (1 + i)z_1 + 2z_2 &= 2 + 2i \\ (1 - i)z_1 - 2iz_2 &= 2 - 2i \end{aligned}$$

A5
$$\begin{aligned} z_1 + 3z_2 + (2 - 3i)z_3 &= 9 + i \\ 2iz_1 + 7iz_2 + (7 + 4i)z_3 &= -2 + 21i \end{aligned}$$

A6
$$\begin{aligned} z_1 + iz_2 - 2z_3 &= -2i \\ iz_1 - iz_3 + (1 + i)z_4 &= 3 \\ -2iz_1 + 2iz_3 + (-2 - 2i)z_4 &= -6 \end{aligned}$$

A7
$$\begin{aligned} iz_1 - z_2 + (-1 + i)z_3 &= -2 + i \\ z_1 + (1 + i)z_2 + z_3 &= 3 + 2i \\ -iz_1 + (1 + i)z_2 + (3 - i)z_3 &= 2 + 2i \end{aligned}$$

A8
$$\begin{aligned} z_1 - 3iz_2 - 4z_3 - 7iz_4 &= -2i \\ 3iz_2 + 3z_3 + 9iz_4 &= 3i \\ 2iz_1 + 8z_2 - 10iz_3 + 20z_4 &= 6 - i \end{aligned}$$

A9
$$\begin{aligned} z_1 + iz_2 + (1 + i)z_3 &= 1 - i \\ -2z_1 + (1 - 2i)z_2 - 2z_3 &= 2i \\ 2iz_1 - 2z_2 - (2 + 3i)z_3 &= -1 + 3i \end{aligned}$$

A10
$$\begin{aligned} z_1 + (1 + i)z_2 + 2z_3 + z_4 &= 1 - i \\ 2z_1 + (2 + i)z_2 + 5z_3 + (2 + i)z_4 &= 4 - i \\ iz_1 + (-1 + i)z_2 + (1 + 2i)z_3 + 2iz_4 &= 1 \end{aligned}$$

Homework Problems

For Problems B1–B9, determine whether the system is consistent. If it is, find the general solution.

B1
$$\begin{aligned} iz_1 + (-1 - 2i)z_2 &= i \\ 2z_1 + (-3 + 4i)z_2 &= 7 \end{aligned}$$

B2
$$\begin{aligned} z_1 + (-2 + 2i)z_2 &= 4 + 5i \\ iz_2 - 2z_2 &= -1 + 4i \end{aligned}$$

B3
$$\begin{aligned} z_1 - 2iz_2 + (1 - i)z_3 &= 2 - i \\ 3z_1 - 6iz_2 + (4 - 2i)z_3 &= 7 - 2i \end{aligned}$$

B4
$$\begin{aligned} z_1 + 3z_2 + z_3 &= 8 \\ -2z_1 - 3z_2 + z_3 &= -10 \end{aligned}$$

B5
$$\begin{aligned} z_1 - z_2 + 2z_3 &= 3 \\ 3iz_1 - 3iz_2 + (1 + 6i)z_3 &= -2 + 9i \\ iz_1 - iz_2 + 3iz_3 &= 1 + i \end{aligned}$$

B6
$$\begin{aligned} z_1 + iz_2 + 2z_3 + (-2 + 2i)z_4 &= 6 + i \\ z_1 + iz_2 + (3 + i)z_3 + (-4 + 2i)z_4 &= 9 + 4i \\ iz_1 - z_2 + 2iz_3 + (-2 - 2i)z_4 &= -1 + 6i \end{aligned}$$

B7
$$\begin{aligned} iz_1 - z_2 + 2iz_3 &= -1 - i \\ (1 + i)z_1 + 2iz_2 + (5 + 2i)z_3 &= -2 - i \\ -2iz_1 - 2iz_2 - (6 + 4i)z_3 &= 3 + 3i \end{aligned}$$

B8
$$\begin{aligned} z_1 + iz_2 + (-1 + 2i)z_3 &= -1 + 2i \\ z_2 + 2z_3 &= 2 + 2i \\ 2z_1 + (-1 + 2i)z_2 + (-6 + 4i)z_3 &= -4 \end{aligned}$$

B9
$$\begin{aligned} z_1 - iz_2 &= 1 + i \\ (1 + i)z_1 + (1 - i)z_2 + (1 - i)z_3 &= 1 + 3i \\ 2z_1 - 2iz_2 + (3 + i)z_3 &= 1 + 5i \end{aligned}$$

9.3 Complex Vector Spaces

The definition of a vector space in Section 4.2 is given in the case where the scalars are real numbers. In fact, the definition makes sense when the scalars are taken from any one system of numbers such that addition, subtraction, multiplication, and division are defined for any pairs of numbers (excluding division by 0) and satisfy the usual commutative, associative, and distributive rules for doing arithmetic. Thus, the vector space axioms make sense if we allow the scalars to be the set of complex numbers. In such cases, we say that we have a **vector space over** $\mathbb{C}$, or a **complex vector space**.

Definition
Complex Vector Space

A set $\mathbb{V}$ with operations of addition and scalar multiplication is called a **complex vector space** if for any $\mathbf{v}, \mathbf{z}, \mathbf{w} \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{C}$ we have:

V1 $\mathbf{z} + \mathbf{w} \in \mathbb{V}$
V2 $\mathbf{z} + \mathbf{w} = \mathbf{w} + \mathbf{z}$
V3 $(\mathbf{z} + \mathbf{w}) + \mathbf{v} = \mathbf{z} + (\mathbf{w} + \mathbf{v})$
V4 There exists a vector $\mathbf{0} \in \mathbb{V}$ such that $\mathbf{z} + \mathbf{0} = \mathbf{z}$ for all $\mathbf{z} \in \mathbb{V}$.
V5 For each $\mathbf{z} \in \mathbb{V}$, there exists $(-\mathbf{z}) \in \mathbb{V}$ such that $\mathbf{z} + (-\mathbf{z}) = \mathbf{0}$.
V6 $\alpha\mathbf{z} \in \mathbb{V}$
V7 $\alpha(\beta\mathbf{z}) = (\alpha\beta)\mathbf{z}$
V8 $(\alpha + \beta)\mathbf{z} = \alpha\mathbf{z} + \beta\mathbf{z}$
V9 $\alpha(\mathbf{z} + \mathbf{w}) = \alpha\mathbf{z} + \alpha\mathbf{w}$
V10 $1\mathbf{z} = \mathbf{z}$

For complex vector spaces, the analog of $\mathbb{R}^n$ is $\mathbb{C}^n$ and the analog of $M_{m\times n}(\mathbb{R})$ is $M_{m\times n}(\mathbb{C})$.

Definition
$\mathbb{C}^n$

The complex vector space $\mathbb{C}^n$ is defined to be the set

$$\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \mid z_1, \dots, z_n \in \mathbb{C} \right\}$$

with addition and scalar multiplication defined in the expected way.

Definition
$M_{m\times n}(\mathbb{C})$

The set $M_{m\times n}(\mathbb{C})$ of all $m \times n$ matrices with complex entries is a complex vector space with standard addition and complex scalar multiplication of matrices.

All of the definitions and theorems regarding linear independence, spanning, subspaces, bases, dimension, coordinates, and determinants remain the same in complex vector spaces, with the exception that the scalars are now allowed to take any complex value.

It is instructive to look carefully at the idea of a basis for complex vector spaces.

EXAMPLE 9.3.1

Find a basis for $\mathbb{C}^1$ as a complex vector space and determine its dimension.

Solution: It is tempting for students to immediately write down $\{1, i\}$ as a basis for $\mathbb{C}^1$. However, this is not a basis since it is linearly dependent. In particular,

$$-i(1) + 1(i) = 0$$

The key is to remember that in a complex vector space we now allow the use of complex scalars. Thus, in fact, a basis for $\mathbb{C}^1$ is $\{1\}$. Indeed, every complex number z can be written in the form

$$z = \alpha 1$$

by taking $\alpha = z$.

Hence, we see that $\mathbb{C}^1$ has a basis consisting of one element, so $\mathbb{C}^1$ is a one-dimensional complex vector space.

In general, we get that the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$ is also the standard basis for $\mathbb{C}^n$. Similarly, the standard basis for $M_{m\times n}(\mathbb{C})$ is the same as that for $M_{m\times n}(\mathbb{R})$.

We can also extend the definition of a linear mapping $L : \mathbb{V} \to \mathbb{W}$ to the case where $\mathbb{V}$ and $\mathbb{W}$ are both vector spaces over the complex numbers, as well as the definition of the matrix of a linear mapping with respect to bases $\mathcal{B}$ and $\mathcal{C}$.

EXAMPLE 9.3.2

Let $\mathbf{z} = \begin{bmatrix} 1 \\ 2i \\ 1-i \end{bmatrix}$ and let $L : \mathbb{C}^3 \to \mathbb{C}^2$ be the linear mapping defined by

$$L(z_1, z_2, z_3) = \Big((1+i)z_1 - 2iz_2 + (1+2i)z_3, 2z_1 + (1-i)z_2 + (3+i)z_3\Big)$$

Find the standard matrix of L and use it to compute $L(\mathbf{z})$.

Solution: We have that

$$L(1,0,0) = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \quad L(0,1,0) = \begin{bmatrix} -2i \\ 1-i \end{bmatrix}, \quad L(0,0,1) = \begin{bmatrix} 1+2i \\ 3+i \end{bmatrix}$$

Hence, the standard matrix of L is

$$[L] = \begin{bmatrix} 1+i & -2i & 1+2i \\ 2 & 1-i & 3+i \end{bmatrix}$$

Therefore,

$$L(\mathbf{z}) = [L]\mathbf{z} = \begin{bmatrix} 1+i & -2i & 1+2i \\ 2 & 1-i & 3+i \end{bmatrix} \begin{bmatrix} 1 \\ 2i \\ 1-i \end{bmatrix} = \begin{bmatrix} 8+2i \\ 8 \end{bmatrix}$$

Complex Conjugate

Since the complex conjugate is so useful in $\mathbb{C}$, we extend the definition of a complex conjugate to vectors in $\mathbb{C}^n$ and matrices in $M_{m\times n}(\mathbb{C})$.

Definition
Complex Conjugate

The **complex conjugate** of $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$ is defined to be $\overline{\mathbf{z}} = \begin{bmatrix} \overline{z_1} \\ \vdots \\ \overline{z_n} \end{bmatrix}$.

The **complex conjugate** of $Z = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \vdots \\ z_{m1} & \cdots & z_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{C})$ is defined to be

$$\overline{Z} = \begin{bmatrix} \overline{z_{11}} & \cdots & \overline{z_{1n}} \\ \vdots & & \vdots \\ \overline{z_{m1}} & \cdots & \overline{z_{mn}} \end{bmatrix}$$

EXAMPLE 9.3.3

Let $\mathbf{z} = \begin{bmatrix} 1+i \\ -2i \\ 3 \\ 1-3i \end{bmatrix}$ and $W = \begin{bmatrix} 1 & 1-3i \\ -i & 2i \end{bmatrix}$. Calculate $\overline{\mathbf{z}}$ and $\overline{W}$.

Solution: We have

$$\overline{\mathbf{z}} = \begin{bmatrix} \overline{1+i} \\ \overline{-2i} \\ \overline{3} \\ \overline{1-3i} \end{bmatrix} = \begin{bmatrix} 1-i \\ 2i \\ 3 \\ 1+3i \end{bmatrix} \quad \text{and} \quad \overline{W} = \begin{bmatrix} \overline{1} & \overline{1-3i} \\ \overline{-i} & \overline{2i} \end{bmatrix} = \begin{bmatrix} 1 & 1+3i \\ i & -2i \end{bmatrix}$$

We will find the following property useful in Section 9.4.

Theorem 9.3.1 If $Z \in M_{m\times n}(\mathbb{C})$ and $\mathbf{w} \in \mathbb{C}^n$, then $\overline{Z\mathbf{w}} = \overline{Z}\,\overline{\mathbf{w}}$.

Since we will frequently need to take both the conjugate and the transpose of a vector in $\mathbb{C}^n$ or for a matrix in $M_{m\times n}(\mathbb{C})$, we invent some notation for this.

Definition
Conjugate Transpose

Let $\mathbf{z} \in \mathbb{C}^n$ and $Z \in M_{m\times n}(\mathbb{C})$. We define the **conjugate transpose** of $\mathbf{z}$ and Z by

$$\mathbf{z}^* = \overline{\mathbf{z}}^T \quad \text{and} \quad Z^* = \overline{Z}^T$$

EXERCISE 9.3.1

Let $Z = \begin{bmatrix} 1+i & 1-2i & i \\ 2 & -i & 3+i \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -1-i \\ 2+i \end{bmatrix}$. Find Z^* and $\mathbf{z}^*$.

The conjugate transpose has the following properties, which follow directly from the properties of the transpose and complex conjugates.

Theorem 9.3.2 If $Z, W \in M_{n\times n}(\mathbb{C})$, $\mathbf{z} \in \mathbb{C}^n$, and $\alpha \in \mathbb{C}$, then

(1) $Z^{**} = Z$
(2) $(Z + W)^* = Z^* + W^*$
(3) $(\alpha Z)^* = \overline{\alpha} Z^*$
(4) $(ZW)^* = W^* Z^*$
(5) $(Z\mathbf{z})^* = \mathbf{z}^* Z^*$

Hermitian Inner Product Spaces

We would like to have an inner product defined for complex vector spaces because the concepts of length, orthogonality, and projection are powerful tools for solving certain problems.

Our first thought would be to determine if we can extend the dot product to $\mathbb{C}^n$. Does this define an inner product on $\mathbb{C}^n$? Let $\mathbf{z} = \vec{x} + i\vec{y}$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$, then we have

$$\begin{aligned}\mathbf{z} \cdot \mathbf{z} &= z_1^2 + \cdots + z_n^2 \\ &= (x_1^2 + \cdots + x_n^2 - y_1^2 - \cdots - y_n^2) + 2i(x_1y_1 + \cdots + x_ny_n)\end{aligned}$$

Observe that $\mathbf{z} \cdot \mathbf{z}$ does not even need to be a real number and so the condition $\mathbf{z} \cdot \mathbf{z} \geq 0$ does not even make sense. Thus *we cannot use the dot product as a rule for defining an inner product in* $\mathbb{C}^n$.

As in the real case, we want $\langle \mathbf{z}, \mathbf{z} \rangle$ to be a non-negative real number so that we can define the length of a vector by $\|\mathbf{z}\| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}$. We recall that if $z \in \mathbb{C}$, then $\overline{z}z = |z|^2 \geq 0$. Hence, it makes sense to choose

$$\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\mathbf{z}} \cdot \mathbf{w}$$

as this gives us

$$\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\mathbf{z}} \cdot \mathbf{z} = \overline{z_1}z_1 + \cdots + \overline{z_n}z_n = |z_1|^2 + \cdots + |z_n|^2$$

which is a non-negative real number.

Definition
Standard Inner Product on $\mathbb{C}^n$

In $\mathbb{C}^n$ the **standard inner product** $\langle\ ,\ \rangle$ is defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\mathbf{z}} \cdot \mathbf{w} = \overline{z_1}w_1 + \cdots + \overline{z_n}w_n, \qquad \text{for } \mathbf{z}, \mathbf{w} \in \mathbb{C}^n$$

Remark

Here we have used the definition of the standard inner product commonly used in science, engineering, and most mathematical software. It is important to note that many mathematics textbooks will use the definition

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{z} \cdot \overline{\mathbf{w}}$$

for the definition of the standard inner product for $\mathbb{C}^n$.

EXAMPLE 9.3.4

Let $\mathbf{u} = \begin{bmatrix} 1+i \\ 2-i \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2+i \\ 3+2i \end{bmatrix}$. Determine $\langle \mathbf{v}, \mathbf{u} \rangle$, $\langle \mathbf{u}, \mathbf{v} \rangle$, and $\langle (2-i)\mathbf{u}, \mathbf{v} \rangle$.

Solution: We have

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{u} \rangle &= \overline{\mathbf{v}} \cdot \mathbf{u} \\
&= \begin{bmatrix} -2-i \\ 3-2i \end{bmatrix} \cdot \begin{bmatrix} 1+i \\ 2-i \end{bmatrix} \\
&= (-2-i)(1+i) + (3-2i)(2-i) \\
&= 3 - 10i \\
\langle \mathbf{u}, \mathbf{v} \rangle &= \begin{bmatrix} 1-i \\ 2+i \end{bmatrix} \cdot \begin{bmatrix} -2+i \\ 3+2i \end{bmatrix} \\
&= 3 + 10i \\
\langle (2-i)\mathbf{u}, \mathbf{v} \rangle &= \overline{(2-i)\mathbf{u}} \cdot \mathbf{v} = \overline{(2-i)\begin{bmatrix} 1+i \\ 2-i \end{bmatrix}} \cdot \begin{bmatrix} -2+i \\ 3+2i \end{bmatrix} \\
&= (2+i)\begin{bmatrix} 1-i \\ 2+i \end{bmatrix} \cdot \begin{bmatrix} -2+i \\ 3+2i \end{bmatrix} \\
&= (2+i)(3+10i) \\
&= -4 + 23i
\end{aligned}$$

Observe that this does not satisfy the properties of the real inner product. In particular, $\langle \mathbf{u}, \mathbf{v} \rangle \neq \langle \mathbf{v}, \mathbf{u} \rangle$ and $\langle \alpha\mathbf{u}, \mathbf{v} \rangle \neq \alpha \langle \mathbf{u}, \mathbf{v} \rangle$.

EXERCISE 9.3.2

Let $\mathbf{u} = \begin{bmatrix} i \\ 1+2i \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2+2i \\ 1-3i \end{bmatrix}$. Determine $\langle \mathbf{u}, \mathbf{v} \rangle$, $\langle 2i\mathbf{u}, \mathbf{v} \rangle$, and $\langle \mathbf{u}, 2i\mathbf{v} \rangle$.

We saw in Chapter 8 that the formula

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$$

was very useful. For the standard inner product on $\mathbb{C}^n$ we get

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{z}^* \mathbf{w}$$

We also get the following theorem.

Theorem 9.3.3

If $A \in M_{n \times n}(\mathbb{C})$ and $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$, then

$$\langle \mathbf{w}, A\mathbf{z} \rangle = \langle A^* \mathbf{w}, \mathbf{z} \rangle$$

Proof: We have

$$\langle \mathbf{w}, A\mathbf{z} \rangle = \mathbf{w}^* A \mathbf{z} = \mathbf{w}^* (A^*)^* \mathbf{z} = (A^* \mathbf{w})^* \mathbf{z} = \langle A^* \mathbf{w}, \mathbf{z} \rangle$$

■

Properties of Complex Inner Products

Example 9.3.4 warns us that for complex vector spaces, we must modify the requirements of symmetry and bilinearity stated for real inner products.

Definition

Complex Inner Product

Complex Inner Product Space

Let $\mathbb{V}$ be a vector space over $\mathbb{C}$. A **complex inner product** on $\mathbb{V}$ is a function $\langle\, ,\rangle : \mathbb{V}\times\mathbb{V}\to\mathbb{C}$ such that

(1) $\langle \mathbf{z},\mathbf{z}\rangle \ \geq 0$ for all $\mathbf{z}\in\mathbb{V}$ and $\langle \mathbf{z},\mathbf{z}\rangle = 0$ if and only if $\mathbf{z}=\mathbf{0}$

(2) $\langle \mathbf{z},\mathbf{w}\rangle \ = \ \overline{\langle \mathbf{w},\mathbf{z}\rangle}$ for all $\mathbf{w},\mathbf{z}\in\mathbb{V}$

(3) For all $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z}\in\mathbb{V}$ and $\alpha\in\mathbb{C}$

$$\begin{aligned}
&\text{(i)} & \langle \mathbf{v}+\mathbf{z},\mathbf{w}\rangle &= \langle \mathbf{v},\mathbf{w}\rangle + \langle \mathbf{z},\mathbf{w}\rangle\\
&\text{(ii)} & \langle \mathbf{z},\mathbf{w}+\mathbf{u}\rangle &= \langle \mathbf{z},\mathbf{w}\rangle + \langle \mathbf{z},\mathbf{u}\rangle\\
&\text{(iii)} & \langle \alpha\mathbf{z},\mathbf{w}\rangle &= \overline{\alpha}\langle \mathbf{z},\mathbf{w}\rangle\\
&\text{(iv)} & \langle \mathbf{z},\alpha\mathbf{w}\rangle &= \alpha\langle \mathbf{z},\mathbf{w}\rangle
\end{aligned}$$

An complex vector space with a complex inner product is called a **complex inner product space**.

EXERCISE 9.3.3 Verify that the standard inner product on $\mathbb{C}^n$ is a complex inner product.

Remarks

1. Property (2) is the **Hermitian** property of the inner product.
2. Property (3) says that the complex inner product is not quite bilinear. However, this property reduces to bilinearity when the scalars are all real.
3. A complex inner product is sometimes called a Hermitian inner product. A complex inner product space may also be called a Hermitian inner product space or a unitary space.

EXAMPLE 9.3.5 The complex vector space $M_{m\times n}(\mathbb{C})$ can be made into a complex inner product space by adding the complex inner product defined by

$$\langle Z,W\rangle = \operatorname{tr}(Z^*W)$$

EXAMPLE 9.3.6 On the complex vector space $C[a,b]$ of complex-valued functions of a real variable x that are continuous on the closed interval $[a,b]$, we often use the complex inner product defined by

$$\langle \mathbf{f},\mathbf{g}\rangle = \int_{\mathbf{a}}^{\mathbf{b}} \overline{\mathbf{f}(\mathbf{x})}\mathbf{g}(\mathbf{x})\,\mathbf{dx}$$

Length and Orthogonality

We can now define length and orthogonality to match the definitions in the real case.

Definition
Length
Unit Vector

Let $\mathbb{V}$ be a Hermitian inner product space. We define the **length** of $\mathbf{z} \in \mathbb{V}$ by

$$\|\mathbf{z}\| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}$$

If $\|\mathbf{z}\| = 1$, then $\mathbf{z}$ is called a **unit vector**.

Definition
Orthogonality

Let $\mathbb{V}$ be a Hermitian inner product space. For any $\mathbf{z}, \mathbf{w} \in \mathbb{V}$ we say that $\mathbf{z}$ and $\mathbf{w}$ are **orthogonal** if $\langle \mathbf{z}, \mathbf{w} \rangle = 0$.

Of course, these satisfy all of our familiar properties of length and orthogonality.

Theorem 9.3.4

If $\mathbb{V}$ is a Hermitian inner product space, then for any $\mathbf{z}, \mathbf{w} \in \mathbb{V}$ and $\alpha \in \mathbb{C}$ we have

(1) $\|\alpha \mathbf{z}\| = |\alpha| \, \|\mathbf{z}\|$
(2) $\frac{1}{\|\mathbf{z}\|}\mathbf{z}$ is a unit vector.
(3) $|\langle \mathbf{z}, \mathbf{w} \rangle| \leq \|\mathbf{z}\| \, \|\mathbf{w}\|$
(4) $\|\mathbf{z} + \mathbf{w}\| \leq \|\mathbf{z}\| + \|\mathbf{w}\|$

Proof: We prove (3) and leave the proof of (2) and (4) as Problem C2 and Problem C3 respectively.

If $\mathbf{w} = \mathbf{0}$, then (3) is immediate, so assume that $\mathbf{w} \neq \mathbf{0}$, and let $\alpha = \dfrac{\overline{\langle \mathbf{z}, \mathbf{w} \rangle}}{\langle \mathbf{w}, \mathbf{w} \rangle}$. Then, we get

$$\begin{aligned}
0 &\leq \langle \mathbf{z} - \alpha \mathbf{w}, \mathbf{z} - \alpha \mathbf{w} \rangle \\
&= \langle \mathbf{z}, \mathbf{z} - \alpha \mathbf{w} \rangle + \langle -\alpha \mathbf{w}, \mathbf{z} - \alpha \mathbf{w} \rangle \\
&= \langle \mathbf{z}, \mathbf{z} \rangle + \langle \mathbf{z}, -\alpha \mathbf{w} \rangle + \langle -\alpha \mathbf{w}, \mathbf{z} \rangle + \langle -\alpha \mathbf{w}, -\alpha \mathbf{w} \rangle \\
&= \langle \mathbf{z}, \mathbf{z} \rangle - \alpha \langle \mathbf{z}, \mathbf{w} \rangle - \overline{\alpha} \langle \mathbf{w}, \mathbf{z} \rangle + \overline{\alpha} \alpha \langle \mathbf{w}, \mathbf{w} \rangle \\
&= \langle \mathbf{z}, \mathbf{z} \rangle - \frac{|\langle \mathbf{z}, \mathbf{w} \rangle|^2}{\langle \mathbf{w}, \mathbf{w} \rangle} - \frac{|\langle \mathbf{z}, \mathbf{w} \rangle|^2}{\langle \mathbf{w}, \mathbf{w} \rangle} + \frac{|\langle \mathbf{z}, \mathbf{w} \rangle|^2}{\langle \mathbf{w}, \mathbf{w} \rangle} \\
&= \|\mathbf{z}\|^2 - \frac{|\langle \mathbf{z}, \mathbf{w} \rangle|^2}{\|\mathbf{w}\|^2}
\end{aligned}$$

and (3) follows. ■

EXERCISE 9.3.4

Let $\mathbb{V}$ be a Hermitian inner product space. Prove that for all $\mathbf{z} \in \mathbb{V}$ and $\alpha \in \mathbb{C}$ we have

$$\|\alpha \mathbf{z}\| = |\alpha| \, \|\mathbf{z}\|$$

Definition
Orthogonal Set
Orthonormal Set

Let $\mathcal{B} = \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ be a set in a Hermitian inner product space. $\mathcal{B}$ is said to be an **orthogonal set** if $\langle \mathbf{z}_\ell, \mathbf{z}_j \rangle = 0$ for all $\ell \neq j$. $\mathcal{B}$ is said to be an **orthonormal set** if it is an orthogonal set and $\|\mathbf{z}_j\| = 1$ for $1 \leq j \leq k$.

The formulas for finding coordinates with respect to an orthogonal basis, for performing the Gram-Schmidt Procedure, and for finding projections and perpendiculars are still valid for Hermitian inner products. However, be careful to remember that a Hermitian product is not symmetric, so you need to have the vectors in the inner products in the correct order.

EXERCISE 9.3.5

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthogonal basis for a Hermitian inner product space $\mathbb{V}$. Prove that if $\mathbf{z} \in \mathbb{V}$, then

$$\mathbf{z} = \frac{\langle \mathbf{v}_1, \mathbf{z}\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \cdots + \frac{\langle \mathbf{v}_n, \mathbf{z}\rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

EXAMPLE 9.3.7

Let $\mathbf{z}_1 = \begin{bmatrix} i \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{z}_2 = \begin{bmatrix} 1 \\ 1 \\ -i \\ i \end{bmatrix}$, and $\mathbf{z}_3 = \begin{bmatrix} 1 \\ i \\ 1 \\ 0 \end{bmatrix}$ and consider $\mathbb{S} = \operatorname{Span}\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ in $\mathbb{C}^4$.

(a) Use the Gram-Schmidt Procedure to find an orthogonal basis for $\mathbb{S}$.

Solution: **First Step:** Let $\mathbf{w}_1 = \mathbf{z}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\mathbf{w}_1\}$.
Second Step: Determine $\operatorname{proj}_{\mathbb{S}_1}(\mathbf{z}_2)$.

$$\mathbf{w}_2 = \mathbf{z}_2 - \frac{\langle \mathbf{w}_1, \mathbf{z}_2\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -i \\ i \end{bmatrix} - \frac{-i}{1}\begin{bmatrix} i \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -i \\ i \end{bmatrix}$$

Third Step: Determine $\operatorname{proj}_{\mathbb{S}_2}(\mathbf{z}_3)$ where $\mathbb{S}_2 = \operatorname{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

$$\operatorname{proj}_{\mathbb{S}_2}(\mathbf{z}_3) = \mathbf{z}_3 - \frac{\langle \mathbf{w}_1, \mathbf{z}_3\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2, \mathbf{z}_3\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{bmatrix} 1 \\ i \\ 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} i \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{2i}{3}\begin{bmatrix} 0 \\ 1 \\ -i \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ i/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

We take $\mathbf{w}_3 = \begin{bmatrix} 0 \\ i \\ 1 \\ 2 \end{bmatrix}$ and get that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis for $\mathbb{S}$.

(b) Let $\mathbf{v} = \begin{bmatrix} 1 \\ -i \\ 1 \\ -1 \end{bmatrix}$. Find $\operatorname{proj}_{\mathbb{S}}(\mathbf{v})$.

Solution: Using the orthogonal basis we found in (a), we get

$$\begin{aligned}\operatorname{proj}_{\mathbb{S}}(\mathbf{v}) &= \frac{\langle \mathbf{w}_1, \mathbf{v}\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 + \frac{\langle \mathbf{w}_2, \mathbf{v}\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 + \frac{\langle \mathbf{w}_3, \mathbf{v}\rangle}{\|\mathbf{w}_3\|^2}\mathbf{w}_3 \\ &= -i\begin{bmatrix} i \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{i}{3}\begin{bmatrix} 0 \\ 1 \\ -i \\ i \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 0 \\ i \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\end{aligned}$$

When working with real inner products, we saw that orthogonal matrices are very important. So, we also consider the complex version of these matrices.

Definition
Unitary Matrix

A matrix in $M_{n\times n}(\mathbb{C})$ is said to be **unitary** if its columns form an orthonormal basis for $\mathbb{C}^n$.

For an orthogonal matrix P, we saw that the defining property is equivalent to the matrix condition $P^{-1} = P^T$. We get the associated result for unitary matrices.

Theorem 9.3.5

If $U \in M_{n\times n}(\mathbb{C})$, then the following are equivalent.

(1) The columns of U form an orthonormal basis for $\mathbb{C}^n$.
(2) $U^{-1} = U^*$.
(3) The rows of U form an orthonormal basis for $\mathbb{C}^n$.

Proof: Let $U = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix}$. We have that

$$(U^*U)_{jk} = \mathbf{u}_j^*\mathbf{u}_k = \langle \mathbf{u}_j, \mathbf{u}_k \rangle$$

Hence, $U^*U = I$ if and only if $\langle \mathbf{u}_j, \mathbf{u}_j \rangle = 1$ for all j, and $\langle \mathbf{u}_j, \mathbf{u}_k \rangle = 0$ for all $j \neq k$. This is true if and only if the columns of U form an orthonormal basis for $\mathbb{C}^n$. The proof that (2) is equivalent to (3) is similar. ■

Remark

Observe that if the entries of A are all real, then A is unitary if and only if it is orthogonal.

EXAMPLE 9.3.8

Are the matrices $U = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$ and $V = \begin{bmatrix} \frac{1}{\sqrt{3}}(1+i) & \frac{1}{\sqrt{6}}(1+i) \\ -\frac{1}{\sqrt{3}}i & \frac{2}{\sqrt{6}}i \end{bmatrix}$ unitary?

Solution: Observe that

$$\left\langle \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix} \right\rangle = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = 1(1) + (-i)i = 2$$

Hence, $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is not a unit vector. Thus, U is not unitary.

For V, we have $V^* = \begin{bmatrix} \frac{1}{\sqrt{3}}(1-i) & \frac{1}{\sqrt{3}}i \\ \frac{1}{\sqrt{6}}(1-i) & -\frac{2}{\sqrt{6}}i \end{bmatrix}$, so

$$V^*V = \begin{bmatrix} \frac{1}{3}(2+1) & \frac{1}{3\sqrt{2}}(2-2) \\ \frac{1}{3\sqrt{2}}(2-2) & \frac{1}{6}(2+4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, V is unitary.

PROBLEMS 9.3

Practice Problems

For Problems, A1–A4, evaluate the expression.

A1 $\begin{bmatrix} -2+i \\ 1 \end{bmatrix} - \begin{bmatrix} 3+4i \\ 1-i \end{bmatrix}$

A2 $2i\begin{bmatrix} 2+5i \\ 3-2i \end{bmatrix}$

A3 $\begin{bmatrix} 2-i \\ 3+i \\ 2-5i \end{bmatrix} + \begin{bmatrix} 3-2i \\ 4+7i \\ -3-4i \end{bmatrix}$

A4 $(-1-2i)\begin{bmatrix} 2-i \\ 3+i \\ 2-5i \end{bmatrix}$

A5 (a) Write the standard matrix of the linear mapping $L : \mathbb{C}^2 \to \mathbb{C}^2$ defined by
$$L(z_1, z_2) = ((1+2i)z_1 + (3+i)z_2, z_1 + (1-i)z_2)$$
(b) Determine $L(2+3i, 1-4i)$.
(c) Find a basis for the range and nullspace of L.

For Problems A6–A9, use the standard inner product on $\mathbb{C}^2$ to calculate $\langle \mathbf{u}, \mathbf{v}\rangle$, $\langle \mathbf{v}, \mathbf{u}\rangle$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$.

A6 $\mathbf{u} = \begin{bmatrix} 2+3i \\ -1-2i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2i \\ 2-5i \end{bmatrix}$

A7 $\mathbf{u} = \begin{bmatrix} 1-i \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$

A8 $\mathbf{u} = \begin{bmatrix} -1+4i \\ 2-i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3+i \\ 1+3i \end{bmatrix}$

A9 $\mathbf{u} = \begin{bmatrix} 1+2i \\ -1-3i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -i \\ -2i \end{bmatrix}$

For Problems A10 and A11, verify that $(ZW)^* = W^*Z^*$.

A10 $Z = \begin{bmatrix} 1 & i \\ 1-i & 2 \end{bmatrix}, W = \begin{bmatrix} 0 & -i \\ 2+i & i \end{bmatrix}$

A11 $Z = \begin{bmatrix} 1+i & 1 \\ -i & i \end{bmatrix}, W = \begin{bmatrix} 2i & i \\ 1 & -i \end{bmatrix}$

For Problems A12–A15, determine whether the matrix is unitary.

A12 $A = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$

A13 $A = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -1/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$

A14 $A = \begin{bmatrix} 1/\sqrt{3} & (1+i)/\sqrt{3} \\ (1+i)/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

A15 $A = \begin{bmatrix} (-1+i)/\sqrt{3} & (1-i)/\sqrt{6} \\ 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$

For Problems A16 and A17, use the Gram-Schmidt Procedure to find an orthogonal basis for $\mathbb{S} = \operatorname{Span}\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ under the standard inner product of $\mathbb{C}^3$.

A16 $\mathbf{z}_1 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 1 \\ -2i \\ -1 \end{bmatrix}, \mathbf{z}_3 = \begin{bmatrix} 0 \\ i \\ 3 \end{bmatrix}$

A17 $\mathbf{z}_1 = \begin{bmatrix} 1+i \\ 1-i \\ 1 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 1+i \\ 2i \\ 3i \end{bmatrix}, \mathbf{z}_3 = \begin{bmatrix} i \\ 2i \\ 2 \end{bmatrix}$

For Problems A18 and A19, compute $\operatorname{proj}_{\mathbb{S}}(\mathbf{z})$ under the standard inner product of $\mathbb{C}^3$.

A18 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix}\right\}, \mathbf{z} = \begin{bmatrix} 1+i \\ 2+i \\ 3+i \end{bmatrix}$

A19 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix}, \begin{bmatrix} -2i \\ -1-i \\ 2-i \end{bmatrix}\right\}, \mathbf{z} = \begin{bmatrix} -4 \\ 3 \\ -1+i \end{bmatrix}$

For Problems A20–A23, let $\mathbf{z}$ and $\mathbf{w}$ be vectors in a complex inner product space such that $\langle \mathbf{z}, \mathbf{w}\rangle = 1+2i$. Evaluate the expression.

A20 $\langle \mathbf{w}, \mathbf{z}\rangle$

A21 $\langle (1+i)\mathbf{z}, \mathbf{w}\rangle$

A22 $\langle \mathbf{w}, (1+2i)\mathbf{z}\rangle$

A23 $\langle i\mathbf{z}, -2i\mathbf{w}\rangle$

A24 (a) Prove that a unitary matrix U satisfies $|\det U| = 1$.
(b) Give a 2×2 unitary matrix U such that $\det U \neq \pm 1$.

Homework Problems

For Problems, B1–B6, evaluate the expression.

B1 $\begin{bmatrix} 1+2i \\ 1-i \end{bmatrix} + \begin{bmatrix} 2-i \\ 3-i \end{bmatrix}$

B2 $(1+i)\begin{bmatrix} 2i \\ -1+3i \end{bmatrix}$

B3 $2\begin{bmatrix} 1 \\ -i \\ 1+i \end{bmatrix} - i\begin{bmatrix} i \\ -i \\ 2 \end{bmatrix}$

B4 $(-1-2i)\begin{bmatrix} 2-i \\ i \\ -3i \end{bmatrix}$

B5 $i\begin{bmatrix} 2-i \\ 3+i \\ 2-5i \end{bmatrix} + (1-i)\begin{bmatrix} i \\ -2i \\ 3 \end{bmatrix}$

B6 $(2-3i)\begin{bmatrix} 1+i \\ -2+2i \\ -1+3i \end{bmatrix}$

B7 (a) Write the standard matrix of the linear mapping $L : \mathbb{C}^2 \to \mathbb{C}^2$ defined by
$$L(z_1, z_2) = ((1-i)z_1 + (1+3i)z_2, 2iz_1 + (2-i)z_2)$$
(b) Determine $L(i, 2+3i)$.
(c) Find a basis for the range and nullspace of L.

B8 (a) Write the standard matrix of the linear mapping $L : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$L(z_1, z_2) = ((1-i)z_1 + -iz_2, -2iz_1 + (-1-i)z_2)$$

(b) Find a basis for the range and nullspace of L.

For Problems B9–B13, use the standard inner product on $\mathbb{C}^2$ to calculate $\langle \mathbf{u}, \mathbf{v}\rangle$, $\langle \mathbf{v}, \mathbf{u}\rangle$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$.

B9 $\mathbf{u} = \begin{bmatrix} 2 \\ i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3i \\ i \end{bmatrix}$

B10 $\mathbf{u} = \begin{bmatrix} 1+2i \\ -i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$

B11 $\mathbf{u} = \begin{bmatrix} 2+i \\ 1-i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1+3i \\ -5 \end{bmatrix}$

B12 $\mathbf{u} = \begin{bmatrix} 1 \\ 1+i \\ -2i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} i \\ 3i \\ 1-2i \end{bmatrix}$

B13 $\mathbf{u} = \begin{bmatrix} 1-2i \\ 1+i \\ i \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1+2i \\ 1-i \\ -i \end{bmatrix}$

For Problems B14 and B15, verify that $(ZW)^* = W^*Z^*$.

B14 $Z = \begin{bmatrix} 1 & 1+2i \\ i & 1-2i \end{bmatrix}, W = \begin{bmatrix} -1-i & i \\ -1 & 1 \end{bmatrix}$

B15 $Z = \begin{bmatrix} 2-3i & 1+2i \\ 0 & -2 \end{bmatrix}, W = \begin{bmatrix} 3+i & -i \\ 0 & 1+i \end{bmatrix}$

For Problems B16–B19, determine whether the matrix is unitary.

B16 $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$

B17 $\begin{bmatrix} (1+2i)/\sqrt{6} & -i/\sqrt{6} \\ (1+i)/\sqrt{6} & (1-i)/\sqrt{6} \end{bmatrix}$

B18 $\begin{bmatrix} (1-2i)/\sqrt{10} & (2+i)/\sqrt{10} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

B19 $\begin{bmatrix} (3-i)/\sqrt{12} & (-1+2i)/\sqrt{30} \\ (1+i)/\sqrt{12} & 5/\sqrt{30} \end{bmatrix}$

For Problems B20 and B21, use the Gram-Schmidt Procedure to find an orthogonal basis for $\mathbb{S} = \operatorname{Span}\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ under the standard inner product of $\mathbb{C}^3$.

B20 $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} -i \\ 1 \\ 2 \end{bmatrix}, \mathbf{z}_3 = \begin{bmatrix} -i \\ 2i \\ 1 \end{bmatrix}$

B21 $\mathbf{z}_1 = \begin{bmatrix} i \\ i \\ i \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 1+i \\ 1-i \\ 1 \end{bmatrix}, \mathbf{z}_3 = \begin{bmatrix} 2-i \\ i \\ 3 \end{bmatrix}$

For Problems B22 and B23, compute $\operatorname{proj}_{\mathbb{S}}(\mathbf{z})$ under the standard inner product of $\mathbb{C}^3$.

B22 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix} i \\ 1 \\ i \end{bmatrix}, \begin{bmatrix} -i \\ 1+i \\ 1 \end{bmatrix}\right\}, \mathbf{z} = \begin{bmatrix} 2+i \\ 2 \\ -2 \end{bmatrix}$

B23 $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix} i \\ 1+i \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix}\right\}, \mathbf{z} = \begin{bmatrix} 1-i \\ i \\ 3i \end{bmatrix}$

For Problems B24–B27, let $\mathbf{z}$ and $\mathbf{w}$ be vectors in a complex inner product space such that $\langle \mathbf{z}, \mathbf{w}\rangle = 2 - i$. Evaluate the expression.

B24 $\langle \mathbf{w}, \mathbf{z}\rangle$

B25 $\langle (1+i)\mathbf{z}, \mathbf{w}\rangle$

B26 $\langle i\mathbf{w}, (1+2i)\mathbf{z}\rangle$

B27 $\langle i\mathbf{z}, (1-i)\mathbf{w}\rangle$

Conceptual Problems

C1 (a) Prove that multiplication by any complex number $\alpha = a + bi$ can be represented as a linear mapping M_α of $\mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(b) Interpret multiplication by an arbitrary complex number as a composition of a contraction or dilation, and a rotation in the plane $\mathbb{R}^2$.

(c) Verify the result by calculating M_α for $\alpha = 3 - 4i$ and interpreting it as in part (b).

C2 Prove Theorem 9.3.4 (2).

C3 Prove Theorem 9.3.4 (4).

C4 Prove if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in a complex inner product space, then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. Is the converse true?

C5 Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also a basis for $\mathbb{C}^3$ (taken as a complex vector space).

C6 Let $A \in M_{n\times n}(\mathbb{C})$. Prove $\det \overline{A} = \overline{\det A}$.

C7 Prove that if A and B are unitary, then AB is unitary.

C8 Define isomorphisms for complex vector spaces and check that the arguments in Section 4.7 are still correct provided that the scalars are always taken to be complex numbers.

9.4 Complex Diagonalization

We now extend everything we did with eigenvalues, eigenvectors, and diagonalization in Chapter 6 to allow the use of complex numbers.

Complex Eigenvalues and Eigenvectors

Definition
Eigenvalue
Eigenvector

Let $A \in M_{n\times n}(\mathbb{C})$. If there exists $\lambda \in \mathbb{C}$ and $\mathbf{z} \in \mathbb{C}^n$ with $\mathbf{z} \neq \mathbf{0}$ such that $A\mathbf{z} = \lambda\mathbf{z}$, then λ is called an **eigenvalue** of A and $\mathbf{z}$ is called an **eigenvector** of A corresponding to λ.

Since the theory of solving systems of equations, inverting matrices, and finding coordinates with respect to a basis is exactly the same for complex vector spaces as the theory for real vector spaces, the basic results on diagonalization are unchanged except that the vector space is now $\mathbb{C}^n$. A complex $n \times n$ matrix A is diagonalized by a matrix P if and only if the columns of P form a basis for $\mathbb{C}^n$ consisting of eigenvectors of A. Since the Fundamental Theorem of Algebra guarantees that every n-th degree polynomial has exactly n roots over $\mathbb{C}$, the only way a matrix cannot be diagonalizable over $\mathbb{C}$ is if it has an eigenvalue with geometric multiplicity less than its algebraic multiplicity.

EXAMPLE 9.4.1

Let $A = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix}$. Find its eigenvectors and diagonalize over $\mathbb{C}$.

Solution: We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -6 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13$$

So, by the quadratic formula, we get that the eigenvalues of A are $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$.

For $\lambda_1 = 2 + 3i$,

$$A - \lambda_1 I = \begin{bmatrix} 3-3i & -6 \\ 3 & -3-3i \end{bmatrix} \sim \begin{bmatrix} 1 & -(1+i) \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to $\lambda_1 = 2 + 3i$ is $\mathbf{z}_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.

For $\lambda_2 = 2 - 3i$,

$$A - \lambda_2 I = \begin{bmatrix} 3+3i & -6 \\ 3 & -3+3i \end{bmatrix} \sim \begin{bmatrix} 1 & -(1-i) \\ 0 & 0 \end{bmatrix}$$

Thus, an eigenvector corresponding to $\lambda_2 = 2 - 3i$ is $\mathbf{z}_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

It follows that A is diagonalized to $\begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$ by $P = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}$.

EXAMPLE 9.4.2

Determine whether $B = \begin{bmatrix} 2 & i \\ i & 4 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$.

Solution: We have

$$C(\lambda) = \det(B - \lambda I) = \begin{vmatrix} 2-\lambda & i \\ i & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

So, the only distinct eigenvalue of B is $\lambda_1 = 3$ with algebraic multiplicity 2. For $\lambda_1 = 3$,

$$B - \lambda_1 I = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

Hence, the geometric multiplicity of λ_1 is 1. Therefore, B is not diagonalizable since the geometric multiplicity of λ_1 is less than its algebraic multiplicity.

EXERCISE 9.4.1

Determine whether $A = \begin{bmatrix} 4 & 1+i \\ 1-i & 3 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. If so, diagonalize it.

EXERCISE 9.4.2

Determine whether $A = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & -5 \\ -2 & 1 & 2 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. If so, diagonalize it.

As usual, these examples teach us more than just how to diagonalize complex matrices. In Example 9.4.1, we see that when a matrix has only real entries, then its eigenvalues should come in complex conjugate pairs. Example 9.4.2 shows that when working with matrices with non-real entries our theory of symmetric matrices for real matrices does not apply. In particular, we had $B^T = B$, but not only did B not have real eigenvalues, it was not even diagonalizable. We will now prove our first observation that non-real eigenvalues of real matrices come in complex conjugate pairs and, moreover, that the corresponding eigenvectors will also be complex conjugates of each other. We will look at how to modify the theory of symmetric matrices to non-real matrices in Section 9.5.

Theorem 9.4.1

Let $A \in M_{n\times n}(\mathbb{R})$. If λ is a non-real eigenvalue of A with corresponding eigenvector $\mathbf{z}$, then $\overline{\lambda}$ is also an eigenvalue of A and has corresponding eigenvector $\overline{\mathbf{z}}$.

Proof: We have $A\mathbf{z} = \lambda\mathbf{z}$. Taking complex conjugates of both sides and using Theorem 9.3.1 gives

$$\overline{A\mathbf{z}} = \overline{\lambda\mathbf{z}} \Rightarrow A\overline{\mathbf{z}} = \overline{\lambda}\overline{\mathbf{z}}$$

since A is real. Hence, $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{\mathbf{z}}$, as required. ■

PROBLEMS 9.4

Practice Problems

For Problems A1–A10, either diagonalize the matrix over $\mathbb{C}$ or show that the matrix is not diagonalizable.

A1 $\begin{bmatrix} 2 & 1+i \\ 1-i & 1 \end{bmatrix}$

A2 $\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}$

A3 $\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

A4 $\begin{bmatrix} i & i \\ 2 & i \end{bmatrix}$

A5 $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

A6 $\begin{bmatrix} 2 & 2 & -1 \\ -4 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

A7 $\begin{bmatrix} -6-3i & -2 & -3-2i \\ 10 & 2 & 5 \\ 8+6i & 3 & 4+4i \end{bmatrix}$

A8 $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$

A9 $\begin{bmatrix} 5 & -1+i & 2i \\ -2-2i & 2 & 1-i \\ 4i & -1-i & -1 \end{bmatrix}$

A10 $\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix}$

A11 $\begin{bmatrix} 0 & -i & 2+i \\ -i & 2-i & 2i \\ -i & 2-2i & 3i \end{bmatrix}$

A12 $\begin{bmatrix} 1+i & 1 & 0 \\ 1 & 1 & -i \\ 1 & 0 & 1 \end{bmatrix}$

Homework Problems

For Problems B1–B8, either diagonalize the matrix over $\mathbb{C}$ or show that the matrix is not diagonalizable.

B1 $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

B2 $\begin{bmatrix} i & 0 \\ i & i \end{bmatrix}$

B3 $\begin{bmatrix} 1+i & -1 \\ i & 0 \end{bmatrix}$

B4 $\begin{bmatrix} 2-i & i \\ -2i & 2+i \end{bmatrix}$

B5 $\begin{bmatrix} 1 & i \\ 2 & -1+2i \end{bmatrix}$

B6 $\begin{bmatrix} -1+i & 0 & -i \\ 1 & i & i \\ -2i & 0 & 2+i \end{bmatrix}$

B7 $\begin{bmatrix} 3-i & -1-i & -1+i \\ -1+i & 3+i & -1-3i \\ 0 & 0 & 2i \end{bmatrix}$

B8 $\begin{bmatrix} -i & -1 & i \\ -1+i & 1+i & 1 \\ -1+i & i & 2 \end{bmatrix}$

B9 $\begin{bmatrix} 1+2i & 3 & -2 \\ 3 & 1+2i & -2 \\ 2 & 2 & -2+2i \end{bmatrix}$

B10 $\begin{bmatrix} 5 & -2 & -i \\ -1 & 6 & i \\ i & -2i & 5 \end{bmatrix}$

B11 $\begin{bmatrix} 2i & -1+i & -1+i \\ 1-i & 2 & 1-i \\ -1-i & -1-i & 0 \end{bmatrix}$

B12 $\begin{bmatrix} 3i & -2i & -1 \\ i & 6i & 1 \\ 1 & 2 & 3i \end{bmatrix}$

Conceptual Problems

C1 Prove that if $\mathbf{z}$ is an eigenvector of $A \in M_{n\times n}(\mathbb{C})$, then $\overline{\mathbf{z}}$ is an eigenvector of $\overline{A}$.

C2 Assume that $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable over $\mathbb{C}$ and has eigenvalues $\lambda_1, \dots, \lambda_n$. Prove that

$$\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$$

(Hint: see Theorem 6.2.1 on page 361.)

C3 Prove if $A \in M_{n\times n}(\mathbb{R})$ with n odd, then A has at least one real eigenvalue.

C4 Let $A \in M_{n\times n}(\mathbb{R})$ with eigenvector $\mathbf{z} = \vec{x} + i\vec{y}$, where $\vec{x}, \vec{y} \in \mathbb{R}^n$, corresponding to the non-real eigenvalue $\lambda = a + bi$.

(a) Prove that $\vec{x} \neq \vec{0}$ and $\vec{y} \neq \vec{0}$.

(b) Prove that $\vec{x} \neq k\vec{y}$ for any $k \in \mathbb{R}$.

(c) Prove that Span$\{\vec{x}, \vec{y}\}$ does not contain an eigenvector of A corresponding to an eigenvalue of A.

C5 (a) Let $A \in M_{2\times 2}(\mathbb{R})$ with eigenvector $\mathbf{z} = \vec{x} + i\vec{y}$, where $\vec{x}, \vec{y} \in \mathbb{R}^n$, corresponding to the non-real eigenvalue $\lambda = a + bi$. Prove if $P = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}$, then P is invertible and

$$P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

The matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is called a **real canonical form** for A.

(b) Let $A \in M_{3\times 3}(\mathbb{R})$ with eigenvector $\mathbf{z} = \vec{x} + i\vec{y}$, where $\vec{x}, \vec{y} \in \mathbb{R}^n$, corresponding to the non-real eigenvalue $\lambda = a + bi$. Find a real canonical form for A.

9.5 Unitary Diagonalization

Since the complex equivalent of an orthogonal matrix is a unitary matrix, the complex equivalent of orthogonal diagonalization is unitary diagonalization.

Definition
Unitarily Similar

Let $A, B \in M_{m\times n}(\mathbb{C})$. If there exists a unitary matrix U such that $U^*AU = B$, then we say that A and B are **unitarily similar**.

If A and B are unitarily similar, then they are similar. Consequently, all of our properties of similarity still apply.

Definition
Unitarily Diagonalizable

A matrix $A \in M_{n\times n}(\mathbb{C})$ is said to be **unitarily diagonalizable** if it is unitarily similar to a diagonal matrix.

The Principal Axis Theorem says that a real matrix A is orthogonally diagonalizable if and only if it is symmetric. We observe that if A is a real symmetric matrix, then the condition $A^T = A$ is equivalent to $A^* = A$. Hence, in the complex case, the condition $A^* = A$ should take the place of the condition $A^T = A$.

Definition
Hermitian Matrix

A matrix $A \in M_{n\times n}(\mathbb{C})$ is called **Hermitian** if $A^* = A$.

EXAMPLE 9.5.1

Which of the following matrices are Hermitian?

$$A = \begin{bmatrix} 2 & 3-i \\ 3+i & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2i \\ -2i & 3-i \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & i & i \\ -i & 0 & i \\ -i & i & 0 \end{bmatrix}$$

Solution: We have $A^* = \begin{bmatrix} 2 & 3-i \\ 3+i & 4 \end{bmatrix} = A$, so A is Hermitian.

$B^* = \begin{bmatrix} 1 & 2i \\ -2i & 3+i \end{bmatrix} \neq B$, so B is not Hermitian.

$C^* = \begin{bmatrix} 0 & i & i \\ -i & 0 & -i \\ -i & -i & 0 \end{bmatrix} \neq C$, so C is not Hermitian.

Observe that if A is Hermitian, then we have $\overline{(A)_{ij}} = A_{ji}$, so the diagonal entries of A must be real, and for $i \neq j$ the ij-th entry must be the complex conjugate of the ji-th entry.

Remark

A linear operator $L : \mathbb{V} \to \mathbb{V}$ is called Hermitian if $\langle \mathbf{x}, L(\mathbf{y})\rangle = \langle L(\mathbf{x}), \mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. A linear operator is Hermitian if and only if its matrix with respect to any orthonormal basis of $\mathbb{V}$ is a Hermitian matrix. Hermitian linear operators play an important role in quantum mechanics.

Theorem 9.5.1 If $A \in M_{n\times n}(\mathbb{C})$ is Hermitian, then

(1) All eigenvalues of A are real.
(2) Eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Proof: We will prove (1). The proof of (2) is left as Problem C1.

Suppose that λ is an eigenvalue of A with corresponding unit eigenvector $\mathbf{z}$. Using Theorem 9.3.3 and the fact that $\langle \mathbf{z}, \mathbf{z}\rangle = 1$ we get

$$\lambda = \lambda\langle \mathbf{z}, \mathbf{z}\rangle = \langle \mathbf{z}, \lambda\mathbf{z}\rangle = \langle \mathbf{z}, A\mathbf{z}\rangle = \langle A^*\mathbf{z}, \mathbf{z}\rangle = \langle A\mathbf{z}, \mathbf{z}\rangle = \langle \lambda\mathbf{z}, \mathbf{z}\rangle = \overline{\lambda}\langle \mathbf{z}, \mathbf{z}\rangle = \overline{\lambda}$$

Thus, λ is real. ■

Since a real symmetric matrix A is Hermitian, Theorem 9.5.1 (1) implies Theorem 8.1.3: that all eigenvalues of a real symmetric matrix are real. Moreover, from this result, we expect to get something very similar to the Principal Axis Theorem for Hermitian matrices. We first consider an example.

EXAMPLE 9.5.2 Unitarily diagonalize $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$.

Solution: The characteristic polynomial of A is

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 1+i \\ 1-i & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1)$$

Hence, the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 1$.
For $\lambda_1 = 4$,

$$A - \lambda_1 I = \begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -(1+i)/2 \\ 0 & 0 \end{bmatrix}$$

Thus, a corresponding eigenvector is $\mathbf{z}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$. For $\lambda_2 = 1$,

$$A - \lambda_2 I = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$$

Thus, a corresponding eigenvector is $\mathbf{z}_2 = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$.

To unitarily diagonalize A, we need an orthonormal basis for $\mathbb{C}^2$ of eigenvectors of A. Hence, we normalize $\mathbf{z}_1$ and $\mathbf{z}_2$ and take

$$U = \begin{bmatrix} (1+i)/\sqrt{6} & (1+i)/\sqrt{3} \\ 2/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$$

to get

$$U^*AU = \operatorname{diag}(4, 1)$$

To prove that every Hermitian matrix is unitarily diagonalizable we will repeat what we did in Section 8.1. To do this we need the following very important extension of the Triangularization Theorem.

Theorem 9.5.2 Schur's Theorem

If $A \in M_{n\times n}(\mathbb{C})$, then A is unitarily similar to an upper triangular matrix T. Moreover, the diagonal entries of T are the eigenvalues of A.

Theorem 9.5.3 Spectral Theorem for Hermitian Matrices

If $A \in M_{n\times n}(\mathbb{C})$ is Hermitian, then it is unitary diagonalizable.

You are asked to prove Theorem 9.5.3 as Problem C2.

In the real case, we found that the only matrices that are orthogonally diagonalizable are symmetric matrices. Because of the power of complex numbers, there are more matrices than just Hermitian matrices that are unitarily diagonalizable.

EXERCISE 9.5.1

Show that $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is unitarily diagonalizable but not Hermitian.

Normal Matrices

To look for a necessary condition for a matrix to be unitarily diagonalizable, we work backwards.

Assume $A \in M_{n\times n}(\mathbb{C})$ is unitarily diagonalizable. Let U be a unitary matrix such that $U^*AU = D$, where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Observe that

$$DD^* = \mathrm{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2) = D^*D$$

Using this and the fact that U is unitary, we get

$$\begin{aligned} AA^* &= (UDU^*)(UDU^*)^* = UDU^*UD^*U^* = UDD^*U^* \\ &= UD^*DU^* = UD^*U^*UDU^* = (UDU^*)^*(UDU^*) = A^*A \end{aligned}$$

Consequently, if A is unitarily diagonalizable, then we must have $AA^* = A^*A$.

Definition
Normal Matrix

A matrix $A \in M_{n\times n}(\mathbb{C})$ is called **normal** if $AA^* = A^*A$.

Theorem 9.5.4 Spectral Theorem for Normal Matrices

A matrix A is normal if and only if it is unitarily diagonalizable.

Of course, normal matrices are very important. We now look at some useful properties of normal matrices.

Theorem 9.5.5 If $A \in M_{n\times n}(\mathbb{C})$ is normal, then

(1) $\|A\mathbf{z}\| = \|A^*\mathbf{z}\|$, for all $\mathbf{z} \in \mathbb{C}^n$.
(2) $A - \lambda I$ is normal for every $\lambda \in \mathbb{C}$.
(3) If $A\mathbf{z} = \lambda\mathbf{z}$, then $A^*\mathbf{z} = \overline{\lambda}\mathbf{z}$.
(4) If $\mathbf{z}_1$ and $\mathbf{z}_2$ are eigenvectors of A corresponding to distinct eigenvalues λ_1 and λ_2 of A, then $\mathbf{z}_1$ and $\mathbf{z}_2$ are orthogonal.

Notice that property (4) shows us that the procedure for unitarily diagonalizing a normal matrix is exactly the same as the procedure for orthogonally diagonalizing a real symmetric matrix, except that the calculations are a little more complex.

EXAMPLE 9.5.3

Unitarily diagonalize $A = \begin{bmatrix} 4 & 1 & -i \\ 1 & 4 & -i \\ i & i & 4 \end{bmatrix}$.

Solution: We find that eigenvalues of A are $\lambda_1 = 3$ with algebraic multiplicity 2 and $\lambda_2 = 6$ with algebraic multiplicity 1. We have

$$A - 3I = \begin{bmatrix} 1 & 1 & -i \\ 1 & 1 & -i \\ i & i & 1 \end{bmatrix} \Rightarrow \mathbf{z}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$$

Since $\mathbf{z}_1$ and $\mathbf{z}_2$ are not orthogonal, we need to apply the Gram-Schmidt Procedure to $\{\mathbf{z}_1, \mathbf{z}_2\}$. Pick $\mathbf{w}_1 = \mathbf{z}_1$ and let $\mathbb{S}_1 = \operatorname{Span}\{\mathbf{w}_1\}$. We find that

$$\operatorname{perp}_{\mathbb{S}_1}(\mathbf{z}_2) = \mathbf{z}_2 - \frac{\langle \mathbf{w}_1, \mathbf{z}_2 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{bmatrix} i/2 \\ i/2 \\ 1 \end{bmatrix}$$

We take $\mathbf{w}_2 = \begin{bmatrix} i \\ i \\ 2 \end{bmatrix}$. We also find that

$$A - 6I = \begin{bmatrix} -2 & 1 & -i \\ 1 & -2 & -i \\ i & i & -2 \end{bmatrix} \Rightarrow \mathbf{z}_3 = \begin{bmatrix} -i \\ -i \\ 1 \end{bmatrix}$$

Hence, taking

$$U = \begin{bmatrix} -1/\sqrt{2} & i/\sqrt{6} & -i/\sqrt{3} \\ 1/\sqrt{2} & i/\sqrt{6} & -i/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

gives

$$U^*AU = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

EXAMPLE 9.5.4

Unitarily diagonalize $A = \begin{bmatrix} 4i & 1+3i \\ -1+3i & i \end{bmatrix}$.

Solution: We have $C(\lambda) = \lambda^2 - 5i\lambda + 6$. Using the quadratic formula, we get eigenvalues $\lambda_1 = 6i$ and $\lambda_2 = -i$. We find that

$$A - 6iI = \begin{bmatrix} -2i & 1+3i \\ -1+3i & -5i \end{bmatrix} \Rightarrow \mathbf{z}_1 = \begin{bmatrix} 3-i \\ 2 \end{bmatrix}$$

$$A + iI = \begin{bmatrix} 5i & 1+3i \\ -1+3i & 2i \end{bmatrix} \Rightarrow \mathbf{z}_2 = \begin{bmatrix} 1+3i \\ -5i \end{bmatrix}$$

Hence, taking

$$U = \begin{bmatrix} (3-i)/\sqrt{14} & (1+3i)/\sqrt{35} \\ 2/\sqrt{14} & -5i/\sqrt{35} \end{bmatrix}$$

gives

$$U^*AU = \begin{bmatrix} 6i & 0 \\ 0 & -i \end{bmatrix}$$

PROBLEMS 9.5

Practice Problems

For Problems A1–A4, determine whether the matrix is normal.

A1 $\begin{bmatrix} 3 & 1-i \\ 1+i & 5 \end{bmatrix}$

A2 $\begin{bmatrix} 2 & -i \\ i & 1+i \end{bmatrix}$

A3 $\begin{bmatrix} 0 & 1-i \\ -1-i & 0 \end{bmatrix}$

A4 $\begin{bmatrix} 1-i & 2i \\ 2 & 3 \end{bmatrix}$

For Problems A5–A12, unitarily diagonalize the matrix.

A5 $\begin{bmatrix} 1+2i & -1 \\ -1 & 1+2i \end{bmatrix}$

A6 $\begin{bmatrix} 5i & -1-i \\ 1-i & 4i \end{bmatrix}$

A7 $\begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}$

A8 $\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$

A9 $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

A10 $\begin{bmatrix} 4 & \sqrt{2}+i \\ \sqrt{2}-i & 2 \end{bmatrix}$

A11 $\begin{bmatrix} 1 & 0 & 1+i \\ 0 & 2 & 0 \\ 1-i & 0 & 0 \end{bmatrix}$

A12 $\begin{bmatrix} i & 0 & 0 \\ 0 & -1 & 1-i \\ 0 & 1+i & 0 \end{bmatrix}$

Homework Problems

For Problems B1–B6, determine whether the matrix is normal.

B1 $\begin{bmatrix} 1+i & 1-i \\ 1-i & i \end{bmatrix}$

B2 $\begin{bmatrix} 2 & 2-i \\ 2+i & -2 \end{bmatrix}$

B3 $\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$

B4 $\begin{bmatrix} i & 2i \\ 2 & i \end{bmatrix}$

B5 $\begin{bmatrix} 0 & 1+i \\ 1+i & 1 \end{bmatrix}$

B6 $\begin{bmatrix} 0 & 3i \\ -3i & 1 \end{bmatrix}$

For Problems B7–B12, unitarily diagonalize the matrix.

B7 $\begin{bmatrix} 1 & 1-2i \\ 1+2i & 5 \end{bmatrix}$

B8 $\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$

B9 $\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$

B10 $\begin{bmatrix} 4i & 1+3i \\ -1+3i & i \end{bmatrix}$

B11 $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

B12 $\begin{bmatrix} 1 & i & -i \\ -i & -1 & i \\ i & -i & 0 \end{bmatrix}$

Conceptual Problems

C1 Let $U \in M_{n\times n}(\mathbb{C})$ be a unitary matrix.
(a) Show that $\|U\mathbf{z}\| = \|\mathbf{z}\|$ for all $\mathbf{z} \in \mathbb{C}^n$.
(b) Show that all of its eigenvalues satisfy $|\lambda| = 1$.
(c) Give a 2×2 unitary matrix such that none of its eigenvalues are real.

C2 Assume that A is Hermitian. Prove that if $\mathbf{v}$ and $\mathbf{z}$ are eigenvectors of A corresponding to distinct eigenvalues, then $\mathbf{v}$ and $\mathbf{z}$ are orthogonal.

C3 Prove that every Hermitian matrix is unitarily diagonalizable.

C4 Prove that if A is Hermitian, then $\det A$ is real.

C5 Let $A \in M_{n\times n}(\mathbb{C})$ satisfy $A^* = iA$.
(a) Prove that A is normal.
(b) Show that every eigenvalue λ of A must satisfy $\lambda = -\overline{\lambda}i$.

C6 Suppose that $A, B \in M_{n\times n}(\mathbb{C})$ are Hermitian matrices and that A is invertible. Determine which of the following are Hermitian.
(a) AB (b) A^2 (c) A^{-1}

C7 A general 2×2 Hermitian matrix can be written as $A = \begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix}$, $a, b, c, d \in \mathbb{R}$.
(a) What can you say about $a, b, c,$ and d if A is unitary as well as Hermitian?
(b) What can you say about $a, b, c,$ and d if A is Hermitian, unitary, and diagonal?
(c) What can you say about the form of a 3×3 matrix that is Hermitian, unitary, and diagonal?

C8 Let $\mathbb{V}$ be a complex inner product space. Prove that a linear operator $L : \mathbb{V} \to \mathbb{V}$ is Hermitian ($\langle \mathbf{x}, L(\mathbf{y})\rangle = \langle L(\mathbf{x}), \mathbf{y}\rangle$) if and only if its matrix with respect to any orthonormal basis of $\mathbb{V}$ is a Hermitian matrix.

CHAPTER REVIEW

Suggestions for Student Review

1 What is the complex conjugate of a complex number? List some properties of the complex conjugate. How does the complex conjugate relate to division of complex numbers? How does it relate to the length of a complex number? (Section 9.1)

2 Define the polar form of a complex number. Explain how to convert a complex number from standard form to polar form. Is the polar form unique? How does the polar form relate to Euler's Formula? (Section 9.1)

3 List some of the similarities and some of the differences between complex vector spaces and real vector spaces. Discuss the differences between viewing $\mathbb{C}$ as a complex vector space and as a real vector space. (Section 9.3)

4 Discuss the standard inner product in $\mathbb{C}^n$. How are the essential properties of an inner product modified in generalizing from the real case to the complex case? (Section 9.3)

5 Define the conjugate transpose of a matrix. List some similarities between the conjugate transpose of a complex matrix and the transpose of a real matrix. (Section 9.3)

6 Explain how diagonalization of matrices over $\mathbb{C}$ differs from diagonalization over $\mathbb{R}$. (Section 9.4)

7 What is a Hermitian matrix? State what you can about diagonalizing a Hermitian matrix. (Section 9.5)

8 What is a normal matrix? List the properties of a normal matrix. (Section 9.5)

Chapter Quiz

For Problems E1–E6, let $z_1 = 3 + 4i$, $z_2 = 1 + 2i$, and $z_3 = 1 - 2i$. Evaluate the expression.

E1 $z_1 + z_2$ **E2** $2z_1 - iz_2$ **E3** $z_1 z_3$

E4 $z_1 z_2 z_3$ **E5** $\frac{z_1}{z_2}$ **E6** $\frac{z_2}{z_3}$

E7 Let $z_1 = 1 - \sqrt{3}i$ and $z_2 = 2 + 2i$.
(a) Find a polar form of z_1 and a polar form of z_2.
(b) Use the polar forms to determine $z_1 z_2$ and $\dfrac{z_1}{z_2}$.

E8 Use the polar form to determine all values of $(i)^{1/2}$.

For Problems E9 and E10, determine whether the system is consistent. If it is, find the general solution.

E9
$$\begin{aligned} z_1 - iz_2 - z_3 &= 1 \\ -2z_1 + 4iz_2 &= -2 + 6i \\ -iz_1 + 2z_2 + 4iz_3 &= 9 \end{aligned}$$

E10
$$\begin{aligned} (1-i)z_1 + (1+i)z_2 + 2z_3 &= 1 + i \\ iz_1 + 0z_2 - z_3 &= 2i \\ (1-i)z_1 + (2-i)z_2 - iz_3 &= 6 + i \end{aligned}$$

For Problems E11–E16, let $\mathbf{u} = \begin{bmatrix} 3-i \\ i \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 4-i \end{bmatrix}$.

Evaluate the expression.

E11 $2\mathbf{u} + (1+i)\mathbf{v}$ **E12** $\overline{\mathbf{u}}$

E13 $\langle \mathbf{u}, \mathbf{v} \rangle$ **E14** $\langle \mathbf{v}, \mathbf{u} \rangle$

E15 $\|\mathbf{v}\|$ **E16** $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$

E17 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ i \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ 1+i \\ 1+i \end{bmatrix}, \begin{bmatrix} 1-2i \\ i \\ 1 \end{bmatrix} \right\}$. Use the Gram-Schmidt Procedure on $\mathcal{B}$ to find an orthogonal basis for $\mathbb{S} = \operatorname{Span} \mathcal{B}$.

E18 Determine the projection of $\mathbf{z} = \begin{bmatrix} i \\ 1+i \\ 2-i \end{bmatrix}$ onto the subspace

$$\mathbb{S} = \operatorname{Span} \left\{ \begin{bmatrix} i \\ -1 \\ 1+i \end{bmatrix}, \begin{bmatrix} 1-i \\ 1+i \\ 2 \end{bmatrix} \right\}$$

E19 Show that $U = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1-i & -i \\ 1 & -1+i \end{bmatrix}$ is unitary.

For Problems E20 and E21, either diagonalize the matrix over $\mathbb{C}$ or show that the matrix is not diagonalizable.

E20 $\begin{bmatrix} 1-3i & -1 \\ -1 & 1-i \end{bmatrix}$ **E21** $\begin{bmatrix} 3+i & 1 & i \\ 1 & 3-i & 1 \\ -1+i & 1+i & 2+i \end{bmatrix}$

E22 Let $A = \begin{bmatrix} 0 & 3+ki \\ 3+i & 3 \end{bmatrix}$.
(a) Determine k such that A is Hermitian.
(b) With the value of k as determined in part (a), unitarily diagonalize A.

APPENDIX A

Answers to Mid-Section Exercises

Section 1.1 Exercises

1 (a) $\begin{bmatrix}1\\0\end{bmatrix}$ (b) $\begin{bmatrix}-2\\-1\end{bmatrix}$ (c) $\begin{bmatrix}-1\\-1\end{bmatrix}$

2 $\vec{x} = t\begin{bmatrix}3\\-1\end{bmatrix}, t \in \mathbb{R}$

3 $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix} + t\begin{bmatrix}-3\\1\end{bmatrix}, t \in \mathbb{R}$

4 $\begin{bmatrix}12\\0\\-14\end{bmatrix}$

5 $\vec{x} = \begin{bmatrix}1\\2\\2\end{bmatrix} + t\begin{bmatrix}0\\-4\\1\end{bmatrix}, t \in \mathbb{R}; \begin{cases}x_1 = 1\\x_2 = 2-4t\\x_3 = 2+t\end{cases}, t \in \mathbb{R}$

6 There are many possible answers $\begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}$ are in the plane. $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not in the plane.

7 $\vec{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{v} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

Section 1.2 Exercises

1 (a) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$, (b) $\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}\right\}$

2 (a) A line through the origin.
(b) If $\{\vec{u}, \vec{v}\}$ is linearly dependent, then a line through the origin. Otherwise, a plane through the origin.
(c) A plane through the origin.
(d) If $\vec{u} = s\vec{v} = t\vec{w}$, then it is a line through the origin. If one of the three vectors is a linear combination of the other two (but they are not all scalar multiples of each other), then it is a plane through the origin. Otherwise, it is all of $\mathbb{R}^3$.

Section 1.3 Exercises

1 $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = 0$, so $\theta = \dfrac{\pi}{2}$ rads

2 $x_1 - 3x_2 - 2x_3 = 1(1) + (-3)(2) + (-2)(3) = -11$

3 $\vec{x} = x_2\begin{bmatrix}-3/2\\1\\0\end{bmatrix} + x_3\begin{bmatrix}1/2\\0\\1\end{bmatrix}, x_2, x_3 \in \mathbb{R}$

4 $\begin{bmatrix}3\\-2\\1\end{bmatrix} \times \begin{bmatrix}2\\3\\7\end{bmatrix} = \begin{bmatrix}-17\\-19\\13\end{bmatrix}$

5 The six cross products are easily checked.

6 $4x_1 - 5x_2 + 3x_3 = -5$

7 Area$= \|\vec{u} \times \vec{v}\| = \left\|\begin{bmatrix}-2\\-1\\-2\end{bmatrix}\right\| = \sqrt{9} = 3$

8 $\vec{x} = \begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}3\\0\\3\end{bmatrix}, t \in \mathbb{R}.$

Section 1.4 Exercises

1 (V5) Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then $-\vec{x} = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix}$ since

$$\vec{x} + (-\vec{x}) = \begin{bmatrix} x_1 + (-x_1) \\ \vdots \\ x_n + (-x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

(V7) $s(t\vec{x}) = s\begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix} = \begin{bmatrix} stx_1 \\ \vdots \\ stx_n \end{bmatrix} = st\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (st)\vec{x}$

2 $S \subseteq \mathbb{R}^2$. $\vec{0} \in S$, since $2(0) = 0$. If $\vec{x}, \vec{y} \in S$, then $s\vec{x} + t\vec{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \end{bmatrix} \in S$ since $2(sx_1 + ty_1) = s(2x_1) + t(2y_1) = sx_2 + ty_2$.
T is not a subspace since $\vec{0} \notin T$.

3 $P \subseteq \mathbb{R}^3$. Taking $a = b = 0$ gives $\vec{0} \in P$.
If $a_1\vec{v}_1 + b_1\vec{v}_2, a_2\vec{v}_1 + b_2\vec{v}_2 \in P$, then

$$\begin{aligned} s(a_1\vec{v}_1 + b_1\vec{v}_2) &+ t(a_2\vec{v}_1 + b_2\vec{v}_2) \\ &= (sa_1 + ta_2)\vec{v}_1 + (sb_1 + tb_2)\vec{v}_2 \in P \end{aligned}$$

4 The standard basis for $\mathbb{R}^4$ is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Consider $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$.
Thus, for every $\vec{x} \in \mathbb{R}^4$ we have a solution $t_i = x_i$ for $1 \leq i \leq 4$. Hence, the set is a spanning set for $\mathbb{R}^4$. Moreover, if we take $\vec{x} = \vec{0}$, we get that the solution is $t_i = 0$ for $1 \leq i \leq 4$, so the set is also linearly independent.

Section 1.5 Exercises

1 $\|\vec{x}\| = \sqrt{1 + 4 + 1} = \sqrt{6}$, $\|\vec{y}\| = \sqrt{\frac{1}{6} + \frac{4}{6} + \frac{1}{6}} = 1$

2 By definition $\hat{x}$ is a scalar multiple of $\vec{x}$ so it is parallel. Using Theorem 1.5.2 (2) we get

$$\|\hat{x}\| = \left\| \frac{1}{\|\vec{x}\|}\vec{x} \right\| = \left| \frac{1}{\|\vec{x}\|} \right| \|\vec{x}\| = \frac{\|\vec{x}\|}{\|\vec{x}\|} = 1$$

3

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{1}{14}\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3/14 \\ 1/14 \\ 2/14 \end{bmatrix} = \begin{bmatrix} 11/14 \\ -29/14 \\ -1/7 \end{bmatrix}$$

4 (L1):

$$\begin{aligned} \text{proj}_{\vec{x}}(s\vec{y} + t\vec{z}) &= \frac{\vec{x} \cdot (s\vec{y} + t\vec{z})}{\|\vec{x}\|^2}\vec{x} \\ &= \frac{\vec{x} \cdot (s\vec{y}) + \vec{x} \cdot (t\vec{z})}{\|\vec{x}\|^2}\vec{x} \\ &= s\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\vec{x} + t\frac{\vec{x} \cdot \vec{z}}{\|\vec{x}\|^2}\vec{x} \\ &= s\,\text{proj}_{\vec{x}}(\vec{y}) + t\,\text{proj}_{\vec{x}}(\vec{z}) \end{aligned}$$

(L2):

$$\begin{aligned} \text{proj}_{\vec{x}}(\text{proj}_{\vec{x}}(\vec{y})) &= \frac{\vec{x} \cdot (\text{proj}_{\vec{x}}(\vec{y}))}{\|\vec{x}\|^2}\vec{x} \\ &= \frac{1}{\|\vec{x}\|^2}\left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}(\vec{x} \cdot \vec{x})\right)\vec{x} \\ &= \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\vec{x} = \text{proj}_{\vec{x}}(\vec{y}) \end{aligned}$$

Section 2.1 Exercises

1 Observe the lines intersect at the point $(-3, 2)$.

2
$$\begin{aligned}(7+2t)-2(2+t)&=3\\(7+2t)+(2+t)+3(-t)&=9\end{aligned}$$

3 There are many possible answers.

(2.1.1) (a)
$$\begin{aligned}x_1+0x_2+0x_3&=-1\\-x_1+x_2+0x_3&=0\\x_1+x_2+0x_3&=0\end{aligned}$$

(2.1.1) (b)
$$\begin{aligned}x_1+x_2+x_3&=1\\x_1+x_2+x_3&=2\\-x_2+x_3&=1\end{aligned}$$

(2.1.2)
$$\begin{aligned}x_1+x_2+x_3&=1\\x_2+x_3&=1\\x_3&=1\end{aligned}$$

(2.1.3)
$$\begin{aligned}x_1+x_2+x_3&=1\\x_2+x_3&=1\\x_1+2x_2+2x_3&=2\end{aligned}$$

4 $x_1 = 15, x_2 = -19/3$.

5 $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}6-2t\\t\\2\end{bmatrix}=\begin{bmatrix}6\\0\\2\end{bmatrix}+t\begin{bmatrix}-2\\1\\0\end{bmatrix},\quad t\in\mathbb{R}$

6 $\left[\begin{array}{ccc|c}2&4&0&12\\1&2&-1&4\end{array}\right]\quad R_2-\tfrac{1}{2}R_1\quad\sim$
$\left[\begin{array}{ccc|c}2&4&0&12\\0&0&-1&-2\end{array}\right]$

7 $\begin{bmatrix}1&2&-1&1\\0&2&1&-1\\0&0&0&5\end{bmatrix}$

Section 2.2 Exercises

1 (a) Not in RREF (b) In RREF
(c) Not in RREF (d) Not in RREF

2 $\begin{bmatrix}1&0&1&1\\0&1&-1&-1\\0&0&0&0\end{bmatrix}$

3 (a) $\operatorname{rank}(A)=2$ (b) $\operatorname{rank}(B)=2$

4 $\vec{x}=r\begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}-2\\-1\\0\\0\\1\end{bmatrix},\quad r,s,t\in\mathbb{R}$

Section 2.3 Exercises

1 Yes, $(-2)\begin{bmatrix}1\\-3\\-3\end{bmatrix}+\frac{19}{4}\begin{bmatrix}2\\-2\\1\end{bmatrix}+\frac{13}{4}\begin{bmatrix}-2\\2\\-3\end{bmatrix}=\begin{bmatrix}1\\3\\1\end{bmatrix}$.

2 The set is linearly independent.

Section 3.1 Exercises

1 $\mathcal{B}$ is linearly dependent. $X \in \operatorname{Span}\mathcal{B}$.

2 For any 2×2 matrix $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ we have

$$x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

Hence, $\mathcal{B}$ spans $M_{2\times 2}(\mathbb{R})$. Moreover, if we take $x_1 = x_2 = x_3 = x_4 = 0$, then the only solution is the trivial solution, so the set is linearly independent.

3 $\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ 9 \\ -3 \end{bmatrix}$

4 $A^T = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 5 \end{bmatrix}$ and $(A^T)^T = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 5 \end{bmatrix} = A$

$3A^T = \begin{bmatrix} 6 & -3 \\ 9 & 0 \\ 3 & 15 \end{bmatrix} = (3A)^T$

5 (a) $\begin{bmatrix} 11 \\ 24 \end{bmatrix}$ (b) $[10]$ (c) $\begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}$

6 $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -2 & -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 15 \\ -5 \end{bmatrix}$

7 (a) $\begin{bmatrix} 11 \\ 24 \end{bmatrix}$ (b) $[10]$ (c) $\begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}$

8 (a) AB is not defined since A has 3 columns and B has 2 rows.

(b) $BA = \begin{bmatrix} 4 & 7 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

(c) $A^T A = \begin{bmatrix} 5 & 8 & 1 \\ 8 & 13 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(d) $BB^T = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

9 (a) Let $\vec{x}_i$ be a solution of $A\vec{x} = \vec{e}_i$, $1 \leq i \leq m$. We can write $\vec{y} = y_1\vec{e}_1 + \cdots + y_m\vec{e}_m$, so

$$\begin{aligned} A(y_1\vec{x}_1 + \cdots + y_m\vec{x}_m) &= y_1A\vec{x}_1 + \cdots + y_mA\vec{x}_m \\ &= y_1\vec{e}_1 + \cdots + y_m\vec{e}_m = \vec{y} \end{aligned}$$

(b) By the System-Rank Theorem, $\operatorname{rank}(A) = m$.

(c) Take $B = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_m \end{bmatrix}$.

(d) $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Section 3.2 Exercises

1 $f_A(-1,1,1,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, f_A(-3,1,0,1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2 $f_A(1,0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $f_A(0,1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $f_A(2,3) = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$
$f_A(2,3) = 2f_A(1,0) + 3f_A(0,1)$

3 (a) f is not linear. $f(1,0) + f(2,0) \neq f(3,0)$

(b) G is linear.

4 $[H] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Section 3.3 Exercises

1 $[R_{\pi/4}] = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}, R_{\pi/4}(1, 1) = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$

2 $[S] = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, S(\vec{x}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

3 $[T] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, T(\vec{x}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

4 $[H] = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

5 $[F] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, F(\vec{x}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

6 x_1x_3-plane: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. x_2x_3-plane $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Section 3.4 Exercises

1 $\operatorname{Range}(L) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$

2 $\operatorname{Null}(L) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

3 $\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}\right\}$

4 Col(A): $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}\right\}$ Null(A^T): { }

Row(A): $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ Null(A): $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right\}$

Section 3.5 Exercises

1 $K^{-1} = \begin{bmatrix} 7/26 & 2/13 \\ 2/13 & 3/13 \end{bmatrix}, x_1 = \frac{45}{13}, x_2 = \frac{35}{13}$

2 $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}, \vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$

3 (a) $[R^{-1}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $[S^{-1}] = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$

Section 3.6 Exercises

1 If $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$E_5 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$,

then $E_5E_4E_3E_2E_1A = R$.

Section 3.7 Exercises

1 $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$

2 $\vec{x}_1 = \begin{bmatrix} 10/3 \\ -11/3 \\ 5 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 17/3 \\ -1/3 \\ 7 \end{bmatrix}$

Section 4.1 Exercises

1 $\mathcal{B}$ is linearly independent. $\mathbf{p} \in \operatorname{Span} \mathcal{B}$.

2 Clearly any polynomial $a + bx + cx^2 + dx^3 \in \operatorname{Span} \mathcal{B}$. If we consider $0 = t_1(1) + t_2x + t_3x^2 + t_4x^3$ we get that $t_1 = t_2 = t_3 = t_4 = 0$.

Section 4.2 Exercises

1 The set $\mathbb{S}$ is not closed under scalar multiplication.

2 Axioms V2, V3, V7, V8, V9, and V10 hold by Theorem 3.1.1. Show the other axioms hold.

3 Axioms V2, V3, V7, V8, V9, and V10 hold by Theorem 4.1.1. Show the other axioms hold.

4 $\mathbb{U} \subseteq P_2(\mathbb{R})$. Taking $a = b = c = 0$, we get $0 \in \mathbb{U}$. If $\mathbf{p}(x) = a + bx + cx^2, \mathbf{q}(x) = d + ex + fx^2 \in \mathbb{U}$, and $s, t \in \mathbb{R}$, then
$s\mathbf{p}(x) + t\mathbf{q}(x) = (sa + td) + (sb + te)x + (sc + tf)x^2$
and
$(sb + te) + (sc + tf) = s(b + c) + t(e + f) = sa + td$
So, $s\mathbf{p} + t\mathbf{q} \in \mathbb{U}$. Hence, $\mathbb{U}$ is a subspace of $P_2(\mathbb{R})$.

5 By definition, $\{\mathbf{0}\}$ is a non-empty subset of $\mathbb{V}$. If $\mathbf{x}, \mathbf{y} \in \{\mathbf{0}\}$ and $s, t \in \mathbb{R}$, then $s\mathbf{x} + t\mathbf{y} = s\mathbf{0} + t\mathbf{0} = \mathbf{0}$. Hence, $\{\mathbf{0}\}$ is a subspace of $\mathbb{V}$ and therefore is a vector space under the same operations as $\mathbb{V}$.

Section 4.3 Exercises

1 $\{1 - x, 2 + 2x + x^2, x + x^2\}$

2 $\dim \mathbb{S} = 3$

3 There are many possible answers. One possibility is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

4 There are many possible answers.
$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the hyperplane.
$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Section 4.4 Exercises

1 $\begin{bmatrix}0\\2\\2\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}6\\4\end{bmatrix}$

2 If $\vec{x} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$, then we have $\vec{x} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$, so $[\vec{x}]_{\mathcal{S}} = \vec{x}$.

3 $Q = \begin{bmatrix}1&1&2\\1&2&2\\2&4&3\end{bmatrix}, P = \begin{bmatrix}2&-5&2\\-1&1&0\\0&2&-1\end{bmatrix}$

Section 4.5 Exercises

1
$$\begin{aligned}
L(s\vec{x} + t\vec{y}) &= L\left(\begin{bmatrix}sx_1+ty_1\\sx_2+ty_2\\sx_3+ty_3\end{bmatrix}\right)\\
&= \begin{bmatrix}sx_1+ty_1 & sx_1+ty_1+sx_2+ty_2+sx_3+ty_3\\ 0 & sx_2+ty_2\end{bmatrix}\\
&= s\begin{bmatrix}x_1 & x_1+x_2+x_3\\ 0 & x_2\end{bmatrix} + t\begin{bmatrix}y_1 & y_1+y_2+y_3\\ 0 & y_2\end{bmatrix}\\
&= sL(\vec{x}) + tL(\vec{y})
\end{aligned}$$

2
$$\begin{aligned}
\mathcal{D}(s\mathbf{p} + t\mathbf{q}) &= (s\mathbf{p} + t\mathbf{q})' = s\mathbf{p}' + t\mathbf{q}'\\
&= s\mathcal{D}(\mathbf{p}) + t\mathcal{D}(\mathbf{q})
\end{aligned}$$

3 A basis for Null(L) is $\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$.

A basis for Range(L) is $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$.

Section 4.6 Exercises

1 $[\text{refl}_{\vec{n}}]_{\mathcal{B}} = \begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

2 $[L]_{\mathcal{B}} = \begin{bmatrix}2&1&0&0\\-2&-2&0&0\\1&2&1&0\\2&1&0&1\end{bmatrix}$

Section 4.7 Exercises

1 If $\mathbf{0} = t_1L(\mathbf{u}_1) + \cdots + t_kL(\mathbf{u}_k) = L(t_1\mathbf{u}_1 + \cdots t_k\mathbf{u}_k)$, then $t_1\mathbf{u}_1 + \cdots + t_k\mathbf{u}_k \in \text{Null}(L)$. Thus, $t_1\mathbf{u}_1 + \cdots + t_k\mathbf{u}_k = \mathbf{0}$ by Theorem 4.7.1. Hence, $t_1 = \cdots = t_k = 0$ as required.

2 Let $\mathbf{v} \in \mathbb{V}$. Since L is onto, there exists $\mathbf{x} \in \mathbb{U}$ such that $L(\mathbf{x}) = \mathbf{v}$. Hence, we have

$$\mathbf{v} = L(\mathbf{x}) = L(t_1\mathbf{u}_1 + \cdots + t_k\mathbf{u}_k) = t_1L(\mathbf{u}_1) + \cdots + t_kL(\mathbf{u}_k)$$

Section 5.1 Exercises

1 (a) $\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 3(1) - 2(2) = -1$

(b) $\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} = 1(-2) - 3(0) = -2$

(c) $\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2(2) - 4(1) = 0$

2 $C_{11} = 3, C_{12} = -8, C_{13} = 4$
$\det A = 1C_{11} + 2C_{12} + 3C_{13} = -1$

3 (a) $\det A = 1C_{11} + 0C_{21} + 3C_{31} + (-2)C_{41} = 112$
(b) $\det A = 0C_{21} + 0C_{22} + (-1)C_{23} + 2C_{24} = 112$
(c) $\det A = 0C_{14} + 2C_{24} + 0C_{34} + 0C_{44} = 112$

Section 5.2 Exercises

1 rA is obtained by multiplying each row of A by r. Thus, by Theorem 5.2.1,

$$\det(rA) = (r)(r)(r)\det A = r^3 \det A$$

2 $\det A = 20$

3 $\det A = 156$

Section 5.3 Exercises

1 $A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T = \begin{bmatrix} 3 & 2 & -2 \\ -1 & -1 & 1 \\ 3 & 3 & -2 \end{bmatrix}$

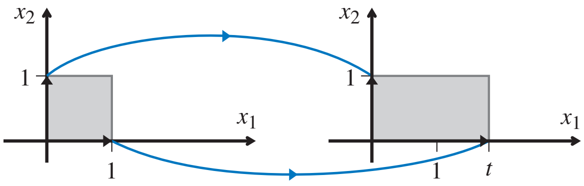
Section 5.4 Exercises

1 We have $S(\vec{e}_1) = \begin{bmatrix} t \\ 0 \end{bmatrix}$ and $S(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence,

$$\text{Area}(S(\vec{e}_1), S(\vec{e}_2)) = \left|\det \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\right| = |t| = t$$

Alternately,

$$\text{Area}(S(\vec{e}_1), S(\vec{e}_2)) = |\det A|\text{Area}(\vec{e}_1, \vec{e}_2) = \left|\det \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\right|\left|\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = t(1) = t$$



Section 6.1 Exercises

1 The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$.

A basis for E_{λ_1} is $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

A basis for E_{λ_2} is Span $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

2 $\lambda_1 = 5$ has algebraic multiplicity 1 and geometric multiplicity 1. $\lambda_2 = -2$ has algebraic multiplicity 2 and geometric multiplicity 1.

3 The only real eigenvalue is $\lambda_1 = 1$ with

$$E_{\lambda_1} = \text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Section 6.2 Exercises

1 $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & -1 \\ 1 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

2 A is not diagonalizable since $\lambda_1 = 2$ has algebraic multiplicity 2, but the geometric multiplicity is 1.

Section 6.3 Exercises

1 $A^{100} = \begin{bmatrix} 2 - 2^{100} & -1 + 2^{100} \\ 2 - 2^{101} & -1 + 2^{101} \end{bmatrix}$

2 (a) A is not a Markov matrix.

(b) B is a Markov matrix. $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

3 We find that $T\vec{s} = \begin{bmatrix} s_2 \\ s_1 \end{bmatrix}, T^2\vec{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, T^3\vec{s} = \begin{bmatrix} s_2 \\ s_1 \end{bmatrix}$, etc.

On the other hand, the fixed-state vector is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Section 7.1 Exercises

1 $\left\{\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{30} \\ -5\sqrt{30} \\ 2\sqrt{30} \end{bmatrix}\right\}$

2 It is easy to verify that $\mathcal{B}$ is orthonormal. We have

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \frac{8}{\sqrt{3}}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} + \frac{5}{\sqrt{6}}\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

3 The result is easily verified.

Section 7.2 Exercises

1 $\mathbb{S}^\perp = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

2 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 5/2 \\ 1 \\ 5/2 \end{bmatrix}, \text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

Section 7.4 Exercises

1 The result is easily verified. We observe that $\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$, so it matches the dot product on the isomorphic vectors in $\mathbb{R}^4$.

2 (a) $\|1\| = \sqrt{3}$, $\|x\| = \sqrt{5}$
(b) $\|1\| = \sqrt{3}$, $\|x\| = \sqrt{2}$

Section 8.1 Exercises

1 $A^T = A$, so A is symmetric. $B^T = \begin{bmatrix} 2 & 3 \\ -3 & 0 \end{bmatrix}$, so B is not symmetric.

2 $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Section 8.2 Exercises

1 (a) $Q(\vec{x}) = 4x_1^2 + x_1x_2 + \sqrt{2}x_2^2$
(b) $Q(\vec{x}) = x_1^2 - 2x_1x_2 + 2x_2^2 + 6x_2x_3 - x_3^2$

2 (a) $\begin{bmatrix} 1 & -1 \\ -1 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3/2 & -1/2 \\ 3/2 & 4 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$

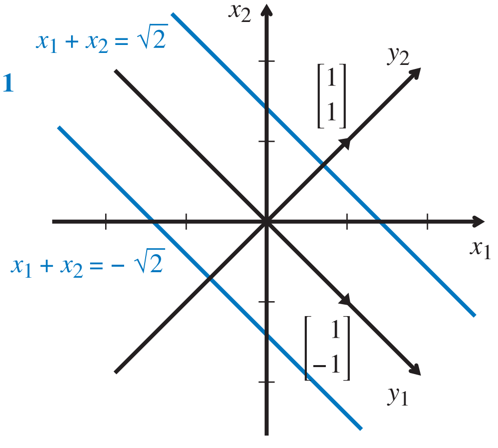
(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

3 $P = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ and $Q(\vec{x}) = -4y_1^2 + y_2^2$.

4 (a) $Q_1(\vec{x})$ is positive definite.
(b) $Q_2(\vec{x})$ is indefinite.

Section 8.3 Exercises

1



Section 8.5 Exercises

1 A has singular values $\sigma_1 = \sqrt{12}, \sigma_2 = \sqrt{2}$.
B has singular values $\sigma_1 = \sqrt{8}, \sigma_2 = \sqrt{3}, \sigma_3 = 0$.

2 $U = \begin{bmatrix} 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix},$

$V = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

3 We have $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. From Example 8.5.7, we have that $B^+ = \begin{bmatrix} 1/10 & 1/10 \\ 1/5 & 1/5 \end{bmatrix}$. Thus, the vector $\vec{x}$ of the smallest length that minimizes $\|B\vec{x} - \vec{b}\|$ is

$$\vec{x} = B^+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 1/5 \end{bmatrix}$$

Section 9.1 Exercises

1 $|-3| = 3, |-1-i| = \sqrt{2}$.

2 (a) $3 + i$ (b) $1 + 6i$ (c) $-2 + 2i$
(d) $5 - 5i$ (e) 13

3 (1) $\overline{\overline{z_1}} = \overline{x - yi} = x + yi = z_1$
(2) $x + iy = z_1 = \overline{z_1} = x - iy$ if and only if $y = 0$.
(4) Let $z_2 = a + ib$. Then

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{x + iy + a + ib} = \overline{x + a + i(y + b)} \\ &= x + a - i(y + b) = x - iy + a - ib \\ &= \overline{z_1} + \overline{z_2} \end{aligned}$$

4 (a) i (b) $1 + i$ (c) $-\frac{1}{26} - \frac{21}{26}i$

5 $|z_1| = 2$. $\theta = \dfrac{\pi}{6} + 2\pi k, k \in \mathbb{Z}$. Hence

$$z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$|z_2| = \sqrt{2}$. $\theta = \dfrac{5\pi}{4} + 2\pi k, k \in \mathbb{Z}$. Hence

$$z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

6
$$\begin{aligned} |\overline{z}| &= |r\cos\theta - ir\sin\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \sqrt{r^2} = |r| = |z| \end{aligned}$$

Using the trigonometric identities $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$ gives

$$\begin{aligned} \overline{z} &= r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) \\ &= r(\cos(-\theta) + i\sin(-\theta)) \end{aligned}$$

Hence, an argument of $\overline{z}$ is $-\theta$.

7 Theorem 9.1.5 says the modulus of a quotient is the quotient of the moduli of the factors, while the argument of the quotient is the difference of the arguments. Taking $z_1 = 1 = 1(\cos 0 + i\sin 0)$ and $z_2 = z$ in Theorem 9.1.5, gives

$$\begin{aligned} \frac{1}{z_2} &= \frac{1}{r}(\cos(0 - \theta) + i\sin(0 - \theta)) \\ &= \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) \end{aligned}$$

8 $(2 - 2i)(-1 + \sqrt{3}i) = 4\sqrt{2}\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$,
$\dfrac{2 - 2i}{-1 + \sqrt{3}i} = \sqrt{2}\left(\cos\dfrac{-11\pi}{12} + i\sin\dfrac{-11\pi}{12}\right)$

9 $(1 - i)^5 = -4 + 4i$, $(-1 - \sqrt{3}i)^5 = -16 + 16\sqrt{3}i$.

Section 9.2 Exercises

1 $\vec{z} = \begin{bmatrix} 1+i \\ -3i \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} i \\ -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{C}$

Section 9.3 Exercises

1 $Z^* = \begin{bmatrix} 1-i & 2 \\ 1+2i & i \\ -i & 3-i \end{bmatrix}, \mathbf{z}^* = \begin{bmatrix} -1+i & 2-i \end{bmatrix}$

2 $\langle \mathbf{u}, \mathbf{v} \rangle = -3-7i, \langle 2i\mathbf{u}, \mathbf{v} \rangle = -14+6i, \langle \mathbf{u}, 2i\mathbf{v} \rangle = 14-6i$

3 The result is easily verified.

4 $||\alpha \mathbf{z}||^2 = \langle \alpha \mathbf{z}, \alpha \mathbf{z} \rangle = \overline{\alpha}\alpha \langle \mathbf{z}, \mathbf{z} \rangle = |\alpha|^2 \, ||\mathbf{z}||^2$

5 We have $\mathbf{z} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Taking the inner product of both sides with $\mathbf{v}_i$ gives

$$\begin{aligned} \langle \mathbf{v}_i, \mathbf{z} \rangle &= \langle \mathbf{v}_i, c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \rangle \\ &= c_1\langle \mathbf{v}_i, \mathbf{v}_1 \rangle + \cdots + c_n\langle \mathbf{v}_i, \mathbf{v}_n \rangle \\ &= c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle \end{aligned}$$

Thus, for any $1 \le i \le n$, we have $c_i = \dfrac{\langle \mathbf{v}_i, \mathbf{z} \rangle}{||\mathbf{v}_i||^2}$.

Section 9.4 Exercises

1 A is diagonalized by $P = \begin{bmatrix} 1+i & -1-i \\ 1 & 2 \end{bmatrix}$ to $D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$.

2 A is diagonalized by $P = \begin{bmatrix} 1 & 1 & 1 \\ i & -i & 2 \\ 1 & 1 & 0 \end{bmatrix}$ to $D = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Section 9.5 Exercises

1 Observe that $A^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq A$, so A is not Hermitian.

A is unitarily diagonalized by $U = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

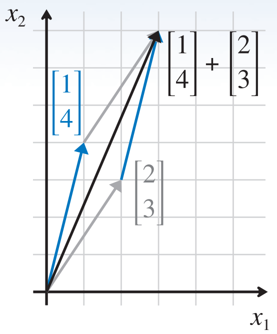
APPENDIX B

Answers to Practice Problems and Chapter Quizzes

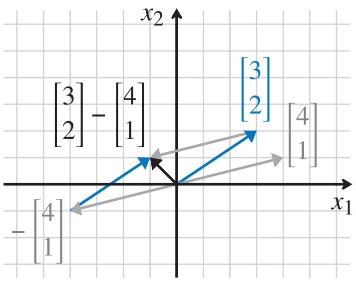
CHAPTER 1

Section 1.1 Practice Problems

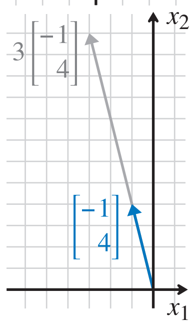
A1



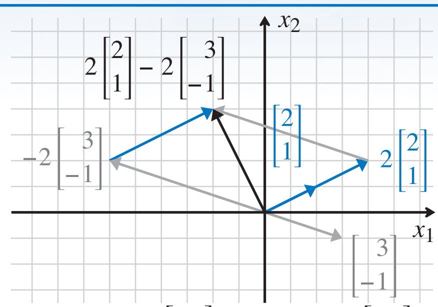
A2



A3



A4



A5 $\begin{bmatrix}3\\1\end{bmatrix}$ **A6** $\begin{bmatrix}-1\\-9\end{bmatrix}$ **A7** $\begin{bmatrix}-6\\4\end{bmatrix}$

A8 $\begin{bmatrix}7/3\\4\end{bmatrix}$ **A9** $\begin{bmatrix}3/2\\0\end{bmatrix}$ **A10** $\begin{bmatrix}5\\4\sqrt{6}\end{bmatrix}$

A11 $\begin{bmatrix}-3\\2\\6\end{bmatrix}$ **A12** $\begin{bmatrix}-1\\2\\-10\end{bmatrix}$ **A13** $\begin{bmatrix}-24\\30\\36\end{bmatrix}$

A14 $\begin{bmatrix}7\\-2\\-5\end{bmatrix}$ **A15** $\begin{bmatrix}7/3\\-4/3\\13/3\end{bmatrix}$ **A16** $\begin{bmatrix}\sqrt{2}-\pi\\\sqrt{2}\\\sqrt{2}+\pi\end{bmatrix}$

A17 (a) $\begin{bmatrix}-4\\7\\-13\end{bmatrix}$ (b) $\begin{bmatrix}-10\\10\\-22\end{bmatrix}$

(c) $\vec{u} = \begin{bmatrix}-1/2\\-7/2\\9/2\end{bmatrix}$ (d) $\vec{u} = \begin{bmatrix}-3\\-6\\6\end{bmatrix}$

A18 (a) $\begin{bmatrix}4\\0\\-1/2\end{bmatrix}$ (b) $\begin{bmatrix}25\\-5\\-10\end{bmatrix}$

(c) $\begin{bmatrix}-1\\-3\\-4\end{bmatrix}$ (d) $\begin{bmatrix}8\\-8/3\\-14/3\end{bmatrix}$

A19 $\vec{PQ} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, $\vec{PR} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{PS} = \begin{bmatrix} -7 \\ -2 \\ 4 \end{bmatrix}$, $\vec{QR} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$, $\vec{SR} = \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix}$, $\vec{PQ} + \vec{QR} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \vec{PS} + \vec{SR}$

A20 $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t\begin{bmatrix} -5 \\ 1 \end{bmatrix}, t \in \mathbb{R}$ **A21** $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} -4 \\ -6 \end{bmatrix}, t \in \mathbb{R}$

A22 $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \\ -11 \end{bmatrix}, t \in \mathbb{R}$ **A23** $\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}$

Other correct answers are possible for problems A24–A28. **A24** $\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -5 \end{bmatrix}, t \in \mathbb{R}$

A25 $\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + t\begin{bmatrix} -6 \\ -2 \end{bmatrix}, t \in \mathbb{R}$

A26 $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}, t \in \mathbb{R}$ **A27** $\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

A28 $\vec{x} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} + t\begin{bmatrix} -3/2 \\ 3/4 \\ -2/3 \end{bmatrix}, t \in \mathbb{R}$

A29 $\begin{cases} x_1 &= -1 + 3t \\ x_2 &= 2 - 5t, \end{cases} t \in \mathbb{R};\ x_2 = -\frac{5}{3}x_1 + \frac{1}{3}.$

A30 $\begin{cases} x_1 &= 1 + t \\ x_2 &= 1 + t, \end{cases} t \in \mathbb{R};\ x_2 = x_1.$

A31 $\begin{cases} x_1 &= 1 + 2t \\ x_2 &= 0 + 0t, \end{cases} t \in \mathbb{R};\ x_2 = 0.$

A32 $\begin{cases} x_1 &= 1 - 2t \\ x_2 &= 3 + 2t, \end{cases} t \in \mathbb{R};\ x_2 = -x_1 + 4.$

A33 (a) Three points P, Q, and R are collinear if $\vec{PQ} = t\vec{PR}$ for some $t \in \mathbb{R}$.
(b) Since $-2\vec{PQ} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \vec{PR}$, the points P, Q, and R must be collinear.
(c) The points S, T, and U are not collinear because $\vec{SU} \neq t\vec{ST}$ for any t.

A34 For V2: $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \end{bmatrix} = \vec{y} + \vec{x}$

For V8: $(s + t)\vec{x} = \begin{bmatrix} (s+t)x_1 \\ (s+t)x_2 \end{bmatrix} = \begin{bmatrix} sx_1 + tx_1 \\ sx_2 + tx_2 \end{bmatrix} = \begin{bmatrix} sx_1 \\ sx_2 \end{bmatrix} + \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} = s\vec{x} + t\vec{x}$

A35 $\vec{F} = \begin{bmatrix} 475 \\ 25\sqrt{3} \end{bmatrix}$

Section 1.2 Practice Problems

A1 $\vec{x} \in \operatorname{Span} \mathcal{B}$
A2 $\vec{x} \in \operatorname{Span} \mathcal{B}$
A3 $\vec{x} \notin \operatorname{Span} \mathcal{B}$
A4 $\vec{x} \in \operatorname{Span} \mathcal{B}$
A5 $\vec{x} \in \operatorname{Span} \mathcal{B}$
A6 $\vec{x} \notin \operatorname{Span} \mathcal{B}$
A7 linearly dependent
A8 linearly independent
A9 linearly independent
A10 linearly dependent
A11 linearly independent
A12 linearly dependent
A13 linearly dependent
A14 linearly independent

A15 A line. $\vec{x} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix}, s \in \mathbb{R}.$

A16 A line. $\vec{x} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$

A17 A line. $\vec{x} = s\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$

A18 Two points in $\mathbb{R}^3$. $\vec{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix}$.

A19 A plane. $\vec{x} = s\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, s, t \in \mathbb{R}.$

A20 The origin. $\vec{x} = \vec{0}$.

A21 Not a basis
A22 A basis
A23 Not a basis
A24 Not a basis
A25 A basis
A26 Not a basis
A27 Not a basis
A28 A basis
A29 A basis
A30 A basis

A31 (a) Show that $\mathcal{B}$ spans $\mathbb{R}^2$ and is linearly independent.
(b) The coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 1$ and $c_2 = 0$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = -1$ and $c_2 = 1$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = -2$ and $c_2 = 3$.

A32 (a) Show that $\mathcal{B}$ spans $\mathbb{R}^2$ and is linearly independent.
(b) The coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 1/2$ and $c_2 = 1/2$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = 1/2$ and $c_2 = -1/2$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = 2$ and $c_2 = -1$.

A33 (a) Show that $\mathcal{B}$ spans $\mathbb{R}^2$ and is linearly independent.
(b) Tthe coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = -1$ and $c_2 = -2$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = 1$ and $c_2 = 1$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = 2$ and $c_2 = 1$.

A34 Assume that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. For a contradiction, assume without loss of generality that $\vec{v}_1 = t\vec{v}_2$. Hence, $\vec{v}_1 - t\vec{v}_2 = \vec{0}$. This contradicts the fact that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.
On the other hand, assume that $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent. Then there exists $c_1, c_2 \in \mathbb{R}$ not both zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Without loss of generality assume that $c_1 \neq 0$. Then $\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2$ and hence $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$.

A35 To prove this, we will prove that both sets are a subset of the other.
Let $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Then there exists $c_1, c_2 \in \mathbb{R}$ such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Since $t \neq 0$ we get

$$\vec{x} = c_1\vec{v}_1 + \frac{c_2}{t}(t\vec{v}_2)$$

so $\vec{x} \in \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$. Thus,

$$\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$$

If $\vec{y} \in \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$, then there exists $d_1, d_2 \in \mathbb{R}$ such that

$$\vec{y} = d_1\vec{v}_1 + d_2(t\vec{v}_2) = d_1\vec{v}_1 + (d_2t)\vec{v}_2 \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$$

Hence, we also have $\operatorname{Span}\{\vec{v}_1, t\vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Therefore, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$.

Section 1.3 Practice Problems

A1 $\sqrt{29}$ **A2** 1 **A3** $\sqrt{2}$
A4 $\sqrt{17}$ **A5** $\sqrt{251}/5$ **A6** 1

A7 $2\sqrt{10}$ **A8** 5
A9 $\sqrt{170}$ **A10** $\sqrt{38}$
A11 Orthogonal **A12** Orthogonal
A13 Not orthogonal **A14** Orthogonal
A15 Orthogonal **A16** Not orthogonal
A17 $k = 6$ **A18** $k = 0, 3$
A19 $k = -3$ **A20** $k \in \mathbb{R}$
A21 $2x_1 + 4x_2 - x_3 = 9$ **A22** $3x_1 + 5x_3 = 26$
A23 $3x_1 - 4x_2 + x_3 = 8$

A24 $\begin{bmatrix} -27 \\ -9 \\ -9 \end{bmatrix}$ **A25** $\begin{bmatrix} -31 \\ -34 \\ 8 \end{bmatrix}$ **A26** $\begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix}$

A27 $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ **A28** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **A29** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

A30 (a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -6 \\ 5 \\ -13 \end{bmatrix}$ (c) $\begin{bmatrix} 6 \\ 9 \\ -15 \end{bmatrix}$
(d) $\begin{bmatrix} -4 \\ 8 \\ -18 \end{bmatrix}$ (e) -14 (f) 14

A31 $x_1 - 4x_2 - 10x_3 = -85$ **A32** $2x_1 - 2x_2 + 3x_3 = -5$
A33 $-5x_1 - 2x_2 + 6x_3 = 15$ **A34** $-17x_1 - x_2 + 10x_3 = 0$

Other correct answers are possible for problems A35–A40.

A35 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}$

A36 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_1, x_3 \in \mathbb{R}$

A37 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$

A38 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} == \begin{bmatrix} 7/3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$

A39 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$

A40 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad x_1, x_3 \in \mathbb{R}$

A41 $39x_1 + 12x_2 + 10x_3 = 140$

A42 $11x_1 - 21x_2 - 17x_3 = -56$

A43 $-12x_1 + 3x_2 - 19x_3 = -14$

A44 $x_2 = 0$

A45 $x_1 + x_2 + 2x_3 = 4$

A46 $14x_1 - 4x_2 - 5x_3 = 9$

A47 $2x_1 - 3x_2 + 5x_3 = 6$ **A48** $x_2 = -2$

A49 $x_1 - x_2 + 3x_3 = 2$

A50 $\vec{x} = \begin{bmatrix} 46/11 \\ 3/11 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ -11 \end{bmatrix}, \quad t \in \mathbb{R}$

A51 $\vec{x} = \begin{bmatrix} 7/2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$

A52 $\vec{x} = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}, \quad t \in \mathbb{R}$

A53 $\vec{x} = t \begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}, \quad t \in \mathbb{R}$

A54 $\sqrt{35}$ **A55** $\sqrt{11}$ **A56** 13

A57 $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that $\vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$. Therefore, $\vec{u}$ lies in the plane through the origin that contains $\vec{v}$ and $\vec{w}$. We can also see this by observing that $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ has volume zero; this can happen only if the three vectors lie in a common plane.

A58 We have

$$\begin{aligned}(\vec{u} - \vec{v}) \times (\vec{u} + \vec{v}) &= \vec{u} \times (\vec{u} + \vec{v}) - \vec{v} \times (\vec{u} + \vec{v}) \\ &= \vec{u} \times \vec{u} + \vec{u} \times \vec{v} - \vec{v} \times \vec{u} - \vec{v} \times \vec{v} \\ &= \vec{0} + \vec{u} \times \vec{v} + \vec{u} \times \vec{v} - \vec{0} \\ &= 2(\vec{u} \times \vec{v})\end{aligned}$$

Section 1.4 Practice Problems

A1 $\begin{bmatrix} 5 \\ 9 \\ 0 \\ 1 \end{bmatrix}$

A2 $\begin{bmatrix} 10 \\ -7 \\ 10 \\ -5 \end{bmatrix}$

A3 $\begin{bmatrix} 6 \\ 0 \\ 4 \\ 5 \\ 3 \end{bmatrix}$

A4 $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -3 \end{bmatrix}$

A5 Subspace **A6** Not a subspace

A7 Subspace **A8** Subspace

A9 Not a subspace **A10** Subspace

A11 Subspace **A12** Not a subspace

A13 Not a subspace **A14** Not a subspace

A15 Subspace **A16** Not a subspace

Other correct answers are possible for problems A17–A20.

A17 $1\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+0\begin{bmatrix}1\\0\\1\\2\end{bmatrix}+0\begin{bmatrix}2\\1\\-1\\-3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

A18 $0\begin{bmatrix}2\\-1\\3\\2\end{bmatrix}-2\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}+1\begin{bmatrix}0\\4\\2\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

A19 $1\begin{bmatrix}1\\1\\0\\2\end{bmatrix}+1\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-1\begin{bmatrix}2\\2\\1\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

A20 $(-3)\begin{bmatrix}1\\1\\-2\\3\end{bmatrix}+2\begin{bmatrix}1\\2\\1\\3\end{bmatrix}+\begin{bmatrix}1\\-1\\-8\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

A21 Linearly dependent **A22** Linearly dependent

A23 Linearly independent **A24** Linearly independent

A25 Show that B is linearly independent and spans P.

A26 Show that B is linearly independent and spans P.

A27 Show that B is linearly independent and spans P.

A28 Show that B is linearly independent and spans P.

A29 A plane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\1\\3\end{bmatrix}\right\}$.

A30 A hyperplane with basis $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

A31 A line with basis $\left\{\begin{bmatrix}3\\1\\-1\\0\end{bmatrix}\right\}$.

A32 A plane with basis $\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$.

A33 If $\vec{x}=\vec{p}+t\vec{d}$ is a subspace of $\mathbb{R}^n$, then it contains the zero vector. Hence, there exists t_1 such that $\vec{0}=\vec{p}+t_1\vec{d}$. Thus, $\vec{p}=-t_1\vec{d}$ and so $\vec{p}$ is a scalar multiple of $\vec{d}$. On the other hand, if $\vec{p}$ is a scalar multiple of $\vec{d}$, say $\vec{p}=t_1\vec{d}$, then we have $\vec{x}=\vec{p}+t\vec{d}=t_1\vec{d}+t\vec{d}=(t_1+t)\vec{d}$. Hence, the set is $\text{Span}\{\vec{d}\}$ and thus is a subspace.

A34 Assume there is a non-empty subset $\mathcal{B}_1=\{\vec{v}_1,\dots,\vec{v}_\ell\}$ of $\mathcal{B}$ that is linearly dependent. Then there exists c_i not all zero such that

$$\vec{0}=c_1\vec{v}_1+\cdots+c_\ell\vec{v}_\ell=c_1\vec{v}_1+\cdots+c_\ell\vec{v}_\ell+0\vec{v}_{\ell+1}+\cdots+0\vec{v}_n$$

which contradicts the fact that $\mathcal{B}$ is linearly independent. Hence, $\mathcal{B}_1$ must be linearly independent.

A35 (a) Assume $\text{Span}\{\vec{v}_1,\dots,\vec{v}_k\}=\text{Span}\{\vec{v}_1,\dots,\vec{v}_{k-1}\}$. Then there exists $b_1,\dots,b_{k-1}\in\mathbb{R}$ such that

$$\vec{v}_k=b_1\vec{v}_1+\cdots+b_{k-1}\vec{v}_{k-1}$$

So, $\vec{v}_k$ is a linear combination of $\vec{v}_1,\dots,\vec{v}_{k-1}$.

(b) If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1,\dots,\vec{v}_{k-1}$, then, there exist $c_1,\dots,c_{k-1}\in\mathbb{R}$ such that $c_1\vec{v}_1+\cdots+c_{k-1}\vec{v}_{k-1}=\vec{v}_k$. For any $\vec{x}\in\text{Span}\{\vec{v}_1,\dots,\vec{v}_k\}$ there exist $d_1,\dots,d_k\in\mathbb{R}$ such that

$$\begin{aligned}\vec{x}&=d_1\vec{v}_1+\cdots+d_{k-1}\vec{v}_{k-1}+d_k\vec{v}_k\\&=d_1\vec{v}_1+\cdots+d_{k-1}\vec{v}_{k-1}+d_k(c_1\vec{v}_1+\cdots+c_{k-1}\vec{v}_{k-1})\\&=(d_1+d_kc_1)\vec{v}_1+\cdots+(d_{k-1}+d_kc_{k-1})\vec{v}_{k-1}\\&\in\text{Span}\{\vec{v}_1,\dots,\vec{v}_{k-1}\}\end{aligned}$$

On the other hand, if $\vec{y}\in\text{Span}\{\vec{v}_1,\dots,\vec{v}_{k-1}\}$, then there exists $a_1,\dots,a_{k-1}\in\mathbb{R}$ such that

$$\begin{aligned}\vec{y}&=a_1\vec{v}_1+\cdots+a_{k-1}\vec{v}_{k-1}\\&=a_1\vec{v}_1+\cdots+a_{k-1}\vec{v}_{k-1}+0\vec{v}_k\end{aligned}$$

Thus, $\vec{y}\in\text{Span}\{\vec{v}_1,\dots,\vec{v}_k\}$.

A36 The linear combination represent how much material is required to produce 100 thingamajiggers and 250 whatchamacallits.

Section 1.5 Practice Problems

A1 -3 **A2** -4 **A3** 0

A4 $\sqrt{6}$ **A5** 1 **A6** $\sqrt{15}$

A7 $\frac{1}{\sqrt{30}}\begin{bmatrix}1\\2\\5\end{bmatrix}$ **A8** $\frac{1}{\sqrt{15}}\begin{bmatrix}3\\-2\\-1\\1\end{bmatrix}$ **A9** $\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$

A10 $\frac{1}{\sqrt{39}}\begin{bmatrix}1\\2\\5\\-3\end{bmatrix}$ **A11** $\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$ **A12** $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}$

A13 $2\sqrt{22} \approx 9.38 \leq \sqrt{26} + \sqrt{30} \approx 10.58$, $16 \leq \sqrt{26(30)} \approx 27.93$

A14 $\sqrt{41} \approx 6.40 \leq \sqrt{6} + \sqrt{29} \approx 7.83$, $3 \leq \sqrt{6(29)} \approx 13.19$

A15 $3x_1 + x_2 + 4x_3 = 0$

A16 $x_2 + 3x_3 + 3x_4 = 1$

A17 $3x_1 - 2x_2 - 5x_3 + x_4 = 4$

A18 $2x_1 - 4x_2 + x_3 - 3x_4 = -19$

A19 $x_1 - 4x_2 + 5x_3 - 2x_4 = 0$

A20 $x_2 + 2x_3 + x_4 + x_5 = 5$

A21 $\begin{bmatrix}2\\1\end{bmatrix}$ **A22** $\begin{bmatrix}3\\-2\\3\end{bmatrix}$ **A23** $\begin{bmatrix}-4\\3\\-5\end{bmatrix}$

A24 $\begin{bmatrix}1\\-1\\2\\-3\end{bmatrix}$ **A25** $\begin{bmatrix}1\\1\\-1\\2\\-1\end{bmatrix}$

A26 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}0\\-5\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3\\0\end{bmatrix}$

A27 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}36/25\\48/25\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-136/25\\102/25\end{bmatrix}$

A28 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}0\\5\\0\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-3\\0\\2\end{bmatrix}$

A29 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-4/9\\8/9\\-8/9\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}40/9\\1/9\\-19/9\end{bmatrix}$

A30 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1\\-1\\2\\-1\end{bmatrix}$

A31 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1/2\\0\\0\\-1/2\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}5/2\\3\\2\\-5/2\end{bmatrix}$

A32 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}0\\0\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3\\-3\end{bmatrix}$

A33 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-2/17\\-3/17\\2/17\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}70/17\\-14/17\\49/17\end{bmatrix}$

A34 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}14/3\\-7/3\\7/3\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}1/3\\4/3\\2/3\end{bmatrix}$

A35 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3/2\\3/2\\-3\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}5/2\\-1/2\\1\end{bmatrix}$

A36 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}1/3\\-2/3\\-1/3\\1\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}5/3\\-1/3\\7/3\\0\end{bmatrix}$

A37 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1/3\\0\\-1/6\\-1/6\end{bmatrix}, \text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-2/3\\2\\-5/6\\13/6\end{bmatrix}$

A38 (a) $\begin{bmatrix}2/7\\6/7\\3/7\end{bmatrix}$ (b) $\begin{bmatrix}220/49\\660/49\\330/49\end{bmatrix}$ (c) $\begin{bmatrix}270/49\\222/49\\-624/49\end{bmatrix}$

A39 (a) $\begin{bmatrix}3/\sqrt{14}\\1/\sqrt{14}\\-2/\sqrt{14}\end{bmatrix}$ (b) $\begin{bmatrix}24/7\\8/7\\-16/7\end{bmatrix}$ (c) $\begin{bmatrix}-3/7\\69/7\\30/7\end{bmatrix}$

A40 $R(5/2, 5/2)$, $\|\text{perp}_{\vec{d}}(\vec{PQ})\| = 5/\sqrt{2}$

A41 $R(58/17, 91/17)$, $\|\text{perp}_{\vec{d}}(\vec{PQ})\| = 6/\sqrt{17}$

A42 $R(17/6, 1/3, -1/6)$, $\|\text{perp}_{\vec{d}}(\vec{PQ})\| = \sqrt{29/6}$

A43 $R(5/3, 11/3, -1/3)$, $\|\text{perp}_{\vec{d}}(\vec{PQ})\| = \sqrt{6}$

A44 $2/\sqrt{26}$ **A45** $13/\sqrt{38}$ **A46** $4/\sqrt{5}$

A47 $\sqrt{6}$ **A48** $3/\sqrt{11}$ **A49** $13/\sqrt{21}$

A50 $5/\sqrt{3}$

A51 $R(1/7, 3/7, -3/7, 4/7)$ **A52** $R(15/14, 13/7, 17/14, 3)$

A53 $R(0, 14/3, 1/3, 10/3)$ **A54** $R(-12/7, 11/7, 9/7, -9/7)$

A55 1 **A56** 126 **A57** 5

A58 35 **A59** $k \approx 3.07$

Chapter 1 Quiz

E1 $\begin{bmatrix}1\\2\end{bmatrix}$

E2 $\dfrac{1}{\sqrt{54}}\begin{bmatrix}2\\-1\\7\end{bmatrix}$

E3 $\begin{bmatrix}4/5\\4/15\\8/15\\-4/15\end{bmatrix}$, $\text{perp}_{\vec{u}}(\vec{v}) = \begin{bmatrix}1/5\\-4/15\\22/15\\49/15\end{bmatrix}$

E4 $\vec{x} = \begin{bmatrix}-2\\1\\-4\end{bmatrix} + t\begin{bmatrix}7\\-3\\5\end{bmatrix}, \quad t \in \mathbb{R}$

E5 $\vec{x} = \begin{bmatrix}3\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}2\\0\\1\end{bmatrix}, x_2, x_3 \in \mathbb{R}.$

E6 $8x_1 - x_2 + 7x_3 = 9$

E7 $\mathcal{B} = \left\{\begin{bmatrix}2\\6\\4\end{bmatrix}, \begin{bmatrix}1\\3\\3\end{bmatrix}\right\}$

E8 The set is linearly independent.

E9 (a) Show B spans $\mathbb{R}^2$ and that it is linearly independent.

(b) $t_1 = \frac{11}{4}, t_2 = -\frac{1}{4}$

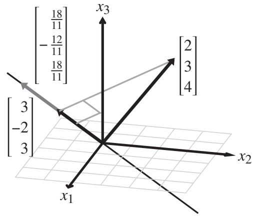
(c) $t_1 = \frac{11}{2}, t_2 = -\frac{1}{2}$

E10 S is not a subspace.

E11 If $d \neq 0$, then $a_1(0) + a_2(0) + a_3(0) = 0 \neq d$, so $\vec{0} \notin S$ and thus, S is not a subspace of $\mathbb{R}^3$.
On the other hand, assume $d = 0$. Show that $s\vec{x} + t\vec{y}$ satisfies the condition of S.

E12 Show that B is linearly independent and spans P.

E13 $Q(18/11, -12/11, 18/11)$



E14 Let $R(5/2, -5/2, -1/2, 3/2)$, $\|\vec{PR}\| = 1$

E15 The volume determined by $\vec{u} + k\vec{v}$, $\vec{v}$, and $\vec{w}$ is

$$\begin{aligned}|(\vec{u} + k\vec{v}) \cdot (\vec{v} \times \vec{w})| &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(\vec{v} \cdot (\vec{v} \times \vec{w}))| \\ &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(0)|\end{aligned}$$

which equals the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

E16 FALSE. The points $P(0,0,0)$, $Q(0,0,1)$, and $R(0,0,2)$ lie in every plane of the form $t_1x_1 + t_2x_2 = 0$ with t_1 and t_2 not both zero.

E17 TRUE. This is the definition of a line reworded in terms of a spanning set.

E18 TRUE. By definition of the plane $\{\vec{v}_1, \vec{v}_2\}$ spans the plane. If $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent, then the set would not satisfy the definition of a plane, so $\{\vec{v}_1, \vec{v}_2\}$ must be linearly independent. Hence, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for the plane.

E19 FALSE. The dot product of the zero vector with itself is 0.

E20 FALSE. Let $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}1\\1\end{bmatrix}$.

Then, $\text{proj}_{\vec{x}}\,\vec{y} = \begin{bmatrix}1\\0\end{bmatrix}$, while $\text{proj}_{\vec{y}}\,\vec{x} = \begin{bmatrix}1/2\\1/2\end{bmatrix}$.

E21 FALSE. If $\vec{y} = \vec{0}$, then $\text{proj}_{\vec{x}}\,\vec{y} = \vec{0}$. Thus, $\{\text{proj}_{\vec{x}}(\vec{y}), \text{perp}_{\vec{x}}(\vec{y})\}$ contains the zero vector so it is linearly dependent.

E22 TRUE. We have

$$\|\vec{u}\times(\vec{v}+3\vec{u})\| = \|\vec{u}\times\vec{v}+3(\vec{u}\times\vec{u})\| = \|\vec{u}\times\vec{v}+\vec{0}\| = \|\vec{u}\times\vec{v}\|$$

so the parallelograms have the same area.

CHAPTER 2

Section 2.1 Practice Problems

A1 $\vec{x} = \begin{bmatrix} 17 \\ 4 \end{bmatrix}$

A2 $\vec{x} = \begin{bmatrix} 13 \\ 0 \\ 6 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}$

A3 $\vec{x} = \begin{bmatrix} 32 \\ -8 \\ 2 \end{bmatrix}$

A4 $\vec{x} = \begin{bmatrix} -1 \\ -3 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

A5 A is in row echelon form.

A6 B is in row echelon form.

A7 C is not in row echelon.

A8 D is not in row echelon.

Other correct answers are possible for problems A9–A14.

A9 $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 13 & -7 \end{bmatrix}$

A10 $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A11 $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

A12 $\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A13 $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

A14 $\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 24 & 1 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

A15 Inconsistent.

A16 Consistent. $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}.$

A17 Consistent. $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R},$

A18 Consistent. $\vec{x} = \begin{bmatrix} 19/2 \\ 0 \\ 5/2 \\ -2 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, t \in \mathbb{R}.$

A19 Consistent. $\vec{x} = s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}.$

A20 Inconsistent.

A21 (a) $\left[\begin{array}{cc|c} 3 & -5 & 2 \\ 1 & 2 & 4 \end{array}\right]$

(b) $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -11 & -10 \end{array}\right]$

(c) Consistent.

(d) $\vec{x} = \begin{bmatrix} 24/11 \\ 10/11 \end{bmatrix}.$

A22 (a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & -3 & 2 & 6 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -7 & 0 & -4 \end{array}\right]$

(c) Consistent.

(d) $\vec{x} = \begin{bmatrix} 27/7 \\ 4/7 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$

A23 (a) $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 1 & 3 & -5 & 11 \\ 2 & 5 & -8 & 19 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(c) Consistent.

(d) $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$

A24 (a) $\left[\begin{array}{ccc|c} -3 & 6 & 16 & 36 \\ 1 & -2 & -5 & -11 \\ 2 & -3 & -8 & -17 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & -2 & -5 & -11 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$

(c) Consistent.

(d) $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$

A25 (a) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 5 & 1 & 10 \\ 4 & 9 & -1 & 19 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(c) Inconsistent.

A26 (a) $\left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 2 & 4 & -6 & 1 & -8 \\ 6 & 13 & -17 & 4 & -21 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 0 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

(c) Consistent.

(d) $\vec{x} = \begin{bmatrix} -7 \\ 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}.$

A27 (a) $\left[\begin{array}{ccccc|c} 0 & 2 & -2 & 0 & 1 & 2 \\ 1 & 2 & -3 & 1 & 4 & 1 \\ 2 & 4 & -5 & 3 & 8 & 3 \\ 2 & 5 & -7 & 3 & 10 & 5 \end{array}\right]$

(b) $\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & 4 & 1 \\ 0 & 2 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3/2 & 2 \end{array}\right]$

(c) Consistent.

(d) $\vec{x} = \begin{bmatrix} -4 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 3/2 \\ -3/2 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$

A28 $y = 3 - x + x^2$

A29 $y = 2 - 3x^2$

A30 $y = 1 + 2x + 3x^2$

A31 $y = -3 - x + 3x^2$

A32 If $a \neq 0$, $b \neq 0$, this system is consistent, and the solution is unique. If $a = 0$, $b \neq 0$, the system is consistent, but the solution is not unique. If $a \neq 0$, $b = 0$, the system is inconsistent. If $a = 0$, $b = 0$, this system is consistent, but the solution is not unique.

A33 If $c \neq 0$, $d \neq 0$, the system is consistent and has no free variables, so the solution is unique. If $c = d = 0$, then the system is consistent and x_3 is a free variable, so there are infinitely many solutions. If $c \neq 0$ and $d = 0$, then the last row is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & c \end{array}\right]$, so the system is inconsistent. If $c = 0$ and $d \neq 0$, then the system is consistent and x_4 is a free variable so there are infinitely many solutions.

A34 600 apples, 400 bananas, and 500 oranges.

A35 75% in algebra, 90% in calculus, and 84% in physics.

Section 2.2 Practice Problems

A1 The matrix is not in RREF.

A2 The matrix is not in RREF.

A3 The matrix is in RREF.

A4 The matrix is in RREF.

A5 The matrix is in RREF.

A6 The matrix is not in RREF.

A7 The matrix is not in RREF.

A8 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$; rank = 2

A9 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; rank = 3

A10 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; rank = 3

A11 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; rank = 3

A12 $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; rank = 2

A13 $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; rank = 3

A14 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$; rank = 3

A15 $\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; rank = 3

A16 $\begin{bmatrix} 1 & 0 & 0 & 0 & -56 \\ 0 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & 23 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$; rank = 4

A17 There is one parameter. The general solution is

$$\vec{x} = t\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

A18 There are two parameters. The general solution is

$$\vec{x} = s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

A19 There are two parameters. The general solution is
$$\vec{x} = s\begin{bmatrix}3\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}, \quad s,t \in \mathbb{R}.$$

A20 There are two parameters. The general solution is
$$\vec{x} = s\begin{bmatrix}-2\\1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\2\\0\\-1\\1\end{bmatrix}, \quad s,t \in \mathbb{R}.$$

A21 There are two parameters. The general solution is
$$\vec{x} = s\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}-4\\0\\5\\1\\0\end{bmatrix}, \quad s,t \in \mathbb{R}.$$

A22 There is one parameter. The general solution is
$$\vec{x} = t\begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix}, t \in \mathbb{R}.$$

A23 $\left[\begin{array}{cc|c}1 & 0 & 24/11\\0 & 1 & 10/11\end{array}\right]$. $\vec{x} = \begin{bmatrix}24/11\\10/11\end{bmatrix}$.

A24 $\left[\begin{array}{ccc|c}1 & 0 & 1 & 27/7\\0 & 1 & 0 & 4/7\end{array}\right]$. $\vec{x} = \begin{bmatrix}27/7\\4/7\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad t \in \mathbb{R}.$

A25 $\left[\begin{array}{ccc|c}1 & 0 & 1 & 2\\0 & 1 & -2 & 3\\0 & 0 & 0 & 0\end{array}\right]$. $\vec{x} = \begin{bmatrix}2\\3\\0\end{bmatrix} + t\begin{bmatrix}-1\\2\\1\end{bmatrix}, \quad t \in \mathbb{R}.$

A26 $\left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\0 & 1 & 0 & -1\\0 & 0 & 1 & 3\end{array}\right]$. $\vec{x} = \begin{bmatrix}2\\-1\\3\end{bmatrix}$.

A27 $\left[\begin{array}{ccc|c}1 & 0 & -7 & 0\\0 & 1 & 3 & 0\\0 & 0 & 0 & 1\end{array}\right]$. Inconsistent.

A28 $\left[\begin{array}{cccc|c}1 & 0 & -5 & 0 & -7\\0 & 1 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 2\end{array}\right]$. $\vec{x} = \begin{bmatrix}-7\\1\\0\\2\end{bmatrix} + t\begin{bmatrix}5\\-1\\1\\0\end{bmatrix}, t \in \mathbb{R}.$

A29 $\left[\begin{array}{ccccc|c}1 & 0 & 0 & 0 & 0 & -4\\0 & 1 & 0 & 0 & -1 & 0\\0 & 0 & 1 & 0 & -3/2 & -1\\0 & 0 & 0 & 1 & 3/2 & 2\end{array}\right]$.
$$\vec{x} = \begin{bmatrix}-4\\0\\-1\\2\\0\end{bmatrix} + t\begin{bmatrix}0\\1\\3/2\\-3/2\\1\end{bmatrix}, \quad t \in \mathbb{R}.$$

A30 $\begin{bmatrix}0 & 2 & -5\\1 & 2 & 3\\1 & 4 & -3\end{bmatrix}$; rank=3; 0 parameters. $\vec{x} = \vec{0}$.

A31 $\begin{bmatrix}3 & 1 & -9\\1 & 1 & -5\\2 & 1 & -7\end{bmatrix}$; rank =2; 1 parameter.
$$\vec{x} = t\begin{bmatrix}2\\3\\1\end{bmatrix}, \quad t \in \mathbb{R}.$$

A32 $\begin{bmatrix}1 & -1 & 2 & -3\\3 & -3 & 8 & -5\\2 & -2 & 5 & -4\\3 & -3 & 7 & -7\end{bmatrix}$; rank=2; 2 parameters.
$$\vec{x} = s\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}7\\0\\-2\\1\end{bmatrix}, \quad s,t \in \mathbb{R}.$$

A33 $\begin{bmatrix}0 & 1 & 2 & 2 & 0\\1 & 2 & 5 & 3 & -1\\2 & 1 & 5 & 1 & -3\\1 & 1 & 4 & 2 & -2\end{bmatrix}$; rank=3; 2 parameters.
$$\vec{x} = s\begin{bmatrix}2\\0\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\-2\\1\\0\\1\end{bmatrix}, \quad s,t \in \mathbb{R}.$$

A34 $\vec{x} = \begin{bmatrix}-3\\1\\2\end{bmatrix}$; $\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

A35 $\vec{x} = \begin{bmatrix}-2\\1\\0\end{bmatrix} + t\begin{bmatrix}-5\\0\\1\end{bmatrix}, t \in \mathbb{R}$; $\vec{x} = t\begin{bmatrix}-5\\0\\1\end{bmatrix}, t \in \mathbb{R}$

A36 $\vec{x} = \begin{bmatrix}-5\\1\\0\\0\end{bmatrix} + s\begin{bmatrix}-9\\5\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\-2\\0\\1\end{bmatrix}, \quad s,t \in \mathbb{R};$
$$\vec{x} = s\begin{bmatrix}-9\\5\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\-2\\0\\1\end{bmatrix}, \quad s,t \in \mathbb{R}$$

A37 $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R};$

$\vec{x} = t \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$

A38 $\vec{x} = \begin{bmatrix} -3 \\ 14 \\ 4 \\ -5 \end{bmatrix}; \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

A39 $\vec{x} = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R};$

$\vec{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$

A40 (a) Row reducing the coefficient matrix gives $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & -2 \\ 1 & 4 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ Since there is a leading one in each row, there cannot be a row in the RREF of $\left[A \mid \vec{b}\right]$ of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$. Hence, the system will be consistent for any $b_1, b_2, b_3 \in \mathbb{R}$.

(b) Since $\operatorname{rank}(A) = m$, there cannot be a row in the RREF of $\left[A \mid \vec{b}\right]$ of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$. So, the system will be consistent for any $\vec{b} \in \mathbb{R}^m$.

(c) $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ makes the system inconsistent.

(d) If rank $A < m$, then the RREF R of A has a row of all zeros. Thus, $[R \mid \vec{e}_m]$ is inconsistent as it has a row of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$. Since elementary row operations are reversible, we can apply the reverse of the row operations needed to row reduce A to R on $[R \mid \vec{e}_m]$ to get $\left[A \mid \vec{b}\right]$ for some $\vec{b} \in \mathbb{R}^n$. Then this system is inconsistent since elementary row operations do not change the solution set. Thus, there exists some $\vec{b} \in \mathbb{R}^m$ such that $\left[A \mid \vec{b}\right]$ is inconsistent.

Section 2.3 Practice Problems

A1 (a) $2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 8 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 5 \\ 4 \\ 6 \\ 7 \end{bmatrix}$ is not in the span.

(c) $3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

A2 (a) $\begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix}$ is not in the span.

(b) $(-2) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 0 \\ 8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the span.

A3 $x_3 = 0$

A4 $x_1 - 2x_2 = 0, x_3 = 0$

A5 $x_1 + 3x_2 - 2x_3 = 0$

A6 $x_1 + 3x_2 + 5x_3 = 0$

A7 $-x_1 - x_2 + x_3 = 0, x_2 + x_4 = 0$

A8 $-4x_1 + 5x_2 + x_3 + 4x_4 = 0$

A9 A basis is the empty set. The dimension is 0.

A10 A basis is $\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. The dimension is 2.

A11 It is a basis for the plane.

A12 It is not a basis for the plane.

A13 It is a basis for the hyperplane.

A14 Linearly independent.

A15 Linearly dependent.

$$-3t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 2t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

A16 Linearly dependent.

$$2t\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} - t\begin{bmatrix}2\\3\\1\\3\\3\end{bmatrix} + t\begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}, \quad t \in \mathbb{R}$$

A17 Linearly independent.

A18 Linearly independent for all $k \neq -3$.

A19 Linearly independent for all $k \neq -5/2$.

A20 It is a basis.

A21 Only two vectors, so it cannot span $\mathbb{R}^3$. Therefore, it is not a basis.

A22 It has four vectors in $\mathbb{R}^3$, so it is linearly dependent. Therefore, it is not a basis.

A23 It is linearly dependent, so it is not a basis.

Section 2.4 Practice Problems

A1 $x_1 = 12, x_2 = 26$, and $x_3 = 28$

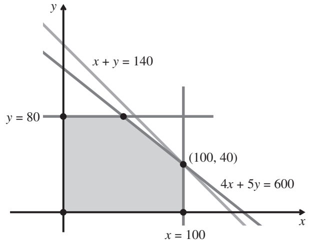
A2
$$\begin{aligned}(R_1 + R_2)i_1 - R_2 i_2 &= E_1\\ -R_2 i_1 + (R_2 + R_3)i_2 - R_3 i_3 &= 0\\ -R_3 i_2 + (R_3 + R_4 + R_8)i_3 - R_8 i_5 &= 0\\ (R_5 + R_6)i_4 - R_6 i_5 &= 0\\ -R_8 i_3 - R_6 i_4 + (R_6 + R_7 + R_8)i_5 &= E_2\end{aligned}$$

A3 $x_1 = 30 - t, x_2 = 10 - t, x_3 = 60 - t, x_4 = t, t \in \mathbb{R}$

A4 $\dfrac{4x^4 + x^3 + x^2 + x + 1}{(x-1)(x^2+1)^2} = \dfrac{2}{x-1} + \dfrac{2x+3}{x^2+1} + \dfrac{-2x-2}{(x^2+1)^2}$

A5 $2Al(OH)_3 + 3H_2CO_3 \to Al_2(CO_3)_3 + 6H_2O$

A6 The maximum value is 140 (occurring at $(100, 40)$).



A7 To simplify writing, let $\alpha = \frac{1}{\sqrt{2}}$.
Total horizontal force: $R_1 + R_2 = 0$.
Total vertical force: $R_V - F_V = 0$.
Total moment about A: $R_1 s + F_V(2\sqrt{2}s) = 0$.
The horizontal and vertical equations at the joints are
$\alpha N_2 + R_2 = 0$ and $N_1 + \alpha N_2 + R_V = 0$;
$N_3 + \alpha N_4 + R_1 = 0$ and $-N_1 + \alpha N_4 = 0$;
$-\alpha N_2 - N_3 + \alpha N_6 = 0$ and $-\alpha N_2 + N_5 + \alpha N_6 = 0$;
$-\alpha N_4 + N_7 = 0$ and $-\alpha N_4 - N_5 = 0$;
$-N_7 - \alpha N_6 = 0$ and $-\alpha N_6 - F_V = 0$.

Chapter 2 Quiz

E1 (a) $\left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 2\\ 2 & -2 & 4 & -1 & 10\\ 1 & -1 & 1 & 0 & 2\\ 1 & 0 & 1 & 0 & 9\end{array}\right]$.

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 13/2\\ 0 & 1 & 0 & 0 & 7\\ 0 & 0 & -2 & 1 & -5\\ 0 & 0 & 0 & 0 & 1\end{array}\right]$. The rank is 4.

(c) The system is inconsistent.

E2 (a) $\left[\begin{array}{cccc|c} 2 & 4 & 1 & -6 & 7\\ 4 & 8 & -3 & 8 & -1\\ -3 & -6 & 2 & -5 & 0\\ 1 & 2 & 1 & -5 & 5\end{array}\right]$.

(b) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2\\ 0 & 0 & 1 & -4 & 3\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$. The rank is 2.

(c) $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}2\\0\\3\\0\end{bmatrix} + x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}1\\0\\4\\1\end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}$

E3 (a) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & -1/3\\ 0 & 0 & 0 & 1 & 1/3\end{bmatrix}$. The rank is 4.

(b) $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1/3 \\ -1/3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}$. The dimension is 1.

E4 $-3x_1 - x_2 + x_3 = 0$

E5 (a) The system is inconsistent for all (a, b, c) of the form $(a, b, 1)$ or $(a, -2, c)$, and is consistent for all (a, b, c) where $b \neq -2$ and $c \neq 1$.

(b) The system has a unique solution if and only if $b \neq -2, c \neq 1$, and $c \neq -1$.

E6 (a) $\vec{x} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 11/4 \\ -11/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$

(b) If there exists a vector $\vec{x} \in \mathbb{R}^5$ which is orthogonal to $\vec{u}, \vec{v}$, and $\vec{w}$, then $\vec{x} \cdot \vec{u} = 0, \vec{x} \cdot \vec{v} = 0$, and $\vec{x} \cdot \vec{w} = 0$, yields a homogeneous system of three linear equations with five variables. Hence, the rank of the matrix is at most three and thus there are at least 2 parameters (# of variables - rank = $5 - 3 = 2$). So, there are in fact infinitely many vectors orthogonal to the three vectors.

E7 The set is linearly independent by Lemma 2.3.3.

E8 The set does not span $\mathbb{R}^3$.

E9 Consider $\vec{x} = t_1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} + t_3 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$. The RREF of the corresponding coefficient matrix is I. Thus, it is a basis for $\mathbb{R}^3$ by Theorem 2.3.5.

E10 False **E11** False **E12** False

E13 False **E14** True **E15** True

CHAPTER 3

Section 3.1 Practice Problems

A1 $\begin{bmatrix} -1 & -6 & 4 \\ 6 & -4 & 2 \end{bmatrix}$

A2 $\begin{bmatrix} 13 & 4 & 11 \\ 6 & 14 & -11 \end{bmatrix}$

A3 Not defined.

A4 Not defined.

A5 $\begin{bmatrix} 10 & -12 \\ 2 & -5 \end{bmatrix}$

A6 $\begin{bmatrix} -7 & 6 & -5 \\ -4 & -13 & 5 \\ -16 & -6 & -2 \end{bmatrix}$

A7 $\begin{bmatrix} 6 & 7/3 \\ 19/2 & 19/3 \\ 21 & 34/3 \end{bmatrix}$

A8 $\begin{bmatrix} 27/2 \\ 49/3 \end{bmatrix}$

A9 10

A10 $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 5 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

A11 $A = \begin{bmatrix} 1 & -4 & 1 & -2 \\ 1 & -1 & 3 & 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

A12 $A = \begin{bmatrix} 1/3 & 3 & -1/4 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2/3 \\ 3 \end{bmatrix}$

A13 $A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 5 & -8 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 4 \\ 17 \end{bmatrix}$

A14 TRUE. Since A has 2 columns, $\vec{x}$ must have 2 entries.

A15 FALSE. Since A has 2 rows, we get that $A\vec{x} \in \mathbb{R}^2$.

A16 FALSE. IF B is the identity matrix, then $AB = A = BA$ by Theorem 3.1.7.

A17 TRUE. By definition of matrix-matrix multiplication $A^T A$ will have n rows (since A^T has n rows) and will have n columns (since A has n columns). Thus, $A^T A$ is $n \times n$.

A18 FALSE. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

A19 FALSE. Take $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

A20 $(A + B)^T = \begin{bmatrix} -3 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}, A^T + B^T = \begin{bmatrix} -3 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$

A21 $(AB)^T = \begin{bmatrix} -21 & -10 \\ 15 & -27 \end{bmatrix}, B^T A^T = \begin{bmatrix} -21 & -10 \\ 15 & -27 \end{bmatrix}$

A22 $\begin{bmatrix} 13 & 31 & 2 \\ 10 & 12 & 10 \end{bmatrix}$

A23 Not defined.

A24 Not defined.

A25 $\begin{bmatrix} 11 & 7 & 3 & 15 \\ 7 & 9 & 11 & 1 \end{bmatrix}$

A26 $x_1 + 2x_2 + x_3$

A27 Not defined.

A28 $\begin{bmatrix} 52 & 139 \\ 62 & 46 \end{bmatrix}$

A29 $\begin{bmatrix} 13 & 10 \\ 31 & 12 \\ 2 & 10 \end{bmatrix}$

A30 $\begin{bmatrix} 5 & -19 \\ 7 & 9 \\ 3 & 11 \\ 15 & 1 \end{bmatrix}$

A31 (a) $A\vec{x} = \begin{bmatrix} 12 \\ 17 \\ 3 \end{bmatrix}, A\vec{y} = \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}, A\vec{z} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 12 & 8 & -2 \\ 17 & 4 & 5 \\ 3 & -4 & 1 \end{bmatrix}$

A32 $\begin{bmatrix} -13 & 16 \\ -27 & 0 \end{bmatrix}$

A33 (a) $A \in \operatorname{Span} \mathcal{B}$.
(b) $\mathcal{B}$ is linearly independent.

A34 Using the second view of matrix-vector multiplication and the fact that the i-th component of $\vec{e}_i$ is 1 and all other components are 0, we get

$$A\vec{e}_i = 0\vec{a}_1 + \cdots + 0\vec{a}_{i-1} + 1\vec{a}_i + 0\vec{a}_{i+1} + \cdots + 0\vec{a}_n = \vec{a}_i$$

Section 3.2 Practice Problems

A1 (a) $f_A : \mathbb{R}^2 \to \mathbb{R}^4$

(b) $f_A(2,-5) = \begin{bmatrix} -19 \\ 6 \\ -23 \\ 38 \end{bmatrix}, f_A(-3,4) = \begin{bmatrix} 18 \\ -9 \\ 17 \\ -36 \end{bmatrix}$

(c) $f_A(1,0) = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, f_A(0,1) = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -6 \end{bmatrix}$

(d) $f_A(\vec{x}) = \begin{bmatrix} -2x_1 + 3x_2 \\ 3x_1 + 0x_2 \\ x_1 + 5x_2 \\ 4x_1 - 6x_2 \end{bmatrix}$

(e) $[f_A] = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix}$

A2 (a) The domain is $\mathbb{R}^4$. The codomain is $\mathbb{R}^3$.

(b) $f_A(2,-2,3,1) = \begin{bmatrix} -11 \\ 9 \\ 7 \end{bmatrix}, f_A(-3,1,4,2) = \begin{bmatrix} -13 \\ -1 \\ 3 \end{bmatrix}$

(c) $f_A(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, f_A(\vec{e}_2) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, f_A(\vec{e}_3) = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix},$

$f_A(\vec{e}_4) = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$

(d) $f_A(\vec{x}) = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ 2x_1 - x_2 + 3x_4 \\ x_1 + 2x_3 - x_4 \end{bmatrix}$

(e) $[f_A] = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix}$

A3 $f : \mathbb{R}^2 \to \mathbb{R}^2$. f is not linear.

A4 $f : \mathbb{R}^3 \to \mathbb{R}^2$. f is not linear.

A5 $g : \mathbb{R}^3 \to \mathbb{R}^3$. g is not linear.

A6 $g : \mathbb{R}^2 \to \mathbb{R}^2$. g is linear.

A7 $h : \mathbb{R}^2 \to \mathbb{R}^3$. h is not linear.

A8 $k : \mathbb{R}^3 \to \mathbb{R}^3$. k is linear.

A9 $\ell : \mathbb{R}^3 \to \mathbb{R}^2$. ℓ is not linear.

A10 $m : \mathbb{R}^1 \to \mathbb{R}^3$. m is not linear.

A11 $L : \mathbb{R}^3 \to \mathbb{R}^2$. L is linear.

A12 $L : \mathbb{R}^2 \to \mathbb{R}^2$. L is not linear.

A13 $M : \mathbb{R}^3 \to \mathbb{R}^2$. M is not linear.

A14 $M : \mathbb{R}^3 \to \mathbb{R}^3$. M is linear.

A15 $N : \mathbb{R}^2 \to \mathbb{R}^3$. N is linear.

A16 $N : \mathbb{R}^3 \to \mathbb{R}^3$. N is not linear.

A17 $\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$

A18 $\begin{bmatrix} 16/41 & 20/41 \\ 20/41 & 25/41 \end{bmatrix}$

A19 $\begin{bmatrix} 4/9 & 4/9 & -2/9 \\ 4/9 & 4/9 & -2/9 \\ -2/9 & -2/9 & 1/9 \end{bmatrix}$ **A20** $\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$

A21 $\begin{bmatrix} 16/17 & -4/17 \\ -4/17 & 1/17 \end{bmatrix}$ **A22** $\begin{bmatrix} 13/14 & -1/7 & -3/14 \\ -1/7 & 5/7 & -3/7 \\ -3/14 & -3/7 & 5/14 \end{bmatrix}$

A23 $L : \mathbb{R}^2 \to \mathbb{R}^2$. $[L] = \begin{bmatrix} -3 & 5 \\ -1 & -2 \end{bmatrix}$

A24 $L : \mathbb{R}^2 \to \mathbb{R}^3$. $[L] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

A25 $L : \mathbb{R}^1 \to \mathbb{R}^3$. $[L] = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

A26 $M : \mathbb{R}^3 \to \mathbb{R}^1$. $[M] = \begin{bmatrix} 1 & -1 & \sqrt{2} \end{bmatrix}$

A27 $M : \mathbb{R}^3 \to \mathbb{R}^2$. $[M] = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & -1 \end{bmatrix}$

A28 $N : \mathbb{R}^3 \to \mathbb{R}^4$. $[N] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A29 $L : \mathbb{R}^3 \to \mathbb{R}^2$. $[L] = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -5 \end{bmatrix}$

A30 $K : \mathbb{R}^4 \to \mathbb{R}^2$. $[K] = \begin{bmatrix} 5 & 0 & 3 & -1 \\ 0 & 1 & -7 & 3 \end{bmatrix}$

A31 $M : \mathbb{R}^4 \to \mathbb{R}^4$. $[M] = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 2 & 0 & -3 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

A32 $\begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix}$ **A33** $\begin{bmatrix} -1 & 13 \\ 11 & -21 \end{bmatrix}$

A34 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ **A35** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

A36 $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **A37** $\begin{bmatrix} -1 & 1/2 & -1 \\ \sqrt{2} & 0 & -1 \end{bmatrix}$

A38 $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ **A39** $\begin{bmatrix} 3 & 2 \\ 5 & -7 \end{bmatrix}$

A40 $\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$ **A41** $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 5 & -1 \end{bmatrix}$

A42 (a) $S : \mathbb{R}^3 \to \mathbb{R}^2$, $T : \mathbb{R}^3 \to \mathbb{R}^2$.

(b) $[S+T] = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$, $[2S-3T] = \begin{bmatrix} 1 & -4 & 9 \\ -8 & -6 & -5 \end{bmatrix}$

A43 (a) $S : \mathbb{R}^4 \to \mathbb{R}^2$, $T : \mathbb{R}^2 \to \mathbb{R}^4$.

(b) $[S \circ T] = \begin{bmatrix} 6 & -19 \\ 10 & -10 \end{bmatrix}$,

$[T \circ S] = \begin{bmatrix} -3 & 5 & 16 & 9 \\ 6 & 8 & 4 & 0 \\ -6 & -8 & -4 & 0 \\ -9 & -17 & -16 & -5 \end{bmatrix}$

A44 $[L \circ M] = \begin{bmatrix} 11 & -4 & 1 \\ 11 & -9 & -6 \\ 3 & -2 & -1 \end{bmatrix}$

A45 $[M \circ L] = [M][L] = \begin{bmatrix} 1 & 9 \\ 8 & 0 \end{bmatrix}$

A46 The composition is not defined.

A47 The composition is not defined.

A48 The composition is not defined.

A49 $[N \circ M] = \begin{bmatrix} 5 & 0 & 3 \\ 6 & 1 & 5 \\ -3 & -3 & -6 \\ 13 & -7 & -2 \end{bmatrix}$

A50 (a) $L(\vec{x}) = (3x_1 + x_2, -x_1 - 5x_2, 4x_1 + 9x_2)$

(b) $L(x_1, x_2) = (3x_1^2 + x_2, -x_1 - 5x_2, 4x_1 + 9x_2)$

Section 3.3 Practice Problems

A1 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

A2 $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

A3 $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

A4 $\begin{bmatrix} 0.309 & -0.951 \\ 0.951 & 0.309 \end{bmatrix}$

A5 $[V] = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $[S] = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

A6 $[V \circ S] = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$

A7 $[S \circ V] = \begin{bmatrix} 1 & 0 \\ 15 & 5 \end{bmatrix}$

A8 $[R_\theta \circ S] = \begin{bmatrix} \cos\theta & -5\sin\theta \\ \sin\theta & 5\cos\theta \end{bmatrix}$

A9 $[S \circ R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ 5\sin\theta & 5\cos\theta \end{bmatrix}$

A10 $[H] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, [V] = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

A11 $[V \circ H] = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

A12 $[H \circ V] = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

A13 $[F \circ H] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

A14 $[H \circ F] = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

A15 $\begin{bmatrix} 4/5 & -3/5 \\ -3/5 & -4/5 \end{bmatrix}$

A16 $\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$

A17 $\begin{bmatrix} -15/17 & 8/17 \\ 8/17 & 15/17 \end{bmatrix}$

A18 $\begin{bmatrix} 8/17 & 15/17 \\ 15/17 & -8/17 \end{bmatrix}$

A19 $\frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

A20 $\frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 8 & 1 & -4 \\ 4 & -4 & 7 \end{bmatrix}$

A21 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

A22 $[\text{refl}_{\vec{n}}] = \begin{bmatrix} 6/7 & -2/7 & 3/7 \\ -2/7 & 3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$

A23 $[\text{inj} \circ D] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

A24 (a) $[P \circ S] = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(b) There is no T.

(c) $[Q \circ S] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Section 3.4 Practice Problems

A1 (a) $\vec{y}_1 \in \text{Range}(L)$; $\vec{x} = \begin{bmatrix} 9 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.

(b) $\vec{y}_2 \in \text{Range}(L)$; $\vec{x} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.

(c) $\vec{v} \in \text{Null}(L)$.

A2 (a) $\vec{y}_1 \notin \text{Range}(L)$.

(b) $\vec{y}_2 \in \text{Range}(L)$; $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) $\vec{v} \notin \text{Null}(L)$.

A3 (a) $\vec{y}_1 \notin \text{Range}(L)$.

(b) $\vec{y}_2 \in \text{Range}(L)$; $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

(c) $\vec{v} \notin \text{Null}(L)$.

A4 $\mathcal{C} = \left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}\right\}$ is a basis for Range(L).
The empty set is a basis for Null(L).

A5 $\mathcal{C} = \left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ is a basis for Range(L).

$\mathcal{B} = \left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$ is a basis for Null(L).

A6 $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \end{bmatrix}\right\}$ is a basis for Range(L).
The empty set is a basis for Null(L).

A7 $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$ is a basis for Range(L).

$\mathcal{B} = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Null(L).

A8 $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is a basis for Range(L).

$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Null(L).

A9 A basis for Range(L) is the empty set.
The standard basis for $\mathbb{R}^3$ is a basis for Null(L).

A10 $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Range(L).

$\mathcal{B} = \left\{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$ is a basis for Null(L).

A11 $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for Range(L).

$\mathcal{B} = \left\{ \begin{bmatrix} -7 \\ 6 \\ 1 \end{bmatrix} \right\}$ is a basis for Null(L).

A12 $C = \left\{ \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \right\}$ is a basis for Range(L).

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for Null(L).

A13 $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Range(L).

The empty set is a basis for Null(L).

A14 $C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for Range(L).

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Null(L).

A15 A basis for Col([L]) is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} \right\}$.

A basis for Null([L]) is the empty set.

A basis for Row([L]) is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

A basis for Null($[L]^T$) is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

A16 A basis for Col([L]) is $\left\{ \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] \right\}$.

A basis for Null([L]) is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for Row([L]) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

A basis for Null($[L]^T$) is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

A17 A basis for Col([L]) is the empty set.
A basis for Null([L]) is the standard basis for $\mathbb{R}^3$.
A basis for Row([L]) is the empty set.
A basis for Null($[L]^T$) is the standard basis for $\mathbb{R}^3$.

A18 A basis for Col([L]) is the standard basis for $\mathbb{R}^2$.

A basis for Null([L]) is $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

A basis for Row([L]) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$.

A basis for Null($[L]^T$) is the empty set.

A19 A basis for Col([L]) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for Null([L]) is the empty set.
A basis for Row([L]) is the standard basis for $\mathbb{R}^4$.

A basis for Null($[L]^T$) is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

A20 A basis for Range(L) is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\}$.

A basis for Null(L) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$.

A21 A basis for Range(L) is $\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

A basis for Null(L) is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$.

A22 A basis for Range(L) is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.
A basis for Null(L) is the empty set.

A23 $[L] = \begin{bmatrix} 1 & 2 \\ -1 & 5 \\ 1 & -3 \end{bmatrix}$

A24 $[L] = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix}$

A25 $[L] = \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}$

A26 One choice is $[L] = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

A27 Since $\vec{r}_0$ is a solution of $A\vec{x} = \vec{b}$, that means $A\vec{r}_0 = \vec{b}$. Then we get

$$A(\vec{r}_0 + \vec{n}) = A\vec{r}_0 + A\vec{n} = \vec{b} + \vec{0} = \vec{b}$$

A28 (a) The number of variables is 4.
(b) The rank of A is 2.
(c) The dimension of the solution space is 2.

A29 (a) The number of variables is 5.
(b) The rank of A is 3.
(c) The dimension of the solution space is 2.

A30 A basis for Row(A) is $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$.

A31 A basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}1\\4\\3\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}\right\}$.

A basis for Null(A^T) is the empty set.

A32 A basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1/2\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}0\\-1/2\\1\end{bmatrix}\right\}$.

A basis for Null(A^T) is the empty set.

A33 A basis for Row(A) is $\left\{\begin{bmatrix}0\\1\\-2\\-1\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}-2\\3\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix}3/2\\1\end{bmatrix}\right\}$.

A34 A basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}1\\4\\3\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}\right\}$.

A basis for Null(A^T) is the empty set.

A35 A basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}8\\5\\-2\end{bmatrix}\right\}$.

A basis for Null(A) is the empty set.
A basis for Null(A^T) is the empty set.

A36 A basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}2\\2\\4\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix}-7/3\\1/3\\1\end{bmatrix}\right\}$.

A37 A basis for Row(A) is $\left\{\begin{bmatrix}1\\2\\4\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$.

A38 A basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\-5\end{bmatrix}, \begin{bmatrix}0\\1\\7\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\0\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-2\\1\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}5\\-7\\1\end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix}2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}$.

A39 A basis for Row(A) is the standard basis for $\mathbb{R}^3$.

A basis for Col(A) is $\left\{\begin{bmatrix}3\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\3\end{bmatrix}, \begin{bmatrix}6\\5\\3\end{bmatrix}\right\}$.

A basis for Null(A) is the empty set.
A basis for Null(A^T) is the empty set.

A40 A basis for Row(A) is the standard basis for $\mathbb{R}^3$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\4\end{bmatrix}\right\}$.

A basis for Null(A) is the empty set.

A basis for Null(A^T) is $\left\{\begin{bmatrix}-1\\3\\-3\\1\end{bmatrix}\right\}$.

A41 A basis for Row(A) is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\2\\1\\3\end{bmatrix}, \begin{bmatrix}0\\0\\2\\4\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-2\\1\end{bmatrix}\right\}$.

A42 A basis for Row(A) is $\left\{\begin{bmatrix}1\\2\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\4\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\right\}$.

A43 $\vec{u} \in \text{Null}(A^T)$

A44 $\vec{v} \notin \text{Col}(A)$

A45 $\vec{w} \notin \text{Null}(A)$

A46 $\vec{z} \in \text{Row}(A)$

A47 $\left\{\begin{bmatrix}-2\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\-1\\0\\-1\\1\end{bmatrix}\right\}$

A48 $\vec{x} = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}-2\\1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-3\\-1\\0\\-1\\1\end{bmatrix}, s, t \in \mathbb{R}$

Section 3.5 Practice Problems

A1 Not invertible.

A2 $\frac{1}{34}\begin{bmatrix}1 & -6\\ 5 & 4\end{bmatrix}$

A3 $\frac{1}{24}\begin{bmatrix}3 & -6\\ 2 & 4\end{bmatrix}$

A4 $\frac{1}{23}\begin{bmatrix}5 & 4\\ -2 & 3\end{bmatrix}$

A5 $\begin{bmatrix}2 & 0 & -1\\ -1 & 1 & -1\\ -1 & 0 & 1\end{bmatrix}$

A6 Not invertible.

A7 Not invertible.

A8 $\begin{bmatrix}1/2 & 1/4 & -7/4\\ 0 & 1/2 & -1/2\\ 0 & 0 & 1\end{bmatrix}$

A9 $\begin{bmatrix}7 & -5 & -1\\ 5 & -4 & -1\\ 4 & -3 & -1\end{bmatrix}$

A10 $\begin{bmatrix}0 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$

A11 $\begin{bmatrix}-1 & 1 & 0\\ -3 & 1 & -1\\ 7/3 & -2/3 & 2/3\end{bmatrix}$

A12 Not invertible.

A13 $\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

A14 Not invertible.

A15 Not invertible.

A16 $\begin{bmatrix}0 & -3 & 1/2 & -2\\ -1 & -2 & 3/2 & -2\\ 0 & 1 & 0 & 1\\ 1 & 3 & -3/2 & 2\end{bmatrix}$

A17 $\begin{bmatrix}6 & 10 & -5/2 & -7/2\\ 1 & 2 & -1/2 & -1/2\\ -2 & -3 & 1 & 1\\ 0 & -3 & 0 & 1\end{bmatrix}$

A18 Not invertible.

A19 $\begin{bmatrix}1 & 0 & -1 & 1 & -2\\ 0 & 1 & 0 & -1 & 2\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$

A20 (a) $\vec{x} = \begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix}-3\\ 0\\ 2\end{bmatrix}$

(c) $\vec{x} = \begin{bmatrix}-2\\ -1\\ 2\end{bmatrix}$ (d) $\vec{x} = \begin{bmatrix}8\\ 0\\ -5\end{bmatrix}$

A21 (a) $A^{-1} = \begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}, B^{-1} = \begin{bmatrix}-5 & 2\\ 3 & -1\end{bmatrix}$

(b) $(AB)^{-1} = \begin{bmatrix}-16 & 9\\ 9 & -5\end{bmatrix} = B^{-1}A^{-1}$

(c) $(3A)^{-1} = \begin{bmatrix}2/3 & -1/3\\ -1 & 2/3\end{bmatrix} = \frac{1}{3}A^{-1}$

(d) $(A^T)^{-1} = \begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$

(e) $(A+B)^{-1} = \frac{1}{3}\begin{bmatrix}7 & -3\\ -6 & 3\end{bmatrix}, A^{-1} + B^{-1} = \begin{bmatrix}-3 & 1\\ 0 & 1\end{bmatrix}$

A22 (a) $[R_{\pi/6}] = \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}$,

$[R_{\pi/6}]^{-1} = [R_{-\pi/6}] = \begin{bmatrix}\sqrt{3}/2 & 1/2\\ -1/2 & \sqrt{3}/2\end{bmatrix}$

(b) $[S] = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}, [S^{-1}] = \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}$

(c) $[R] = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, [R^{-1}] = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = [R]$

(d) $[(R\circ S)^{-1}] = \begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix}, [(S\circ R)^{-1}] = \begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}$

A23 Let $\vec{v}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Then there exists $\vec{u}, \vec{x} \in \mathbb{R}^n$ such that $\vec{x} = M(\vec{y})$ and $\vec{u} = M(\vec{v})$. Then $L(\vec{x}) = \vec{y}$ and $L(\vec{u}) = \vec{v}$. Since L is linear $L(s\vec{x} + t\vec{u}) = sL(\vec{x}) + tL(\vec{u}) = s\vec{y} + t\vec{v}$. Hence, M is linear since

$$M(s\vec{y} + t\vec{v}) = s\vec{x} + t\vec{u} = sM(\vec{y}) + tM(\vec{v})$$

A24 $\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$

A25 $\begin{bmatrix}1/5 & 0\\ 0 & 1\end{bmatrix}$

A26 $\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$

A27 $\begin{bmatrix}1 & 0\\ -4 & 1\end{bmatrix}$

A28 $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$

A29 $\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

A30 $L^{-1}(x_1, x_2) = \left(\frac{5}{7}x_1 - \frac{3}{7}x_2, -\frac{1}{7}x_1 + \frac{2}{7}x_2\right)$

A31 $L^{-1}(x_1, x_2, x_3) = (-3x_1 - 3x_2 + 2x_3, -2x_1 + x_3, 2x_1 + x_2 - x_3)$

A32 $X = \begin{bmatrix}-18 & -15 & 4\\ 13 & 11 & -3\end{bmatrix}$

A33 $X = \begin{bmatrix}-1 & 3\\ -1 & 4\\ -8 & 19\end{bmatrix}$

A34 If A is invertible, then A^T is invertible by the Invertible Matrix Theorem. Let $A^T = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ and consider

$$\vec{0} = c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A^T\vec{c}$$

Since A^T is invertible, this system has unique solution

$$\vec{c} = (A^T)^{-1}\vec{0} = \vec{0}$$

Thus, $c_1 = \cdots = c_n = 0$, so the columns of A^T form a linearly independent set.

A35 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, A and B are both invertible, but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

A36 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ gives $XC = AX$, but $A \neq C$.

Section 3.6 Practice Problems

A1 $E = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $EA = \begin{bmatrix} 6 & -13 & -17 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix}$

A2 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}$

A3 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ -4 & -2 & 0 \end{bmatrix}$

A4 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 2 & 3 \\ -6 & 18 & 24 \\ 4 & 2 & 0 \end{bmatrix}$

A5 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 8 & 10 & 12 \end{bmatrix}$

A6 $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

A7 $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

A8 $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A9 $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A10 $E = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A11 $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A12 $E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A13 $E = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A14 Elementary. $5R_1$

A15 Not elementary.

A16 Elementary. $R_1 + 2R_2$

A17 Elementary. $R_3 - 4R_2$

A18 Not elementary.

A19 Not elementary.

A20 Elementary. $R_1 \updownarrow R_3$

A21 Not elementary

A22 Elementary. $1R_1$

A23 (a) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}1&0&4\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}$

A24 (a) $E_1 = \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}, E_2 = \begin{bmatrix}1&0&0\\0&1&-3\\0&0&1\end{bmatrix},$

$E_3 = \begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}, E_4 = \begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix}-7&-2&4\\6&1&-3\\-2&0&1\end{bmatrix}$

(c) $A = \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$

A25 (a) $E_1 = \begin{bmatrix}1/2&0&0\\0&1&0\\0&0&1\end{bmatrix}, E_2 = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix},$

$E_3 = \begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}, E_4 = \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix},$

$E_5 = \begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix}-1/2&2&2\\1&1&0\\-1/2&1&1\end{bmatrix}$

(c) $A = \begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}$
$\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}$

A26 (a) $E_1 = \begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}, E_2 = \begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix},$

$E_3 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, E_4 = \begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix},$

$E_5 = \begin{bmatrix}1&0&0\\0&1&-1\\0&0&1\end{bmatrix}, E_6 = \begin{bmatrix}1&0&0\\0&1/3&0\\0&0&1\end{bmatrix}$

(b) $A^{-1} = E_6E_5E_4E_3E_2E_1 = \begin{bmatrix}-2&1&0\\0&1/3&-1/3\\1&-1&1\end{bmatrix}$

(c) $A = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}$

A27 (a) $E_1 = \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}, E_2 = \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix},$

$E_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}, E_4 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1/2\end{bmatrix},$

$E_5 = \begin{bmatrix}1&0&-4\\0&1&0\\0&0&1\end{bmatrix}$

(b) $A^{-1} = E_5E_4E_3E_2E_1 = \begin{bmatrix}5&4&-2\\1&1&0\\-1/2&-1/2&1/2\end{bmatrix}$

(c) $A = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}1&0&4\\0&1&0\\0&0&1\end{bmatrix}$

A28 (a) $E_1 = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}, E_2 = \begin{bmatrix}1&0&0\\0&1&0\\4&0&1\end{bmatrix},$

$E_3 = \begin{bmatrix}1&0&0\\0&-1/4&0\\0&0&1\end{bmatrix}, E_4 = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},$

$E_5 = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix}1/2&-1/4&0\\4&0&1\\-1/2&-1/4&0\end{bmatrix}$

(c) $A = \begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-4&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-4&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$

Section 3.7 Practice Problems

A1 (a) An elementary matrix is lower triangular if and only if it corresponds to an elementary row operation of the form $R_i + aR_j$ where $i > j$ or of the form aR_i. In the former case, the inverse elementary matrix corresponds to $R_i - aR_j$ and hence will be lower triangular. In the latter case, the elementary matrix corresponds to $\frac{1}{a}R_i$ and hence is also lower triangular.

(b) Assume $A, B \in M_{n\times n}(\mathbb{R})$ are both lower triangular matrices. Then, by definition, we have $a_{ij} = 0$ and $b_{ij} = 0$ whenever $i < j$. Hence, for any $i < j$ we have

$$\begin{aligned}(AB)_{ij} &= a_{i1}b_{1j} + \cdots + a_{ii}b_{ij} + a_{i(i+1)}b_{(i+1)j} + \cdots + a_{in}b_{nj}\\ &= a_{i1}(0) + \cdots + a_{ii}(0) + 0b_{(i+1)j} + \cdots + 0b_{nj}\\ &= 0\end{aligned}$$

So, (AB) is lower triangular.

A2 $\begin{bmatrix} 1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix} -2 & -1 & 5\\ 0 & 2 & -12\\ 0 & 0 & 8\end{bmatrix}$

A3 $\begin{bmatrix} 1 & 0 & 0\\ 3 & 1 & 0\\ 2 & 3/2 & 1\end{bmatrix}\begin{bmatrix} 1 & -2 & 4\\ 0 & 4 & -8\\ 0 & 0 & -1\end{bmatrix}$

A4 $\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1/3 & 1\end{bmatrix}\begin{bmatrix} 2 & -4 & 5\\ 0 & 9 & -3\\ 0 & 0 & 1\end{bmatrix}$

A5 $\begin{bmatrix} 1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 0 & 1/2 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 5 & 3 & 4\\ 0 & 4 & 5 & 11\\ 0 & 0 & -7/2 & -13/2\\ 0 & 0 & 0 & 0\end{bmatrix}$

A6 $\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 3 & -1 & 1 & 0\\ 0 & -4/3 & 17/9 & 1\end{bmatrix}\begin{bmatrix} 1 & -2 & 1 & 1\\ 0 & -3 & -2 & 1\\ 0 & 0 & -3 & -3\\ 0 & 0 & 0 & 7\end{bmatrix}$

A7 $\begin{bmatrix} 1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ -3/2 & 3/2 & 1 & 0\\ -1 & -2 & 2 & 1\end{bmatrix}\begin{bmatrix} -2 & -1 & 2 & 0\\ 0 & 1 & 2 & 2\\ 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

A8 $\begin{bmatrix} 1 & 0\\ -1/3 & 1\end{bmatrix}\begin{bmatrix} 3 & -2\\ 0 & 13/3\end{bmatrix}$;
$\vec{x}_1 = \begin{bmatrix} 1\\ 0\end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 38/13\\ 31/13\end{bmatrix}$

A9 $\begin{bmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ -1 & 4 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 3\\ 0 & 1 & 3\\ 0 & 0 & -4\end{bmatrix}$;
$\vec{x}_1 = \begin{bmatrix} -3\\ -4\\ 2\end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 5\\ 2\\ -1\end{bmatrix}$

A10 $\begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & -2\\ 0 & -4 & 2\\ 0 & 0 & 3\end{bmatrix}$;
$\vec{x}_1 = \begin{bmatrix} 3\\ 3\\ 2\end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -4\\ -2\\ -3\end{bmatrix}$

A11 $\begin{bmatrix} 1 & 0 & 0\\ -3 & 1 & 0\\ -3 & 2 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 1\\ 0 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}$;
$\vec{x}_1 = \begin{bmatrix} 3\\ 2\\ 0\end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -3\\ -3\\ -1\end{bmatrix}$

A12 $\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -3 & 2 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix} -1 & 2 & -3 & 0\\ 0 & -1 & 3 & 1\\ 0 & 0 & -12 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$;
$\vec{x}_1 = \begin{bmatrix} -1\\ 1\\ 3\\ -1\end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -5\\ -3\\ -2\\ 0\end{bmatrix}$

Chapter 3 Quiz

E1 $\begin{bmatrix} -14 & 1 & -17 \\ -1 & 10 & -39 \end{bmatrix}$

E2 Not defined.

E3 $\begin{bmatrix} -3 & -38 \\ 0 & -23 \\ -8 & -42 \end{bmatrix}$

E4 (a) $f_A(\vec{u}) = \begin{bmatrix} -11 \\ 0 \end{bmatrix}$, $f_A(\vec{v}) = \begin{bmatrix} -16 \\ 17 \end{bmatrix}$

(b) $\begin{bmatrix} -16 & -11 \\ 17 & 0 \end{bmatrix}$

E5 (a) $[R] = [R_{\pi/3}] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $[M] = [\text{refl}_{(-1,-1,2)}] = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

(c) $[R \circ M] = \frac{1}{6}\begin{bmatrix} 2+\sqrt{3} & -1-2\sqrt{3} & 2-2\sqrt{3} \\ 2\sqrt{3}-1 & -\sqrt{3}+2 & 2\sqrt{3}+2 \\ 4 & 4 & -2 \end{bmatrix}$

E6 $\vec{x} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $s, t \in \mathbb{R}$, and

$$\vec{x} = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 0 \\ 7 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, s, t \in \mathbb{R}.$$

E7 (a) $\vec{u} \notin \text{Col}(B)$, $\vec{v} \in \text{Col}(B)$

(b) $\vec{x} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ (c) $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

E8 A basis for Row(A) is $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}\right\}$.

A basis for Col(A) is $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}\right\}$.

A basis for Null(A) is $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$.

A basis for Null(A^T) is $\left\{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

E9 $A^{-1} = \begin{bmatrix} 2/3 & 0 & 0 & 1/3 \\ 1/6 & 0 & 1/2 & -1/6 \\ 0 & 1 & 0 & 0 \\ -1/3 & 0 & 0 & 1/3 \end{bmatrix}$

E10 The matrix is invertible only for $p \neq 1$. The inverse is $\frac{1}{1-p}\begin{bmatrix} 1 & p & -p \\ -1 & 1-2p & p \\ -1 & -1 & 1 \end{bmatrix}$.

E11 By definition, the range of L is a subset of $\mathbb{R}^m$. We have $L(\vec{0}) = \vec{0}$, so $\vec{0} \in \text{Range}(L)$. If $\vec{x}, \vec{y} \in \text{Range}(L)$, then there exists $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $L(\vec{u}) = \vec{x}$ and $L(\vec{v}) = \vec{y}$. Hence, $L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v}) = \vec{x} + \vec{y}$, so $\vec{x} + \vec{y} \in \text{Range}(L)$. Similarly, $L(t\vec{u}) = tL(\vec{u}) = t\vec{x}$, so $t\vec{x} \in \text{Range}(L)$. Thus, L is a subspace of $\mathbb{R}^m$.

E12 Consider $c_1L(\vec{v}_1)+\cdots+c_kL(\vec{v}_k) = \vec{0}$. Since L is linear, we get $L(c_1\vec{v}_1+\cdots+c_k\vec{v}_k) = \vec{0}$. Thus, $c_1\vec{v}_1+\cdots+c_k\vec{v}_k \in \text{Null}(L)$ and so $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$. This implies that $c_1 = \cdots = c_k = 0$ since $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent. Therefore, $\{L(\vec{v}_1), \ldots, L(\vec{v}_k)\}$ is linearly independent.

E13 (a) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix}$

E14 $K = I_3$

E15 KM cannot equal MK for any matrix K.

E16 The range cannot be spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ because this vector is not in $\mathbb{R}^3$.

E17 The matrix of L is any multiple of $\begin{bmatrix} 1 & -2/3 \\ 1 & -2/3 \\ 2 & -4/3 \end{bmatrix}$.

E18 This contradicts the Rank Theorem, so there can be no such mapping L.

E19 This contradicts Theorem 3.5.2, so there can be no such matrix.

CHAPTER 4

Section 4.1 Practice Problems

A1 $-1 - 6x + 4x^2 + 6x^3$

A2 $-3 + 6x - 6x^2 - 3x^3 - 12x^4$

A3 $-1 + 9x - 11x^2 - 17x^3$

A4 $-3 + 2x + 6x^2$

A5 $7 - 2x - 5x^2$

A6 $\frac{7}{3} - \frac{4}{3}x + \frac{13}{3}x^2$

A7 $\sqrt{2} - \pi + \sqrt{2}x + (\sqrt{2} + \pi)x^2$

A8 $0 = 0(1+x^2+x^3)+0(2+x+x^3)+0(-1+x+2x^2+x^3)$.

A9 $2 + 4x + 3x^2 + 4x^3$ is not in the span.

A10
$$\begin{aligned}-x + 2x^2 + x^3 &= 2(1 + x^2 + x^3) + (-1)(2 + x + x^3)\\ &\quad + 0(-1 + x + 2x^2 + x^3)\end{aligned}$$

A11
$$\begin{aligned}-4 - x + 3x^2 &= 1(1 + x^2 + x^3) + (-2)(2 + x + x^3)\\ &\quad + 1(-1 + x + 2x^2 + x^3)\end{aligned}$$

A12
$$\begin{aligned}-1 - 7x + 5x^2 + 4x^3 &= (-3)(1 + x^2 + x^3)\\ &\quad + 3(2 + x + x^3) + 4(-1 + x + 2x^2 + x^3)\end{aligned}$$

A13 $2 + x + 5x^3$ is not in the span.

A14 The set is linearly independent.

A15 The set is linearly dependent.
$$\begin{aligned}0 &= (-3t)(1 + x + x^2) + tx + t(x^2 + x^3)\\ &\quad + t(3 + 2x + 2x^2 - x^3),\ t \in \mathbb{R}\end{aligned}$$

A16 The set is linearly independent.

A17 The set is linearly dependent.
$$\begin{aligned}0 &= (-2t)(1 + x + x^3 + x^4) + t(2 + x - x^2 + x^3 + x^4)\\ &\quad + t(x + x^2 + x^3 + x^4),\ t \in \mathbb{R}\end{aligned}$$

A18 Consider
$$a_1 + a_2x + a_3x^2 = t_1 1 + t_2(x - 1) + t_3(x - 1)^2$$
The augmented matrix is $\left[\begin{array}{ccc|c} 1 & -1 & 1 & a_1 \\ 0 & 1 & -2 & a_2 \\ 0 & 0 & 1 & a_3 \end{array}\right]$.
Since there is a leading 1 in each row, the system is consistent for all polynomials $a_1 + a_2x + a_3x^2$. Thus, $\mathcal{B}$ spans $P_2(\mathbb{R})$. Since there is a leading 1 in each column, there is a unique solution and so $\mathcal{B}$ is also linearly independent. Therefore, it is a basis for $P_2(\mathbb{R})$.

Section 4.2 Practice Problems

A1 Subspace

A2 Subspace

A3 Subspace

A4 Not a subspace

A5 Not a subspace

A6 Subspace

A7 Not a subspace

A8 Subspace

A9 Not a subspace

A10 Subspace

A11 Subspace

A12 Subspace

A13 Subspace

A14 Subspace

A15 Not a subspace

A16 Subspace

A17 Subspace

A18 Not a subspace

A19 Subspace

A20 Not a subspace

Section 4.3 Practice Problems

A1 Basis

A2 Not a basis

A3 Not a basis

A4 Not a basis

A5 Basis

A6 Not a basis

A7 Not a basis

A8 Basis

A9 $\dim \operatorname{Span} \mathcal{B} = 2$

A10 $\dim \operatorname{Span} \mathcal{B} = 3$

A11 $\dim \operatorname{Span} \mathcal{B} = 3$

A12 $\dim \operatorname{Span} \mathcal{B} = 2$

A13 $\dim \operatorname{Span} \mathcal{B} = 4$

A14 $\dim \operatorname{Span} \mathcal{B} = 3$

A15 $\dim \operatorname{Span} \mathcal{B} = 3$

A16 $\dim \operatorname{Span} \mathcal{B} = 4$

A17 Other correct answers are possible.

(a) $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\-1\\-1\end{bmatrix}\right\}$

A18 Other correct answers are possible.

(a) $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$

A19 $\{1 + x + x^3, 1 + x^2, x^2, x\}$

A20 $\left\{\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$

A21 $\left\{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\right\}$

A22 $\{x, 1 - x^2\}$, $\dim \mathbb{S} = 2$

A23 $\{1 + x^3, x - 2x^3, x^2\}$, $\dim \mathbb{S} = 3$

A24 $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$, $\dim \mathbb{S} = 3$

A25 $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$, $\dim \mathbb{S} = 2$

A26 $\{x^2 - 5x + 6\}$, $\dim \mathbb{S} = 1$

A27 $\left\{\begin{bmatrix}-1 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$, $\dim \mathbb{S} = 2$

Section 4.4 Practice Problems

A1 (a) Show that it is linearly independent and spans the plane.

(b) $\begin{bmatrix}3\\2\\1\end{bmatrix}$ and $\begin{bmatrix}5\\2\\3\end{bmatrix}$ are not in the plane. $\begin{bmatrix}3\\2\\2\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}3\\-2\end{bmatrix}$

A2 (a) $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix}4\\1\\1\end{bmatrix}$, $[\mathbf{r}]_{\mathcal{B}} = \begin{bmatrix}2\\-1\\3\end{bmatrix}$

(b) $[2 - 4x + 10x^2]_{\mathcal{B}} = \begin{bmatrix}6\\0\\4\end{bmatrix}$

A3 $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}1\\2\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix}-19/11\\-29/11\end{bmatrix}$

A4 $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}2\\1\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix}32/11\\35/11\end{bmatrix}$

A5 $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}9\\-6\\-1\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix}5\\-3\\-2\end{bmatrix}$

A6 $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}3\\-2\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix}$

A7 $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix}1\\2\end{bmatrix}$, $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix}5/11\\3/11\end{bmatrix}$

A8 $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix}5\\1\\-2\end{bmatrix}$, $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix}2\\2\\-2\end{bmatrix}$

A9 $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix}5\\-3\\4\end{bmatrix}$, $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$

A10 $[A]_{\mathcal{B}} = \begin{bmatrix}-2\\3\\1\end{bmatrix}$, $[B]_{\mathcal{B}} = \begin{bmatrix}-2\\3\\-1\end{bmatrix}$

A11 $[A]_{\mathcal{B}} = \begin{bmatrix}1\\1\end{bmatrix}$, $[B]_{\mathcal{B}} = \begin{bmatrix}3\\-2\end{bmatrix}$

A12 $[A]_{\mathcal{B}} = \begin{bmatrix}-1/2\\1/2\\3/4\end{bmatrix}$, $[B]_{\mathcal{B}} = \begin{bmatrix}1\\2\\-1/2\end{bmatrix}$

A13 $[A]_{\mathcal{B}} = \begin{bmatrix}1\\2\\-4\end{bmatrix}$, $[B]_{\mathcal{B}} = \begin{bmatrix}1\\-1\\-2\end{bmatrix}$

A14 $Q = \begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}$, $P = \begin{bmatrix}1 & 0\\-1/2 & 1/2\end{bmatrix}$

A15 $Q = \begin{bmatrix}3 & 0 & -2\\4 & 1 & -3\\1 & 0 & 3\end{bmatrix}$, $P = \begin{bmatrix}3/11 & 0 & 2/11\\-15/11 & 1 & 1/11\\-1/11 & 0 & 3/11\end{bmatrix}$

A16 $Q = \begin{bmatrix}1 & 1 & 1\\0 & -2 & -4\\0 & 0 & 4\end{bmatrix}$, $P = \begin{bmatrix}1 & 1/2 & 1/4\\0 & -1/2 & -1/2\\0 & 0 & 1/4\end{bmatrix}$

A17 $Q = \begin{bmatrix}1 & 0 & 1\\2 & 1 & 3\\1 & 1 & 0\end{bmatrix}$, $P = \begin{bmatrix}3/2 & -1/2 & 1/2\\-3/2 & 1/2 & 1/2\\-1/2 & 1/2 & -1/2\end{bmatrix}$

A18 $Q = \begin{bmatrix}1 & 1 & 0\\-2 & 0 & 1\\5 & -2 & 1\end{bmatrix}$, $P = \begin{bmatrix}2/9 & -1/9 & 1/9\\7/9 & 1/9 & -1/9\\4/9 & 7/9 & 2/9\end{bmatrix}$

A19 $Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

A20 $Q = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & 1 \end{bmatrix}, P = \begin{bmatrix} 1/3 & 2/9 & -8/9 \\ 0 & -1/3 & 1/3 \\ 1/3 & -1/9 & 4/9 \end{bmatrix}$

A21 $Q = \begin{bmatrix} -1 & 5 \\ 1 & -1 \end{bmatrix}, P = \begin{bmatrix} 1/4 & 5/4 \\ 1/4 & 1/4 \end{bmatrix}$

A22 $Q = \begin{bmatrix} 19 & 13 \\ 4 & 3 \end{bmatrix}, P = \begin{bmatrix} 3/5 & -13/5 \\ -4/5 & 19/5 \end{bmatrix}$

A23 $Q = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1/4 & -3/4 \\ 1/2 & -3/4 & 3/4 \end{bmatrix}, P = \begin{bmatrix} 3 & 3 & -1 \\ 3 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$

Section 4.5 Practice Problems

A1 Show that $L(s\mathbf{x} + t\mathbf{y}) = sL(\mathbf{x}) + tL(\mathbf{y})$.

A2 Show that $L(s\mathbf{x} + t\mathbf{y}) = sL(\mathbf{x}) + tL(\mathbf{y})$.

A3 Show that $\operatorname{tr}(s\mathbf{x} + t\mathbf{y}) = s\operatorname{tr}(\mathbf{x}) + t\operatorname{tr}(\mathbf{y})$.

A4 Show that $T(s\mathbf{x} + t\mathbf{y}) = sT(\mathbf{x}) + tT(\mathbf{y})$.

A5 Not linear

A6 Linear

A7 Not linear

A8 Linear

A9 $\mathbf{y} \in \operatorname{Range}(L)$. One choice is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

A10 $\mathbf{y} \in \operatorname{Range}(L)$. $\vec{x} = (2-t)1 + (3-t)x + tx^2$

A11 $\mathbf{y} \notin \operatorname{Range}(L)$.

A12 $\mathbf{y} \notin \operatorname{Range}(L)$.

Other correct answers are possible for problems A13–A18.

A13 A basis for Range(L) is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

A basis for Null(L) is $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$.

A14 A basis for Range(L) is $\{1 + x, x\}$.

A basis for Null(L) is $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$.

A15 A basis for Range(L) is $\{1 + x^2\}$.
A basis for Null(L) is $\{2 + x\}$.

A16 A basis for Range(L) is $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

A basis for Null(L) is the empty set.

A17 A basis for Range(tr) is $\{1\}$.

A basis for Null(tr) is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$.

A18 A basis for Range(L) is the standard basis for $M_{2\times 2}(\mathbb{R})$. A basis for Null($L$) is the empty set.

A19 They are not inverses of each other.

A20 They are inverses of each other.

A21 $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = a_3 + (2a_2 + a_3)x + (a_1 + a_3)x^2$

A22 $L\left(a + bx + cx^2\right) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

A23 One possible answer is $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ a \\ d \\ 0 \end{bmatrix}$

Section 4.6 Practice Problems

A1 $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}, [L(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

A2 $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 4 \\ -1 & -1 & 5 \end{bmatrix}, [L(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} 12 \\ -4 \\ -11 \end{bmatrix}$

A3 $[L]_{\mathcal{B}} = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$

A4 $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

A5 $[L]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, [L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

A6 $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ -7 & 5 \end{bmatrix}, [L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

A7 $[L]_{\mathcal{B}} = \begin{bmatrix} 11 & 16 \\ -4 & -3 \end{bmatrix}$, $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

A8 $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}$

A9 $[L]_{\mathcal{B}} = \begin{bmatrix} 4 & 4 & 1 \\ -3 & -4 & -4 \\ 2 & 3 & 4 \end{bmatrix}$, $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} -15 \\ 2 \\ 4 \end{bmatrix}$

A10 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, $[\text{refl}]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

A11 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, $[\text{perp}_{(1,-2)}]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

A12 $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, $[\text{proj}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A13 $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, $[\text{refl}]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A14 (a) $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (b) $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

(c) $L(1,2,4) = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$

A15 (a) $\begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$ (b) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & -3 \end{bmatrix}$

(c) $L(-1,7,6) = \begin{bmatrix} 11 \\ 6 \\ -14 \end{bmatrix}$

A16 $[L]_{\mathcal{B}} = \begin{bmatrix} 11 & 16 \\ -4 & -3 \end{bmatrix}$ **A17** $[L]_{\mathcal{B}} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$

A18 $[L]_{\mathcal{B}} = \begin{bmatrix} -40 & -118 \\ 18 & 52 \end{bmatrix}$ **A19** $[L]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$

A20 $[L]_{\mathcal{B}} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ **A21** $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

A22 $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -3 & -2 \end{bmatrix}$ **A23** $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

A24 $[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ **A25** $[T]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix}$

Section 4.7 Practice Problems

A1 One-to-one, onto

A2 One-to-one, not onto

A3 Not one-to-one, onto

A4 Not one-to-one, not onto

A5 One-to-one, onto

A6 Not one-to-one, not onto

A7 Not one-to-one, not onto

A8 Not one-to-one, not onto

A9 One-to-one, onto

A10 One-to-one, not onto

A11 Not one-to-one, not onto

A12 One-to-one, onto

A13 They are not isomorphic.

A14 Define $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

A15 Define $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

A16 Define $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

A17 If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for the plane P, then $L(a\vec{v}_1 + b\vec{v}_2) = \begin{bmatrix} a \\ b \end{bmatrix}$.

A18 $L(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}) = a_1 + a_2 x$

A19 They are not isomorphic.

A20 They are not isomorphic.

A21 $L((x-1)(a_1 x + a_0)) = \begin{bmatrix} a_2 & a_1 \\ 0 & a_0 \end{bmatrix}$

Chapter 4 Quiz

E1 It is a vector space.

E2 It is a vector space.

E3 It is not a vector space.

E4 It is a vector space.

E5 $\mathbb{S}$ is not a subspace.

E6 $\left\{\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}\right\}$ is a basis for $\mathbb{S}$.

E7 $\left\{1 + x^2\right\}$ is a basis for $\mathbb{S}$.

E8 $\left\{1 + x^2, 2 + x\right\}$ is a basis for $\mathbb{S}$.

E9 Not a basis since it is linearly dependent.

E10 Not a basis since it is linearly dependent.

E11 Not a basis since it does not span $M_{2\times 2}(\mathbb{R})$.

E12 (a) $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\3\\1\\0\end{bmatrix}\right\}$, $\dim \mathbb{S} = 3$.

(b) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\-1\\1\end{bmatrix}$

E13 $Q = \begin{bmatrix}-5/3 & 5/3\\ 4/3 & -1/3\end{bmatrix}$, $P = \begin{bmatrix}1/5 & 1\\ 4/5 & 1\end{bmatrix}$

E14 (a) $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

(c) $[L]_{\mathcal{B}} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$

(d) $[L]_{\mathcal{S}} = P[L]_{\mathcal{B}}P^{-1} = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$

E15 $[L]_{\mathcal{B}} = \begin{bmatrix}0 & 2/3 & 11/3\\ -1 & 2/3 & -10/3\\ 0 & 1/3 & 1/3\end{bmatrix}$

E16 We have $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k \in \text{Null}(L)$ since

$$\mathbf{0} = L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k) = L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k)$$

Thus, $t_1\mathbf{v}_1+\cdots+t_k\mathbf{v}_k = \mathbf{0}$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, this gives $t_1 = \cdots = t_k = 0$. Hence, $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is also linearly independent.

E17 FALSE. $\mathbb{R}^n$ is an n-dimensional subspace of $\mathbb{R}^n$.

E18 TRUE. The dimension of $P_2(\mathbb{R})$ is 3, so a set of 4 polynomials in $P_2(\mathbb{R})$ must be linearly dependent.

E19 FALSE. The number of components in a coordinate vector is the number of vectors in the basis. So, if $\mathcal{B}$ is a basis for a 4-dimensional subspace, then the $\mathcal{B}$-coordinate vector would have only 4 components.

E20 TRUE. Both ranks equal the dimension of Range(L).

E21 FALSE. If $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is a linear mapping, then the range of L is a subspace of $P_2(\mathbb{R})$, but the column space of $[L]_{\mathcal{B}}$ is a subspace of $\mathbb{R}^3$. Hence, they cannot equal.

E22 FALSE. The mapping $L : \mathbb{R} \to \mathbb{R}^2$ given by $L(x_1) = (x_1, 0)$ is one-to-one, but $\dim \mathbb{R} \neq \dim \mathbb{R}^2$.

CHAPTER 5

Section 5.1 Practice Problems

A1 38 **A2** −5 **A3** 0
A4 0 **A5** 64 **A6** 17
A7 0 **A8** 0 **A9** −20
A10 3 **A11** 3 **A12** −5
A13 5 **A14** 196 **A15** −136

A16 −26 **A17** 98 **A18** 0
A19 20 **A20** −72 **A21** 72
A22 18 **A23** −90 **A24** 48
A25 18 **A26** 76 **A27** 420
A28 −1 **A29** 1 **A30** −3
A31 1

Section 5.2 Practice Problems

A1 $\det A = 30$, so A is invertible.

A2 $\det A = 1$, so A is invertible.

A3 $\det A = 8$, so A is invertible.

A4 $\det A = 0$, so A is not invertible.

A5 $\det A = -1120$, so A is invertible.

A6 $\det A = 12$, so A is invertible.

A7 0 **A8** 0 **A9** 36

A10 6 **A11** 14 **A12** -12

A13 -5 **A14** 716 **A15** -65

A16 32 **A17** 0 **A18** 448

A19 50

A20 $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$

A21 $\det A = 4-2p$, so A is invertible for all $p \neq 2$.

A22 $\det A = p^2+2 > 0$, so A is invertible for all p.

A23 $\det A = p$, so A is invertible for all $p \neq 0$.

A24 $\det A = 6p+32$, so A is invertible for all $p \neq -\frac{16}{3}$.

A25 $\det A = 3p-14$, so A is invertible for all $p \neq \frac{14}{3}$.

A26 $\det A = -5p-20$, so A is invertible for all $p \neq -4$.

A27 $\det A = 2p-116$, so A is invertible for all $p \neq 58$.

A28 $\det A = 0$, so there is no value of p.

A29 $\det A = 13$, $\det B = 14$, $\det AB = 182$

A30 $\det A = -2$, $\det B = 56$, $\det AB = -112$

A31 $\lambda = 1, 7$ **A32** $\lambda = -\sqrt{7}, \sqrt{7}$

A33 $\lambda = -1, 0, 2$ **A34** $\lambda = 0, 2, 6$

A35 $\lambda = -1, 1$ **A36** $\lambda = 2, 8$

A37 Since rA is the matrix where each of the n rows of A must been multiplied by r, we can use Theorem 5.2.1 n times to get $\det(rA) = r^n \det A$.

A38 We have AA^{-1} is I, so

$$1 = \det I = \det AA^{-1} = (\det A)(\det A^{-1})$$

by Theorem 5.2.7. Since $\det A \neq 0$, we get $\det A^{-1} = \dfrac{1}{\det A}$.

A39 By Theorem 5.2.7., we have

$$1 = \det I = \det A^3 = (\det A)^3$$

Taking cube roots of both sides gives $\det A = 1$.

Section 5.3 Practice Problems

A1 $\begin{bmatrix} 8 \\ -10 \\ 5 \\ 1 \end{bmatrix}$ **A2** $\begin{bmatrix} 9 \\ -10 \\ 4 \\ -6 \end{bmatrix}$ **A3** $\begin{bmatrix} 4 \\ -2 \\ 2 \\ -2 \end{bmatrix}$ **A4** $\begin{bmatrix} 4 \\ -9 \\ -3 \\ -1 \end{bmatrix}$

A5 (a) $\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & -5 \end{bmatrix}$

(b) $A \operatorname{adj}(A) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $\det A = -2$,

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & 0 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & -5 \end{bmatrix}$$

A6 (a) We get $\operatorname{adj}(A) = \begin{bmatrix} 7 & -14 & -13 \\ 0 & 7 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A \operatorname{adj}(A) = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, $\det A = 7$,

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 7 & -14 & -13 \\ 0 & 7 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

A7 (a) $\operatorname{adj}(A) = \begin{bmatrix} ce & 0 & -dc \\ 0 & ae-bd & 0 \\ -cb & 0 & ac \end{bmatrix}$

(b) $A \operatorname{adj}(A) = \begin{bmatrix} ace-bcd & 0 & 0 \\ 0 & ace-bcd & 0 \\ 0 & 0 & ace-bcd \end{bmatrix}$,

$\det A = ace - bcd$,

$$A^{-1} = \frac{1}{ace-bcd}\begin{bmatrix} ce & 0 & -dc \\ 0 & ae-bd & 0 \\ -cb & 0 & ac \end{bmatrix}$$

A8 (a) $\operatorname{adj}(A) = \begin{bmatrix} b & -2b & -13 \\ 0 & ab & 5a \\ 0 & 0 & a \end{bmatrix}$

(b) $A\operatorname{adj}(A) = \begin{bmatrix} ab & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & ab \end{bmatrix}$, $\det A = ab$,

$A^{-1} = \frac{1}{ab}\begin{bmatrix} b & -2b & -13 \\ 0 & ab & 5a \\ 0 & 0 & a \end{bmatrix}$

A9 (a) $\operatorname{adj}(A) = \begin{bmatrix} -1 & 3+t & -3 \\ 5 & 2-3t & -2-2t \\ -2 & -11 & -6 \end{bmatrix}$

(b) $A\operatorname{adj}(A) = \begin{bmatrix} -2t-17 & 0 & 0 \\ 0 & -2t-17 & 0 \\ 0 & 0 & -2t-17 \end{bmatrix}$,

$\det A = -2t - 17$,

$A^{-1} = \frac{1}{-2t-17}\begin{bmatrix} -1 & 3+t & -3 \\ 5 & 2-3t & -2-2t \\ -2 & -11 & -6 \end{bmatrix}$

A10 (a) $\operatorname{adj}(A) = \begin{bmatrix} -1 & t & -t^2+2 \\ 1 & -t & t-3 \\ -t+1 & -1 & 3t-2 \end{bmatrix}$

(b) $A\operatorname{adj}(A) = \begin{bmatrix} -t^2+t-1 & 0 & 0 \\ 0 & -t^2+t-1 & 0 \\ 0 & 0 & -t^2+t-1 \end{bmatrix}$,

$\det A = -t^2 + t - 1$,

$A^{-1} = \frac{1}{-t^2+t-1}\begin{bmatrix} -1 & t & -t^2+2 \\ 1 & -t & t-3 \\ -t+1 & -1 & 3t-2 \end{bmatrix}$

A11 $\frac{1}{-2}\begin{bmatrix} 10 & -3 \\ -4 & 1 \end{bmatrix}$

A12 $\frac{1}{7}\begin{bmatrix} -1 & 5 \\ -2 & 3 \end{bmatrix}$

A13 $\frac{1}{8}\begin{bmatrix} -3 & 21 & 11 \\ -1 & 7 & 5 \\ 3 & -13 & -7 \end{bmatrix}$

A14 $\frac{1}{4}\begin{bmatrix} -1 & -8 & -4 \\ -2 & -12 & -4 \\ -2 & -8 & -4 \end{bmatrix}$

A15 $\begin{bmatrix} 3 & -1 & -1 \\ 2 & 0 & -1 \\ -7 & 2 & 3 \end{bmatrix}$

A16 $\frac{1}{4}\begin{bmatrix} -1 & -2 & -2 \\ -8 & -12 & -8 \\ -4 & -4 & -4 \end{bmatrix}$

A17 $\frac{1}{-56}\begin{bmatrix} 4 & -16 & 2 \\ -12 & 20 & 8 \\ 0 & -28 & -14 \end{bmatrix}$

A18 $\vec{x} = \begin{bmatrix} 51/19 \\ -4/19 \end{bmatrix}$

A19 $\vec{x} = \begin{bmatrix} 7/5 \\ -11/15 \end{bmatrix}$

A20 $\vec{x} = \begin{bmatrix} 21/11 \\ -26/11 \\ 2 \end{bmatrix}$

A21 $\vec{x} = \begin{bmatrix} 3/5 \\ -12/5 \\ -8/5 \end{bmatrix}$

A22 $\vec{x} = \begin{bmatrix} -5/9 \\ -8/3 \\ 23/9 \end{bmatrix}$

A23 $\vec{x} = \begin{bmatrix} 3 \\ 0 \\ -5/3 \end{bmatrix}$

Section 5.4 Practice Problems

A1 (a) $\text{Area}(\vec{u}, \vec{v}) = 9$ (b) $\text{Area}(A\vec{u}, A\vec{v}) = 36$

A2 (a) $\text{Area}(\vec{u}, \vec{v}) = 18$ (b) $\text{Area}(A\vec{u}, A\vec{v}) = 0$

A3 (a) $\text{Area}(\vec{u}, \vec{v}) = 11$ (b) $\text{Area}(A\vec{u}, A\vec{v}) = 286$

A4 $A\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\text{Area}(A\vec{u}, A\vec{v}) = 8$

A5 (a) 63 (b) 42 (c) 2646

A6 (a) 41 (b) 78 (c) 3198

A7 Area = 2

A8 Collinear

A9 Area = 5/2

A10 Area = 22

A11 Collinear

A12 (a) 5 (b) 245

A13 The n-volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_n$ is $\left|\det\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\right|$. Since adding a multiple of one column to another does not change the determinant, we get that

$$\left|\det\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\right| = \left|\det\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n + t\vec{v}_1 \end{bmatrix}\right|$$

which is the volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_{n-1}, \vec{v}_n + t\vec{v}_1$.

Chapter 5 Quiz

E1 The determinant is -13, thus the matrix is invertible.

E2 The determinant is 18, thus the matrix is invertible.

E3 The determinant is 0, thus the matrix is not invertible.

E4 The determinant is -12, thus the matrix is invertible.

E5 The determinant is 180, thus the matrix is invertible.

E6 The determinant is 120, thus the matrix is invertible.

E7 The matrix is invertible for all $k \neq -\frac{7}{8} \pm \frac{\sqrt{145}}{8}$.

E8 $\det B = 21$

E9 $\det C = 7$

E10 $\det(2A) = 224$

E11 $\det A^{-1} = \frac{1}{7}$

E12 $\det(A^T A) = 49$

E13 $\lambda = -1, 0, 5$

E14 (a) $\operatorname{adj}(A) = \begin{bmatrix} 2 & -6 & 2 \\ -4 & 6 & -1 \\ 2 & -6 & -1 \end{bmatrix}$

(b) $A \operatorname{adj}(A) = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$, $\det A = -6$

(c) $(A^{-1})_{31} = -\frac{1}{3}$.

E15 $x_2 = -\frac{1}{2}$

E16 (a) 33 (b) 792

E17 $\vec{x} = \begin{bmatrix} -8 \\ 7 \\ -1 \\ -2 \end{bmatrix}$

E18 The points are collinear.

E19 The area of the triangle is 7.

E20 The area of the triangle is 1/2.

CHAPTER 6

Section 6.1 Practice Problems

A1 $\vec{v}_1$ is an eigenvector corresponding to the eigenvalue $\lambda = 7$, and $\vec{v}_2$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

A2 $\vec{v}_1$ is not an eigenvector, but $\vec{v}_2$ is an eigenvector corresponding to the eigenvalue $\lambda = -4$.

A3 $\vec{v}_1$ is not an eigenvector, but $\vec{v}_2$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$.

A4 $\vec{v}_1$ is an eigenvector corresponding to the eigenvalue $\lambda = -6$, but $\vec{v}_2$ is not an eigenvector.

A5 $\vec{v}_1$ is not an eigenvector, but $\vec{v}_2$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

A6 $\lambda_1 = 2, \lambda_2 = 3$. $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$,

$E_{\lambda_2} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

A7 $\lambda = 1$. $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

A8 $\lambda_1 = 2, \lambda_2 = 3$. $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$,

$E_{\lambda_2} = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

A9 $\lambda_1 = -1, \lambda_2 = 4$. $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\}$,

$E_{\lambda_2} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

A10 $\lambda_1 = 5, \lambda_2 = -2$. $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$,

$E_{\lambda_2} = \operatorname{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

A11 $\lambda_1 = 0, \lambda_2 = -3$. $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$,

$E_{\lambda_2} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

A12 $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. $E_{\lambda_1} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$,

$E_{\lambda_2} = \left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right\}$, $E_{\lambda_3} = \left\{\begin{bmatrix} 13 \\ 7 \\ 1 \end{bmatrix}\right\}$.

A13 $\lambda_1 = -2, \lambda_2 = 2, \lambda_3 = -1$. $E_{\lambda_1} = \text{Span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$, $E_{\lambda_2} = \left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$, $E_{\lambda_3} = \left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

A14 $\lambda_1 = 2$. $E_{\lambda_1} = \left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

A15 $\lambda_1 = 1, \lambda_2 = -1$. $E_{\lambda_1} = \left\{\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}\right\}$, $E_{\lambda_2} = \left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

A16 $\lambda_1 = 2$ has algebraic multiplicity 1 and geometric multiplicity 1. $\lambda_2 = 3$ has algebraic multiplicity 1 and geometric multiplicity 1.

A17 $\lambda_1 = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

A18 $\lambda_1 = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

A19 $\lambda_1 = 1$ has algebraic multiplicity 1 and geometric multiplicity 1. $\lambda_2 = 2$ has algebraic multiplicity 1 and geometric multiplicity 1. $\lambda_3 = -2$ has algebraic multiplicity 1 and geometric multiplicity 1.

A20 $\lambda_1 = 0$ has algebraic multiplicity 2 and geometric multiplicity 2. $\lambda_2 = 6$ has algebraic multiplicity 1 and geometric multiplicity 1.

A21 $\lambda_1 = 2$ has algebraic multiplicity 2 and geometric multiplicity 2. $\lambda_2 = 5$ has algebraic multiplicity 1 and geometric multiplicity 1.

A22 $\lambda_1 = 1$ has algebraic multiplicity 1 and geometric multiplicity 1. $\lambda_2 = 3$ has algebraic multiplicity 1 and geometric multiplicity 1. $\lambda_3 = -1$ has algebraic multiplicity 1 and geometric multiplicity 1.

A23 $\lambda = 6$ **A24** $\lambda = 6$

A25 $\lambda = 8$ **A26** $\lambda = 10$

Section 6.2 Practice Problems

A1 $P^{-1}AP = \begin{bmatrix} 14 & 0 \\ 0 & -7 \end{bmatrix}$

A2 P does not diagonalize A.

A3 P does not diagonalize A.

A4 $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$

A5 $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

A6 $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

A7 $P = \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix}, D = \begin{bmatrix} 7 & 0 \\ 0 & -8 \end{bmatrix}$

A8 A is not diagonalizable.

A9 $P = \begin{bmatrix} -1 & 9 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} -8 & 0 \\ 0 & 5 \end{bmatrix}$

A10 $P = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$

A11 $P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}$

A12 A is not diagonalizable.

A13 A is not diagonalizable.

A14 $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}$

A15 $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$

A16 $P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A17 A is not diagonalizable.

A18 A is not diagonalizable.

A19 $P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

A20 A is not diagonalizable.

A21 $P = \begin{bmatrix} -2 & 1 & 1 \\ -2 & -2 & 1 \\ 5 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

A22 $P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A23 A is not diagonalizable.

Section 6.3 Practice Problems

A1 $A^3 = \begin{bmatrix} 90 & -63 \\ -126 & 153 \end{bmatrix}$

A2 $A^{100} = \frac{1}{3}\begin{bmatrix} -5 \cdot 2^{100} + 8 & -5 \cdot 2^{101} + 10 \\ 2^{102} - 4 & 2^{103} - 5 \end{bmatrix}$

A3 $A^{100} = -\frac{1}{5}\begin{bmatrix} -6 + 4^{100} & -2 + 2 \cdot 4^{100} \\ 3 - 3 \cdot 4^{100} & 1 - 6 \cdot 4^{100} \end{bmatrix}$

A4 $A^{200} = \frac{1}{5}\begin{bmatrix} 6 - 4^{200} & -2 + 2 \cdot 4^{200} \\ 3 - 3 \cdot 4^{200} & -1 + 6 \cdot 4^{200} \end{bmatrix}$

A5 $A^{200} = \begin{bmatrix} -3(2^{200}) + 4(3^{200}) & 3(2^{201}) - 2(3^{201}) \\ -2^{201} + 2(3^{200}) & 2^{202} - 3^{201} \end{bmatrix}$

A6 $A^{100} = I$

A7 $A^{100} = \begin{bmatrix} 5 & -1 & 2 \\ 4 & 0 & 2 \\ -8 & 2 & -3 \end{bmatrix}$

A8 $A^{100} = \begin{bmatrix} -1 + 2^{101} & 1 - 2^{100} & 0 \\ -2 + 2^{101} & 2 - 2^{100} & 0 \\ 2 - 2^{101} & -1 + 2^{100} & 1 \end{bmatrix}$

A9 Not a Markov matrix.

A10 The fixed state is $\begin{bmatrix} 6/13 \\ 7/13 \end{bmatrix}$.

A11 The fixed state is $\begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix}$.

A12 The fixed state is $\begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix}$.

A13 Not a Markov matrix.

A14 The fixed state is $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

A15 (a) In the long run, 25% of the population will be rural dwellers and 75% will be urban dwellers.
(b) After five decades, approximately 33% of the population will be rural dwellers and 67% will be urban dwellers.

A16 $T = \frac{1}{10}\begin{bmatrix} 8 & 3 & 3 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{bmatrix}$. In the long run, 60% of the cars will be at the airport, 20% of the cars will be at the train station, and 20% of the cars will be at the city centre.

A17 (a) $\begin{bmatrix} J_{m+1} \\ T_{m+1} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} J_m \\ T_m \end{bmatrix}$. In the long run, $\frac{8}{15} =$ 53% of the customers will deal with Johnson and $\frac{7}{15} =$ 47% will deal with Thomson.
(b) $T^m \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 8 \\ 7 \end{bmatrix} + \frac{1}{15}(4.25)(-0.5)^m \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

A18 $\begin{bmatrix} y \\ z \end{bmatrix} = ae^{-5t}\begin{bmatrix} -1 \\ 4 \end{bmatrix} + be^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A19 $\begin{bmatrix} y \\ z \end{bmatrix} = ae^{-0.5t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + be^{0.3t}\begin{bmatrix} 7 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A20 $\begin{bmatrix} y \\ z \end{bmatrix} = ae^{-9t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + be^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A21 $\begin{bmatrix} y \\ z \end{bmatrix} = ae^{0.6t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + be^{0.2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A22 $\begin{bmatrix} y \\ z \end{bmatrix} = a\begin{bmatrix} -1 \\ 2 \end{bmatrix} + be^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A23 $\begin{bmatrix} y \\ z \end{bmatrix} = ae^{8t}\begin{bmatrix} 6 \\ 5 \end{bmatrix} + be^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A24 $\begin{bmatrix} y \\ z \end{bmatrix} = ae^{5t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + be^{11t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A25 $\begin{bmatrix} y \\ z \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \end{bmatrix} + be^{t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad a, b \in \mathbb{R}$

A26 $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + be^{-2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + ce^{-4t}\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}, \quad a, b, c \in \mathbb{R}$

A27 $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ae^{3t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + be^{6t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + ce^{-9t}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad a, b, c \in \mathbb{R}$

Chapter 6 Quiz

E1 (a) (i) $\begin{bmatrix}3\\1\\0\end{bmatrix}$ is not an eigenvector of A.

(ii) $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is an eigenvector with eigenvalue 1.

(iii) $\begin{bmatrix}4\\1\\0\end{bmatrix}$ is an eigenvector with eigenvalue 1.

(iv) $\begin{bmatrix}2\\1\\-1\end{bmatrix}$ is an eigenvector with eigenvalue -1.

(b) $P = \begin{bmatrix}1 & 4 & 2\\0 & 1 & 1\\1 & 0 & -1\end{bmatrix}$, $D = \operatorname{diag}(1, 1, -1)$.

E2 $\lambda_1 = 2$ has algebraic multiplicity 2 and geometric multiplicity 2. $\lambda_2 = 4$ has algebraic multiplicity 1 and geometric multiplicity 1.

E3 $\lambda_1 = 1$ has algebraic multiplicity 3 and geometric multiplicity 1.

E4 $P = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$, $D = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$

E5 A is not diagonalizable since the geometric multiplicity of $\lambda_1 = 2$ is less than its algebraic multiplicity.

E6 $P = \begin{bmatrix}-1 & -1 & 2\\1 & 0 & 2\\0 & 2 & 1\end{bmatrix}$, $D = \begin{bmatrix}9 & 0 & 0\\0 & 9 & 0\\0 & 0 & 0\end{bmatrix}$

E7 $P = \begin{bmatrix}-1 & 1 & 1\\-3 & -1 & 1\\1 & 2 & 1\end{bmatrix}$, $D = \begin{bmatrix}0 & 0 & 0\\0 & -4 & 0\\0 & 0 & -2\end{bmatrix}$

E8 $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, so the dominant eigenvalue is $\lambda = 20$.

E9 $A^{100} = \begin{bmatrix}-3+4\cdot 2^{100} & -6+6\cdot 2^{100}\\2-2\cdot 2^{100} & 4-3\cdot 2^{100}\end{bmatrix}$

E10 (a) The solution space $A\vec{x} = \vec{0}$ is one-dimensional, since it is the eigenspace corresponding to the eigenvalue 0.

(b) 2 cannot be an eigenvalue, as we already have three eigenvalues for the 3×3 matrix A. Hence, there are no vectors that satisfy $A\vec{x} = 2\vec{x}$, so the solution space is zero dimensional in this case.

(c) Since $A\vec{x} = \vec{0}$ is one-dimensional, we could apply the Rank-Nullity Theorem, to get that

$$\operatorname{rank}(A) = n - \operatorname{nullity}(A) = 3 - 1 = 2$$

E11 The invariant state is $\vec{x} = \begin{bmatrix}1/4\\1/4\\1/2\end{bmatrix}$.

E12 $\begin{bmatrix}y\\z\end{bmatrix} = ae^{-0.1t}\begin{bmatrix}-1\\1\end{bmatrix} + be^{0.4t}\begin{bmatrix}2\\3\end{bmatrix}, \quad a, b \in \mathbb{R}$

E13 Since A is invertible, 0 is not an eigenvalue of A (see Section 6.2 Problem C8). Then, if $A\vec{x} = \lambda\vec{x}$ we get $\vec{x} = \lambda A^{-1}\vec{x}$, so $A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$.

E14 Since A is diagonalizable, we have that there exists an invertible matrix P such that

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_1) = \lambda_1 I$$

Hence,

$$A = P(\lambda_1 I)P^{-1} = \lambda_1 PP^{-1} = \lambda_1 I$$

E15 Let λ be the eigenvalue corresponding to $\vec{v}$. Hence, $A\vec{v} = \lambda\vec{v}$. For any $t \neq 0$, we have that

$$A(t\vec{v}) = tA\vec{v} = t(\lambda\vec{v}) = \lambda(t\vec{v})$$

Thus, $t\vec{v}$ is also an eigenvector of A.

E16 If A is diagonalizable, then there exists an invertible matrix P such that $P^{-1}AP = D$ where D is diagonal. Since A and B are similar, there exists an invertible matrix Q such that $Q^{-1}BQ = A$. Thus, we have that

$$D = P^{-1}(Q^{-1}BQ)P = (P^{-1}Q^{-1})B(QP) = (QP)^{-1}B(QP)$$

Thus, by definition, B is also diagonalizable.

E17 This is false. The matrix $A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ is invertible, but not diagonalizable.

E18 This is false. The zero matrix is diagonal and hence diagonalizable, but is not invertible.

E19 This is true. It is Theorem 6.2.4.

E20 This is false. If $\vec{v} = \vec{0}$, then $\vec{v}$ is not an eigenvector.

E21 This is true. If the RREF of $A - \lambda I$ is I, then $(A - \lambda I)\vec{v} = \vec{0}$ has the unique solution $\vec{v} = \vec{0}$. Hence, by definition, λ is not an eigenvalue.

E22 This is true. If $L(\vec{v}) = \lambda\vec{v}$ where $\vec{v} \neq \vec{0}$, then we have $\lambda\vec{v} = L(\vec{v}) = [L]\vec{v}$ as required.

CHAPTER 7

Section 7.1 Practice Problems

A1 $\left\{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix}\right\}$

A2 The set is not orthogonal.

A3 $\left\{\begin{bmatrix} 4/\sqrt{18} \\ -1/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{17} \\ 0 \\ 4/\sqrt{17} \end{bmatrix}, \begin{bmatrix} 4/\sqrt{306} \\ 17/\sqrt{306} \\ 1/\sqrt{306} \end{bmatrix}\right\}$

A4 The set is not orthogonal.

A5 $\left\{\begin{bmatrix} 1/\sqrt{11} \\ 3/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{22} \\ -2/\sqrt{22} \\ -3/\sqrt{22} \end{bmatrix}\right\}$

A6 $\left\{\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}\right\}$

A7 $\left\{\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right\}$

A8 The set is not orthogonal.

A9 (a) $\vec{w} = \frac{8}{9}\vec{v}_1 + \frac{19}{9}\vec{v}_2 + \frac{5}{9}\vec{v}_3$

(b) $\vec{x} = \frac{1}{9}\vec{v}_1 + \frac{2}{9}\vec{v}_2 - \frac{2}{9}\vec{v}_3$

(c) $\vec{y} = -\frac{8}{9}\vec{v}_1 - \frac{1}{9}\vec{v}_2 + \frac{4}{9}\vec{v}_3$

A10 (a) $\vec{w} = \frac{8}{3}\vec{v}_1 + \frac{19}{3}\vec{v}_2 + \frac{5}{3}\vec{v}_3$

(b) $\vec{x} = 7\vec{v}_1 - 2\vec{v}_2 - 3\vec{v}_3$

(c) $\vec{y} = \frac{22}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_2 + \frac{4}{3}\vec{v}_3$

A11 (a) $\vec{x} = 2\vec{v}_1 - \frac{5}{2}\vec{v}_2 - \frac{5}{2}\vec{v}_3 - \frac{1}{2}\vec{v}_4$

(b) $\vec{y} = -\frac{5}{4}\vec{v}_1 + \frac{3}{4}\vec{v}_2 + \frac{7}{2}\vec{v}_3 + 3\vec{v}_4$

(c) $\vec{w} = \frac{5}{4}\vec{v}_1 + \frac{1}{4}\vec{v}_2 - \frac{3}{2}\vec{v}_3 + 0\vec{v}_4$

A12 It is orthogonal.

A13 It is not orthogonal. The columns of the matrix are not orthogonal.

A14 It is not orthogonal. The columns are not unit vectors.

A15 It is orthogonal.

A16 It is not orthogonal. The third column is not orthogonal to the first or second column.

A17 It is orthogonal.

A18 $\left\{\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ -1/\sqrt{2} \end{bmatrix}\right\}$

A19 By Theorem 7.1.1, $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent. Thus, it is a basis for $\mathbb{R}^n$ by Theorem 2.3.6.

Section 7.2 Practice Problems

A1 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

A2 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 11/2 \\ 9 \\ 11/2 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$

A3 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 0 \\ \sqrt{5} \\ 0 \end{bmatrix}$

A4 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

A5 $\text{proj}_{\mathbb{S}}\, \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

A6 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} -5/7 \\ 6/7 \\ 32/7 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 12/7 \\ -6/7 \\ 3/7 \end{bmatrix}$

A7 $\text{proj}_{\mathbb{S}}\, \vec{x} = \begin{bmatrix} 2 \\ 9/2 \\ 5 \\ 9/2 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 0 \\ -3/2 \\ 0 \\ 3/2 \end{bmatrix}$

A8 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} 5/2 \\ 5 \\ 5/2 \\ 5 \end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix} -1/2 \\ -2 \\ 5/2 \\ 1 \end{bmatrix}$

A9 $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$

A10 $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$

A11 $\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$

A12 $\left\{\begin{bmatrix}-1\\-3\\-5\\12\end{bmatrix}\right\}$

A13 $\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\-3\\0\\1\end{bmatrix}\right\}$

A14 $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

A15 $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}\right\}$

A16 $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}9/14\\13/14\\-12/7\end{bmatrix}\right\}$

A17 $\left\{\begin{bmatrix}2\\1\\2\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\4\\-1\end{bmatrix}\right\}$

A18 $\left\{\begin{bmatrix}4\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\-3\end{bmatrix}\right\}$

A19 $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}$

A20 $\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-2\\3\\1\end{bmatrix},\begin{bmatrix}1\\-2\\-2\\1\end{bmatrix}\right\}$

A21 $\left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

A22 $\left\{\begin{bmatrix}2/3\\1/3\\-2/3\end{bmatrix},\begin{bmatrix}4/\sqrt{45}\\2/\sqrt{45}\\5/\sqrt{45}\end{bmatrix},\begin{bmatrix}1/\sqrt{5}\\-2/\sqrt{5}\\0\end{bmatrix}\right\}$

A23 $\left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\\1/\sqrt{3}\end{bmatrix},\begin{bmatrix}-2/\sqrt{15}\\1/\sqrt{15}\\3/\sqrt{15}\\1/\sqrt{15}\end{bmatrix},\begin{bmatrix}2/\sqrt{10}\\-1/\sqrt{10}\\2/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix}\right\}$

A24 $\left\{\begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\0\\1/\sqrt{3}\end{bmatrix},\begin{bmatrix}1/\sqrt{3}\\0\\-1/\sqrt{3}\\1/\sqrt{3}\\0\end{bmatrix},\begin{bmatrix}-1/\sqrt{15}\\3/\sqrt{15}\\1/\sqrt{15}\\2/\sqrt{15}\\0\end{bmatrix}\right\}$

A25 $\vec{y} = \frac{4}{13}\begin{bmatrix}2\\3\end{bmatrix}$

A26 $\vec{y} = \begin{bmatrix}5/2\\-1\\-3/2\end{bmatrix}$

A27 $\vec{y} = \begin{bmatrix}-1\\2\end{bmatrix}$

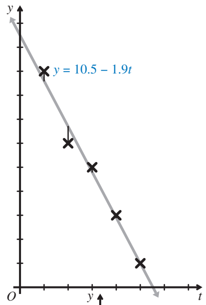
A28 $\vec{y} = \begin{bmatrix}9/2\\5/2\\6\end{bmatrix}$

A29 $\vec{y} = \begin{bmatrix}-1/5\\-1/5\\2/5\\3/5\end{bmatrix}$

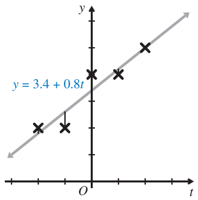
A30 $\vec{y} = \begin{bmatrix}10/7\\3/7\\-10/7\\1/7\end{bmatrix}$

Section 7.3 Practice Problems

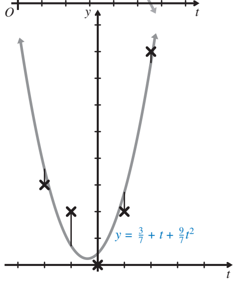
A1



A2



A3



A4 $y = 2 - \frac{3}{2}t$

A5 $y = -\frac{1}{3} - \frac{7}{6}t$

A6 $y = \frac{21}{10} + \frac{7}{10}t$

A7 $y = \frac{17}{10} + \frac{3}{5}t$

A8 $y = 1 - \frac{3}{2}t + \frac{3}{2}t^2$

A9 $y = -2 + \frac{1}{4}t + \frac{9}{8}t^2$

A10 $y = -2 + \frac{1}{4}t + \frac{9}{8}t^2$

A11 $y = \frac{5}{3} - \frac{3}{2}t - \frac{1}{6}t^2$

A12 $y = 1 - \frac{1}{2}t + t^2$

A13 $y = -\frac{3}{2}t + \frac{5}{2}t^2$

A14 $y = 2x - 3x^3$

A15 $y = \frac{17}{7} - \frac{5}{7}t^2$

A16 $y = -\frac{9}{5}t + \frac{7}{5}2^t$

A17 $\vec{x} = \begin{bmatrix}3\\1\end{bmatrix}$

A18 $\vec{x} = \begin{bmatrix}41/9\\3\end{bmatrix}$

A19 $\vec{x} = \begin{bmatrix}5/11\\17/11\end{bmatrix}$

A20 $\vec{x} = \begin{bmatrix}6/7\\10/7\end{bmatrix}$

A21 $\vec{x} = \begin{bmatrix}3/10\\0\\1/10\end{bmatrix}$

A22 $\vec{x} = \begin{bmatrix}-3\\16/11\end{bmatrix}$

A23 $\vec{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

A24 $\vec{x} = \begin{bmatrix}5/2\\0\end{bmatrix}$

Section 7.4 Practice Problems

A1 16

A2 1

A3 $\sqrt{2}$

A4 3

A5 $\sqrt{15}$

A6 -46

A7 -84

A8 $\sqrt{22}$

A9 $9\sqrt{59}$

A10 It is not positive definite.

A11 It is not bilinear.

A12 It is an inner product.

A13 It is not positive definite.

A14 It is not bilinear.

A15 (a) $\left\{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}\right\}$

(b) $\operatorname{proj}_{\mathbb{S}}\left(\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 4 & 5/3 \\ -2/3 & 7/3 \end{bmatrix}$

A16 (a) $\left\{\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$

(b) $\operatorname{proj}_{\mathbb{S}}\left(\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 4 & 5/3 \\ -2/3 & 7/3 \end{bmatrix}$

A17 (a) $\left\{\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\right\}$

(b) $\operatorname{proj}_{\mathbb{S}}\left(\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 14/9 & -2/3 \\ 4/9 & -2/9 \end{bmatrix}$

A18 (a) $\left\{\begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\right\}$

(b) $\operatorname{proj}_{\mathbb{S}}\left(\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 \\ 0 & -1 \end{bmatrix}$

A19 (a) $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}$ (b) $\operatorname{proj}_{\mathbb{S}}(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

A20 (a) $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right\}$ (b) $\operatorname{proj}_{\mathbb{S}}(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

A21 (a) $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ (b) $\operatorname{proj}_{\mathbb{S}}(\vec{e}_1) = \begin{bmatrix} 2/3 \\ 2/3 \\ 0 \end{bmatrix}$

A22 (a) $\mathcal{B} = \left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}\right\}$ (b) $\operatorname{proj}_{\mathbb{S}}(\vec{e}_1) = \begin{bmatrix} 8/9 \\ -2/9 \\ 2/9 \end{bmatrix}$

A23 $\operatorname{proj}_{\mathbb{S}}(\mathbf{p}) = \frac{0}{9}(1+2x-x^2) = 0, \operatorname{proj}_{\mathbb{S}}(\mathbf{q}) = \frac{11}{9}(1+2x-x^2)$

A24 $\operatorname{proj}_{\mathbb{S}}(\mathbf{p}) = \frac{2}{7}(1+3x), \operatorname{proj}_{\mathbb{S}}(\mathbf{q}) = \frac{-15}{21}(1+3x)$

A25 $\operatorname{proj}_{\mathbb{S}}(\mathbf{p}) = \frac{5}{3} + x, \operatorname{proj}_{\mathbb{S}}(\mathbf{q}) = \frac{1}{3}$

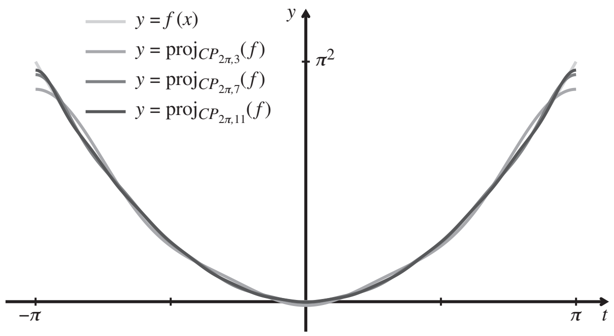
A26 $\operatorname{proj}_{\mathbb{S}}(\mathbf{p}) = 1 + \frac{1}{2}x - \frac{1}{2}x^2, \operatorname{proj}_{\mathbb{S}}(\mathbf{p}) = \frac{5}{2}x + \frac{5}{2}x^2$

A27 Since $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is orthogonal, we have $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$. Hence,

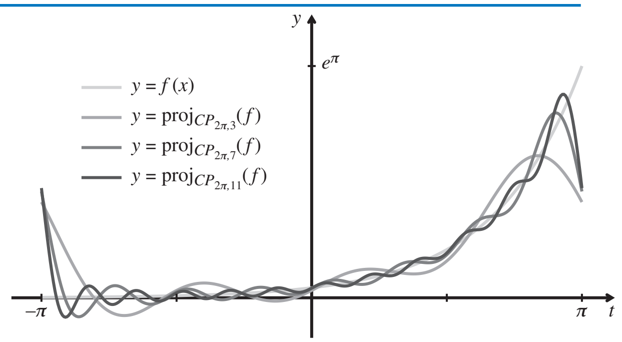
$$\begin{aligned} \|\mathbf{v}_1 + \cdots + \mathbf{v}_k\|^2 &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + \cdots + \langle \mathbf{v}_k, \mathbf{v}_k \rangle \\ &= \|\mathbf{v}_1\|^2 + \cdots + \|\mathbf{v}_k\|^2 \end{aligned}$$

Section 7.5 Practice Problems

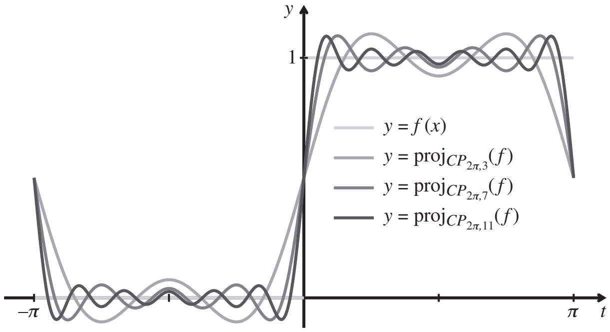
A1



A2



A3



A4 $CP_{2\pi,3}(f) = \frac{1}{2} - \frac{1}{2}\cos 2x$

A5 $CP_{2\pi,3}(f) = 1 + (2\pi^2 - 14)\sin x + \left(-\pi^2 + \frac{5}{2}\right)\sin 2x + \left(\frac{2}{3}\pi^2 - \frac{10}{9}\right)\sin 3x$

A6 $CP_{2\pi,3}(f) = \frac{\pi^2}{3} + 1 - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x$

A7 $CP_{2\pi,3}(f) = \pi - 2\sin x + \sin 2x - \frac{2}{3}\sin 3x$

A8 $CP_{2\pi,3}(f) = \frac{\pi}{4} - \frac{2}{\pi}\cos x + \sin x - \frac{1}{2}\sin 2x - \frac{2}{9\pi}\cos 3x + \frac{1}{3}\sin 3x$

A9 $CP_{2\pi,3}(f) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$

A10 $CP_{2\pi,3}(f) = \frac{1}{\pi} + \frac{2\sin 1}{\pi}\cos x + \frac{\sin 2}{\pi}\cos 2x + \frac{2\sin 3}{3\pi}\cos 3x$

Chapter 7 Quiz

E1 Neither. The vectors are not of unit length, and the first and third vectors are not orthogonal.

E2 Neither. The vectors are not orthogonal.

E3 The set is orthonormal.

E4 $\left\{\begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}-12\\-6\\10\end{bmatrix}\right\}$

E5 $\vec{x} = \frac{2}{3}\vec{v}_1 + \frac{5}{3}\vec{v}_2 + \frac{4}{3}\vec{v}_3$

E6 $\text{proj}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix}2\\2\\-1\end{bmatrix}$, $\text{perp}_{\mathbb{S}}(\vec{x}) = \begin{bmatrix}-1\\1\\0\end{bmatrix}$

E7 It is not positive definite.

E8 It is an inner product.

E9 (a) $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}\right\}$.

(b) $(5/4, -7/4, -5/4, 3/4)$

E10 $\left\{\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}4\\-1\\-1\end{bmatrix}\right\}$

E11 $y = \frac{39}{11} + \frac{20}{11}t$

E12 $y = 1 + \frac{7}{2}t^2$

E13 $\begin{bmatrix}2\\0\end{bmatrix}$

E14 $\begin{bmatrix}5/6\\-1/9\end{bmatrix}$

E15 PR is orthogonal since $(PR)^T(PR) = R^TP^TPR = R^TIR = R^TR = I$

E16 One counter example is $P = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$.

E17 Let $P = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}$. Since P has orthonormal columns, then we have

$$(P^TP)_{ij} = \vec{v}_i \cdot \vec{v}_j = 0, \quad \text{for } i \neq j (P^TP)_{ii} \quad = \vec{v}_i \cdot \vec{v}_i = 1$$

Therefore, $P^TP = I$.

E18 The statement may be false when the basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ is not an orthogonal basis for $\mathbb{S}$.

E19 We proved that $\text{perp}_{\mathbb{S}}(\vec{x}) \in \mathbb{S}^\perp$. By definition, $\text{proj}_{\mathbb{S}}(\vec{x}) \in \mathbb{S}$ and so, since $\vec{x} \in \mathbb{S}$ we also have that $\vec{x} - \text{proj}_{\mathbb{S}}(\vec{x}) \in \mathbb{S}$. Hence, $\text{perp}_{\mathbb{S}}(\vec{x})$ is in $\mathbb{S}$ and $\mathbb{S}^\perp$. So, by Theorem 7.2.2(2), $\text{perp}_{\mathbb{S}}(\vec{x}) = \vec{0}$.

E20 Assume that $\vec{z}$ is another vector in $\mathbb{S}$ such that $\|\vec{x}-\vec{z}\| < \|\vec{x} - \vec{v}\|$ for all $\vec{v} \in \mathbb{S}$, $\vec{v} \neq \vec{z}$. But, $\vec{y} \in \mathbb{S}$, so that would imply that $\|\vec{x}-\vec{z}\| < \|\vec{x}-\vec{y}\|$. But, since $\vec{z} \in \mathbb{S}$ we must also have that $\|\vec{x}-\vec{y}\| < \|\vec{x}-\vec{z}\|$. Hence, $\vec{z}$ cannot exist.

E21 (a) $\mathcal{B} = \{1, 2 + 3x - 3x^2\}$

(b) $\text{proj}_{\mathbb{S}}(1 + x + x^2) = -1 + \frac{1}{2}x + \frac{3}{2}x^2$

CHAPTER 8

Section 8.1 Practice Problems

A1 Symmetric

A2 Symmetric

A3 Not symmetric

A4 Symmetric

A5 $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$

A6 $P = \begin{bmatrix} -1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}, D = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}$

A7 $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$

A8 $P = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

A9 $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix},$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

A10 $P = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

A11 $P = \begin{bmatrix} 1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 0 & 5/\sqrt{45} & 2/3 \\ 2/\sqrt{5} & -2/\sqrt{45} & 1/3 \end{bmatrix}, D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$

A12 $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix},$

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A13 $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix},$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

A14 $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A15 $P = \begin{bmatrix} 1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix},$

$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

A16 $P = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{30} & -1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix},$

$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Section 8.2 Practice Problems

A1 $x_1^2 + 6x_1x_2 - x_2^2$

A2 $3x_1^2 - 4x_1x_2$

A3 $x_1^2 - 2x_2^2 + 6x_2x_3 - x_3^2$

A4 $-2x_1^2 + 2x_1x_2 + 2x_1x_3 + x_2^2 - 2x_2x_3$

A5 Positive definite

A6 Positive definite

A7 Indefinite

A8 Negative definite

A9 Positive definite

A10 Indefinite

A11 (a) $A = \begin{bmatrix} 1 & -3/2 \\ -3/2 & 1 \end{bmatrix}$

(b) $Q(\vec{x}) = \frac{5}{2}y_1^2 - \frac{1}{2}y_2^2, P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(c) $Q(\vec{x})$ is indefinite.

A12 (a) $A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$

(b) $Q(\vec{x}) = y_1^2 + 6y_2^2, P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

(c) $Q(\vec{x})$ is positive definite.

A13 (a) $A = \begin{bmatrix} -7 & 2 \\ 2 & -4 \end{bmatrix}$

(b) $Q(\vec{x}) = -8y_1^2 - 3y_2^2$, $P = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

(c) $Q(\vec{x})$ is negative definite.

A14 (a) $A = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix}$

(b) $Q(\vec{x}) = 1y_1^2 - 5y_2^2$, $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(c) $Q(\vec{x})$ is negative definite.

A15 (a) $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$

(b) $Q(\vec{x}) = 10y_1^2 - 5y_2^2$, $P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

(c) $Q(\vec{x})$ is indefinite.

A16 (a) $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & 3 \\ 3 & 3 & -3 \end{bmatrix}$

(b) $Q(\vec{x}) = 2y_1^2 + 3y_2^2 - 6y_3^2$,

$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$

(c) $Q(\vec{x})$ is indefinite.

A17 (a) $A = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -5 & -1 \\ 0 & -1 & -4 \end{bmatrix}$

(b) $Q(\vec{x}) = -3y_1^2 - 6y_2^2 - 4y_3^2$,

$P = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$

(c) $Q(\vec{x})$ is negative definite.

(a) $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

(b) $Q(\vec{x}) = 6y_1^2 + 3y_2^2 + 2y_3^2$,

$P = \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$

(c) $Q(\vec{x})$ is positive definite.

(a) $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

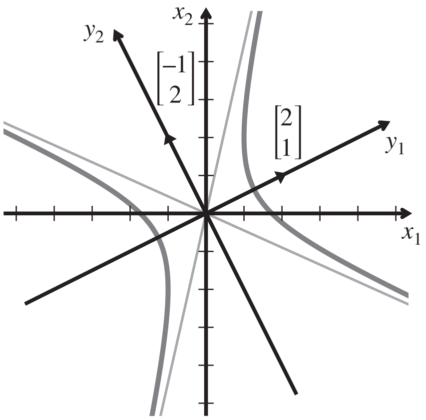
(b) $Q(\vec{x}) = 7y_1^2 + 7y_2^2 - 2y_3^2$,

$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$

(c) $Q(\vec{x})$ is indefinite.

Section 8.3 Practice Problems

A1



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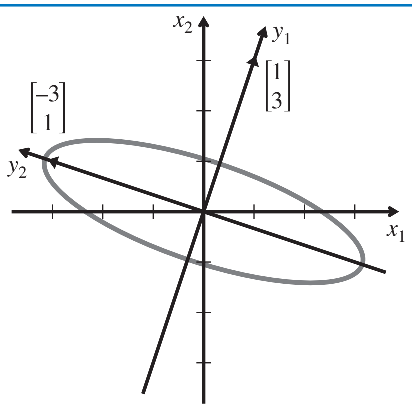

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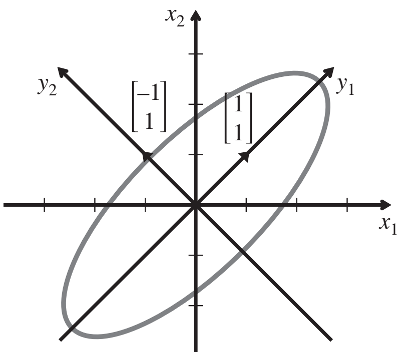

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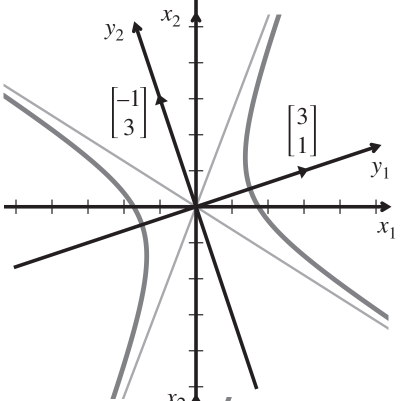
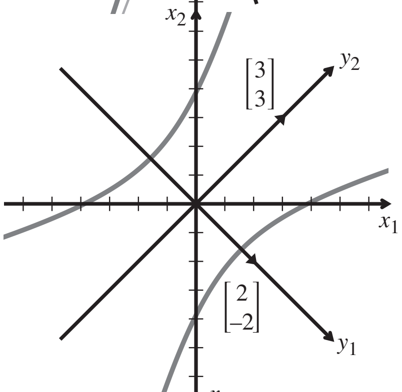
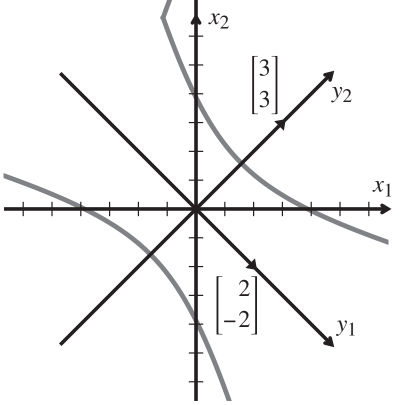


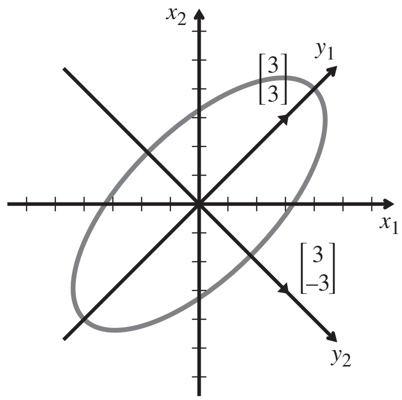
A5



A6



A7



A8 The graph of $\vec{x}^T A\vec{x} = 1$ is a set of two parallel lines. The graph of $\vec{x}^T A\vec{x} = -1$ is the empty set.

A9 The graph of $\vec{x}^T A\vec{x} = 1$ is a hyperbola. The graph of $\vec{x}^T A\vec{x} = -1$ is a hyperbola.

A10 The graph of $\vec{x}^T A\vec{x} = 1$ is a hyperboloid of two sheets. The graph of $\vec{x}^T A\vec{x} = -1$ is a hyperboloid of one sheet.

A11 The graph of $\vec{x}^T A\vec{x} = 1$ is a hyperbolic cylinder. The graph of $\vec{x}^T A\vec{x} = -1$ is a hyperbolic cylinder.

A12 The graph of $\vec{x}^T A\vec{x} = 1$ is a hyperboloid of one sheet. The graph of $\vec{x}^T A\vec{x} = -1$ is a hyperboloid of two sheets.

A13 The graph of $\vec{x}^T A\vec{x} = 1$ is an elliptic cylinder. The graph of $\vec{x}^T A\vec{x} = -1$ is the empty set.

A14 The graph of $Q(\vec{x}) = 0$ is two intersecting lines. The graph of $Q(\vec{x}) = -1$ is a hyperbola opening in the y_2-direction.

A15 The graph of $Q(\vec{x}) = 1$ is two parallel lines. The graph of $Q(\vec{x}) = 0$ is a plane. The graph of $Q(\vec{x}) = -1$ is the empty set.

A16 The graph of $Q(\vec{x}) = 1$ is an elliptic cylinder. The graph of $Q(\vec{x}) = 0$ is a line. The graph of $Q(\vec{x}) = -1$ is the empty set.

A17 The graph of $Q(\vec{x}) = 1$ is an ellipsoid. The graph of $Q(\vec{x}) = 0$ is the point $(0, 0, 0)$. The graph of $Q(\vec{x}) = -1$ is the empty set.

Section 8.5 Practice Problems

A1 $\sigma_1 = 2, \sigma_2 = 2$

A2 $\sigma_1 = 3, \sigma_2 = 1$

A3 $\sigma_1 = \sqrt{15}$

A4 $U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix},$
$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

A5 $U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix},$
$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

A6 $U = \begin{bmatrix} 11/\sqrt{95} & 3/\sqrt{26} & 1/\sqrt{30} \\ 7/\sqrt{195} & -4/\sqrt{26} & 2/\sqrt{30} \\ 5/\sqrt{195} & -1/\sqrt{26} & -5/\sqrt{30} \end{bmatrix},$
$\Sigma = \begin{bmatrix} \sqrt{15} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$

A7 $U = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix},$
$\Sigma = \begin{bmatrix} \sqrt{15} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

A8 $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix},$
$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

A9 $U = \begin{bmatrix} 0 & -1 & 0 \\ 2/\sqrt{5} & 0 & -1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix},$
$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

A10 $U = \begin{bmatrix} 1/\sqrt{3} & 0 & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix},$
$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}, V = I$

A11 $U = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$
$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

A12 $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix},$
$V = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

A13 $U = \begin{bmatrix} 3/\sqrt{90} & -1/\sqrt{15} & -1/\sqrt{3} & 3/\sqrt{18} \\ 4/\sqrt{90} & 2/\sqrt{15} & -1/\sqrt{3} & -2/\sqrt{18} \\ -4/\sqrt{90} & 3/\sqrt{15} & 0 & 2/\sqrt{18} \\ 7/\sqrt{90} & 1/\sqrt{15} & 1/\sqrt{3} & 1/\sqrt{18} \end{bmatrix},$
$\Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

A14 $A^+ = \begin{bmatrix} 1/3 & -1/3 \\ 1/6 & 1/3 \end{bmatrix}$

A15 $A^+ = \begin{bmatrix} 2/15 & 2/15 & 2/15 \\ -1/15 & -1/15 & -1/15 \end{bmatrix}$

A16 $A^+ = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \\ 1/4 & -1/4 \end{bmatrix}$

Chapter 8 Quiz

E1 $P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

E2 $2x_1^2 + 8x_1x_2 + 5x_2^2$

E3 $x_1^2 - 4x_1x_3 - 3x_2^2 + 8x_2x_3 + x_3^2$

E4 (a) $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

(b) $Q(\vec{x}) = 7y_1^2 + 3y_2^2$, $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(c) $Q(\vec{x})$ is positive definite.

(d) $Q(\vec{x}) = 1$ is an ellipse, and $Q(\vec{x}) = 0$ is the origin.

E5 (a) $A = \begin{bmatrix} 2 & -3 & -3 \\ -3 & -3 & 2 \\ -3 & 2 & -3 \end{bmatrix}$

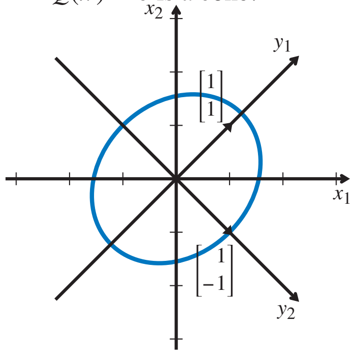
(b) $Q(\vec{x}) = -5y_1^2 + 5y_2^2 - 4y_3^2$,

$P = \begin{bmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

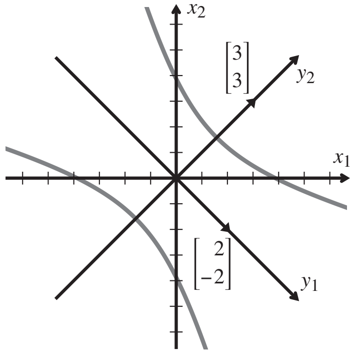
(c) $Q(\vec{x})$ is indefinite.

(d) $Q(\vec{x}) = 1$ is a hyperboloid of two sheets, and $Q(\vec{x}) = 0$ is a cone.

E6



E7



E8 $U = \begin{bmatrix} /1\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$,

$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

E9 $U = \begin{bmatrix} 5/\sqrt{35} & 0 & 2/\sqrt{14} \\ 1/\sqrt{35} & 3/\sqrt{10} & -1/\sqrt{14} \\ -3/\sqrt{35} & 1/\sqrt{10} & 3/\sqrt{14} \end{bmatrix}$,

$\Sigma = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

E10 $A^+ = V\Sigma^+U^T = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$

E11 Since A is positive definite, we have that

$$\langle \vec{x}, \vec{x} \rangle = \vec{x}^T A \vec{x} \geq 0$$

and $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$.
Since A is symmetric, we have that

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \vec{x}^T A \vec{y} = \vec{x} \cdot A\vec{y} = A\vec{y} \cdot \vec{x} = (A\vec{y})^T \vec{x} \\ &= \vec{y}^T A^T \vec{x} = \vec{y}^T A \vec{x} = \langle \vec{y}, \vec{x} \rangle \end{aligned}$$

For any $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

$$\begin{aligned} \langle \vec{x}, s\vec{y} + t\vec{z} \rangle &= \vec{x}^T A(s\vec{y} + t\vec{z}) = \vec{x}^T A(s\vec{y}) + \vec{x}^T A(t\vec{z}) \\ &= s\vec{x}^T A\vec{y} + t\vec{x}^T A\vec{z} = s\langle \vec{x}, \vec{y} \rangle + t\langle \vec{x}, \vec{z} \rangle \end{aligned}$$

Thus, $\langle \vec{x}, \vec{y} \rangle$ is an inner product on $\mathbb{R}^n$.

E12 Since A is a 4×4 symmetric matrix, there exists an orthogonal matrix P that diagonalizes A. Since the only eigenvalue of A is 3, we must have $P^TAP = 3I$. Then multiply on the left by P and on the right by P^T and we get

$$A = P(3I)P^T = 3PP^T = 3I$$

E13 $A = \begin{bmatrix} -1/13 & -18/13 \\ -18/13 & 14/13 \end{bmatrix}$

E14 If A is invertible, then $\text{rank}(A) = n$ by the Invertible Matrix Theorem. Hence, A has n non-zero singular values by Theorem 8.5.1. But, since an $n\times n$ matrix has exactly n singular values, the matrix A cannot have 0 as a singular value.
On the other hand, if A is not invertible, then $\text{rank}(A) < n$ by the Invertible Matrix Theorem. Hence, A has less than n non-zero singular values by Theorem 8.5.1. So, A has 0 as a singular value.

E15 The statement is true. We have $P^TAP = B$, so

$$B^2 = (P^TAP)(P^TAP) = P^TAIAP = P^TA^2P$$

E16 The statement is true. We have $P^TAP = B$, so

$$B^T = (P^TAP)^T = P^TA^T(P^T)^T = P^TAP = B$$

E17 The statement is false. Take $A = I$.

E18 This is true by the Principal Axis Theorem.

E19 The statement is false since $Q(\vec{0}) = 0$.

E20 The statement is false. Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

E21 The statement is false. Take $A = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$.

E22 If $A = U\Sigma V^T$, then

$$\begin{aligned} |\det A| &= |\det(U\Sigma V^T)| = |\det U||\det \Sigma||\det V^T| \\ &= 1(|\det \Sigma|)(1) = \sigma_1 \dots \sigma_n \end{aligned}$$

CHAPTER 9

Section 9.1 Practice Problems

A1 $\bar{z} = 3 + 5i, |z| = \sqrt{34}$

A2 $\bar{z} = 2 - 7i, |z| = \sqrt{53}$

A3 $\bar{z} = 4i, |z| = 4$

A4 $\bar{z} = -1 + 2i, |z| = \sqrt{5}$

A5 $\bar{z} = 2, |z| = 2$

A6 $\bar{z} = -3 - 2i, |z| = \sqrt{13}$

A7 $z = \sqrt{18}e^{-i\frac{3\pi}{4}}$, $\operatorname{Arg} z = -\frac{3\pi}{4}$

A8 $z = 2e^{-i\frac{\pi}{6}}$, $\operatorname{Arg} z = -\frac{\pi}{6}$

A9 $z = 2e^{i\frac{5\pi}{6}}$, $\operatorname{Arg} z = \frac{5\pi}{6}$

A10 $z = 4e^{-i\frac{2\pi}{3}}$, $\operatorname{Arg} z = -\frac{2\pi}{3}$

A11 $z = 3e^{i\pi}$, $\operatorname{Arg} z = \pi$

A12 $z = \sqrt{8}e^{i\frac{3\pi}{4}}$, $\operatorname{Arg} z = \frac{3\pi}{4}$

A13 $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

A14 $1 - \sqrt{3}i$

A15 $1 + \sqrt{3}i$

A16 $-\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$

A17 $\sqrt{3} - i$

A18 $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

A19 $5 + 7i$

A20 $-3 - 4i$

A21 $-7 + 2i$

A22 $-14 + 5i$

A23 $9 + 7i$

A24 $-10 - 10i$

A25 $2 + 25i$

A26 -2

A27 $\operatorname{Re}(z) = 3, \operatorname{Im}(z) = -6$

A28 $\operatorname{Re}(z) = 17, \operatorname{Im}(z) = -1$

A29 $\operatorname{Re}(z) = 24/37, \operatorname{Im}(z) = 4/37$

A30 $\operatorname{Re}(z) = 0, \operatorname{Im}(z) = 1$

A31 $\frac{2}{13} - \frac{3}{13}i$

A32 $-\frac{4}{13} - \frac{19}{13}i$

A33 $-\frac{2}{17} + \frac{25}{17}i$

A34 $z_1z_2 = 2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$, $\frac{z_1}{z_2} = \frac{2}{\sqrt{2}}\left(\cos\frac{-\pi}{12} + i\sin\frac{-\pi}{12}\right)$

A35 $z_1z_2 = 2\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$, $\frac{z_1}{z_2} = \sqrt{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

A36 $z_1z_2 = 4i$, $\frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

A37 $z_1z_2 = 6\sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$, $\frac{z_1}{z_2} = \frac{\sqrt{18}}{2}\left(\cos\frac{-\pi}{12} + i\sin\frac{-\pi}{12}\right)$

A38 -4

A39 $-54 - 54i$

A40 $-8 - 8\sqrt{3}i$

A41 $512(\sqrt{3} + i)$

A42 The roots are $\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)$, $0 \le k \le 4$.

A43 The roots are $2\left[\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right)\right]$, $0 \le k \le 3$.

A44 The roots are $2^{1/3}\left[\cos\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right) + i\sin\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right)\right]$, $0 \le k \le 2$.

A45 The roots are $17^{1/6}\left[\cos\left(\frac{\theta + 2k\pi}{3}\right) + i\sin\left(\frac{\theta + 2k\pi}{3}\right)\right]$, $0 \le k \le 2$, where $\theta = \arctan(4)$.

Section 9.2 Practice Problems

A1 $\mathbf{z} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$

A2 The system is inconsistent.

A3 $\mathbf{z} = \begin{bmatrix} 1-2i \\ 2+3i \end{bmatrix}$

A4 $\mathbf{z} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1+i \\ 1 \end{bmatrix}, t \in \mathbb{C}.$

A5 $\mathbf{z} = \begin{bmatrix} i \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ i \\ 1 \end{bmatrix}, t \in \mathbb{C}$

A6 $\mathbf{z} = \begin{bmatrix} -3i \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -i \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1+i \\ -1-i \\ 0 \\ 1 \end{bmatrix}, t, s \in \mathbb{C}$

A7 $\mathbf{z} = \begin{bmatrix} 2 \\ 1 \\ i \end{bmatrix}$

A8 The system is inconsistent.

A9 $\mathbf{z} = \begin{bmatrix} 1-i \\ \frac{4}{5}+\frac{2}{5}i \\ -\frac{1}{5}-\frac{3}{5}i \end{bmatrix}$

A10 $\mathbf{z} = \begin{bmatrix} 5+i \\ -2+2i \\ -i \\ 0 \end{bmatrix} + t\begin{bmatrix} -1+2i \\ 0 \\ -i \\ 1 \end{bmatrix}, t \in \mathbb{C}$

Section 9.3 Practice Problems

A1 $\begin{bmatrix} -5-3i \\ i \end{bmatrix}$

A2 $\begin{bmatrix} 5-3i \\ 7+8i \\ -1-9i \end{bmatrix}$

A3 $\begin{bmatrix} -10+4i \\ 4+6i \end{bmatrix}$

A4 $\begin{bmatrix} -4-3i \\ -1-7i \\ -12+i \end{bmatrix}$

A5 (a) $[L] = \begin{bmatrix} 1+2i & 3+i \\ 1 & 1-i \end{bmatrix}$

(b) $L(2+3i, 1-4i) = \begin{bmatrix} 3-4i \\ -1-2i \end{bmatrix}$

(c) A basis for Range(L) is $\left\{\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}\right\}$.

A basis for Null(L) is $\left\{\begin{bmatrix} -1+i \\ 1 \end{bmatrix}\right\}$.

A6 $\langle \mathbf{u}, \mathbf{v}\rangle = 2+5i, \langle \mathbf{v}, \mathbf{u}\rangle = 2-5i, \|\mathbf{u}\| = \sqrt{18}$, $\|\mathbf{v}\| = \sqrt{33}$

A7 $\langle \mathbf{u}, \mathbf{v}\rangle = -6i, \langle \mathbf{v}, \mathbf{u}\rangle = 6i, \|\mathbf{u}\| = \sqrt{22}, \|\mathbf{v}\| = \sqrt{20}$

A8 $\langle \mathbf{u}, \mathbf{v}\rangle = 3-i, \langle \mathbf{v}, \mathbf{u}\rangle = 3+i, \|\mathbf{u}\| = \sqrt{11}, \|\mathbf{v}\| = 2$

A9 $\langle \mathbf{u}, \mathbf{v}\rangle = 4+i, \langle \mathbf{v}, \mathbf{u}\rangle = 4-i, \|\mathbf{u}\| = \sqrt{15}, \|\mathbf{v}\| = \sqrt{5}$

A10 $(ZW)^* = \begin{bmatrix} -1-2i & 4-2i \\ -1+i & -1-i \end{bmatrix} = W^*Z^*$

A11 $(ZW)^* = \begin{bmatrix} -1-2i & 2-i \\ -1 & 2 \end{bmatrix} = W^*Z^*$

A12 Not unitary

A13 Unitary

A14 Unitary

A15 Unitary

A16 $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4i \\ -1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -i \\ 3/2 \end{bmatrix}\right\}$

A17 $\mathcal{B} = \left\{\begin{bmatrix} 1+i \\ 1-i \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1+i \\ 2i \end{bmatrix}, \begin{bmatrix} i \\ -1+i \\ 1-i \end{bmatrix}\right\}$

A18 $\text{proj}_{\mathbb{S}}(\mathbf{z}) = \begin{bmatrix} \frac{2}{3}+i \\ 2+\frac{1}{3}i \\ 3+\frac{4}{3}i \end{bmatrix}$

A19 $\text{proj}_{\mathbb{S}}(\mathbf{z}) = \begin{bmatrix} -1+\frac{3}{4}i \\ \frac{9}{4} \\ -\frac{7}{4}-\frac{5}{4}i \end{bmatrix}$

A20 $1-2i$

A21 $3+i$

A22 5

A23 $-2-4i$

A24 (a) We have

$$1 = \det I = \det(U^*U) = \det(U^*)\det U$$
$$= \overline{\det U}\det U = |\det U|^2$$

Therefore, $|\det U| = 1$.

(b) The matrix $U = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$ is unitary and $\det U = i$.

Section 9.4 Practice Problems

A1 $P = \begin{bmatrix} 1+i & 1+i \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

A2 $P = \begin{bmatrix} 3+4i & 3-4i \\ -5 & -5 \end{bmatrix}, D = \begin{bmatrix} 4i & 0 \\ 0 & -4i \end{bmatrix}$

A3 The matrix is not diagonalizable.

A4 $P = \begin{bmatrix} 1+i & -1-i \\ 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 1+2i & 0 \\ 0 & -1 \end{bmatrix}$

A5 If $\sin\theta = 0$, then $P = I$. Otherwise, $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ and

$$D = \begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}.$$

A6 $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & i & -i \\ 2 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}$

A7 The matrix is not diagonalizable.

A8 $P = \begin{bmatrix} 1 & -i & i \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$

A9 $P = \begin{bmatrix} 1-i & -i & -2i \\ 4 & 0 & -1+i \\ 0 & 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

A10 $P = \begin{bmatrix} 0 & 1+2i & 1-2i \\ 1 & 3+i & 3-i \\ 1 & 5 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$

A11 The matrix is not diagonalizable.

A12 $P = \begin{bmatrix} i & 1 & -1 \\ 0 & 1-i & 1+i \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Section 9.5 Practice Problems

A1 Normal

A2 Not normal

A3 Normal

A4 Not normal

A5 $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 2i & 0 \\ 0 & 2+2i \end{bmatrix}$

A6 $U = \begin{bmatrix} (1-i)/\sqrt{6} & (-1+i)/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ to $D = \begin{bmatrix} 3i & 0 \\ 0 & 6i \end{bmatrix}$

A7 $U = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 3+5i & 0 \\ 0 & 3-5i \end{bmatrix}$

A8 $U = \begin{bmatrix} (1+i)/\sqrt{6} & (-1-i)/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

A9 $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

A10 $U = \begin{bmatrix} (\sqrt{2}+i)/\sqrt{12} & (\sqrt{2}+i)/2 \\ -3/\sqrt{12} & 1/2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$

A11 $U = \begin{bmatrix} 0 & (1+i)/\sqrt{3} & (1+i)/\sqrt{6} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix},$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

A12 $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1+i)/\sqrt{3} & (1-i)/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix},$

$$D = \begin{bmatrix} i & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Chapter 9 Quiz

E1 $4+6i$

E2 $8+7i$

E3 $11-2i$

E4 $15+20i$

E5 $\frac{11}{5}-\frac{2}{5}i$

E6 $-\frac{3}{5}+\frac{4}{5}i$

E7 (a) $z_1 = 2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right), z_2 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(b) $z_1z_2 = 4\sqrt{2}\left(\cos\frac{-\pi}{12} + i\sin\frac{-\pi}{12}\right)$,

$\frac{z_1}{z_2} = \frac{1}{\sqrt{2}}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right)$

E8 $w_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, w_1 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

E9 The system is inconsistent.

E10 $\mathbf{z} = \begin{bmatrix} 2 \\ 1+2i \\ 0 \end{bmatrix} + t\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}, t \in \mathbb{C}$

E11 $\begin{bmatrix} 7-i \\ 3+5i \\ 9+3i \end{bmatrix}$

E12 $\begin{bmatrix} 3+i \\ -i \\ 2 \end{bmatrix}$

E13 $11-4i$

E14 $11+4i$

E15 $\sqrt{27}$

E16 $\frac{1}{15}\begin{bmatrix} 29-23i \\ 4+11i \\ 22-8i \end{bmatrix}$

E17 $\left\{\begin{bmatrix} 1 \\ i \\ i \end{bmatrix}, \begin{bmatrix} 2i \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2}+\frac{1}{2}i \\ \frac{1}{2}-\frac{1}{2}i \end{bmatrix}\right\}$

E18 $\text{proj}_{\mathbb{S}}(\mathbf{z}) = \begin{bmatrix} 1/2 \\ i/2 \\ 2-i \end{bmatrix}$

E19 Show $UU^* = I$.

E20 A is not diagonalizable.

E21 $P = \begin{bmatrix} i & -1 & 1-i \\ 1 & 0 & -1-i \\ 0 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2+i \end{bmatrix}$

E22 (a) A is Hermitian if and only if $k = -1$.

(b) $U = \begin{bmatrix} (3-i)/\sqrt{14} & (3-i)/\sqrt{35} \\ -2/\sqrt{14} & 5/\sqrt{35} \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$

Index

Index of Notations

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ISBN 978-0-13-468263-1